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MICROWAVE CONDUCTIVITY OF BOUNDED PLASMA*

BY

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ABSTRACT

An expression for microwave conductivity of bounded plasma was derived in the absence of magnetic field and then another similar expression in the presence of it, introducing the depolarization effect in each case. It was shown that this effect is to replace ω in specific inductive capacity for unbounded plasma by $\omega' = \omega\{1 - L(\omega_p/\omega)^2\}$. The results were as follows: (i) plasma resonance occurs at $\omega = \sqrt{L}\,\omega_p$, (ii) magneto-plasma resonance occurs at two positions of electron density at each magnetic field, both for the transversal wave ($\boldsymbol{k}_0 // \boldsymbol{B}$) and for the hybrid wave ($\boldsymbol{k}_0 \perp \boldsymbol{B}$).

1. Introduction

Microwave conductivities of plasma in the absence of or in the presence of magnetic field have been studied by various workers (**1, 2**), since Margenau's first report was published (**3**). The expressions for conductivity have so far been derived by solving the Boltzmann transport equation for distribution function of electrons moving under the Lorentz force. The experimental results on the interaction between plasma and microwave have been analysed in the theories (**4, 5**).

In these theories, the plasma was considered to extend through infinite space (unbounded plasma). It is not only theoretically interesting but also experimentally important to study the interaction of bounded plasma with microwave, since the experimental plasma generated in our laboratories is actually bounded.

Investigation on bounded plasmas with no magnetic field was made first by Tonks (**6**) and he reported "plasma resonance" which depends on both electron density and geometrical configuration of plasma. Thereafter, the plasma resonance for the cylindrical plasma has been studied theoretically and experimentally by many researchers (**7~11**). Moreover, the plasma resonance for plasma confined between parallel plates was theoretically investigated in detail by Wolff (**12**), by solving the Boltzmann transport equation for spatial distribution of electron density.

In studies on cyclotron resonance of electron and hole in semiconductor, Kittel *et al.* (**13**) considered the influence of boundaries, and introducing the

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depolarization effect they discussed "magnetoplasma resonance" which depends on both electron density and geometrical configuration.

In this report, the microwave conductivity of bounded plasma in the absence of as well as in the presence of magnetic field was studied by introducing the depolarization effect, similarly to the research on semiconductor by C. Kittel *et al.*

2. Depolarization effect of bounded plasma

For simplicity we consider the plasma of uniform electron density with axial symmetry whose axis, when it is in the presence of magnetic field ($\boldsymbol{B}$), should coincide with the direction of $\boldsymbol{B}$. Thus, a spherical or a circular cylindrical plasma is considered (see Fig. 1), in which the electric vector of electromagnetic wave lies and is uniform in transversal plane to $\boldsymbol{B}$, and the frequency is so high that ions can be assumed to be at rest. Then, the electron gas is subject

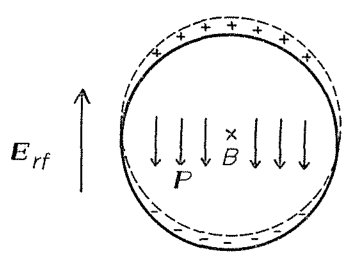

Fig. 1. Depolarization of bounded plasma.

to rigid-body displacement in transversal plane from equilibrium position and surface charges appear on boundary surfaces of bounded plasma. The induced field opposite to the external field exerts the force upon the electron gas, tending to return it to its original position. The effect of this restoring force of plasma may be taken into account by introducing a depolarizing field, $(-L\boldsymbol{P}/\varepsilon_0)$, which is uniform in transversal plane. Then, the internal electric field is:

$$\boldsymbol{E}_i = \boldsymbol{E} - L\boldsymbol{P}/\varepsilon_0\,, \tag{1}$$

where L is the depolarizing factor associated with the plasma shape and given by:

$$L = \begin{cases} 1/3 \text{ for spherical plasma,} \\ 1/2 \text{ for circular cylindrical plasma.} \end{cases}$$

The polarization $\boldsymbol{P}$ due to displacement of electrons is:

$$\boldsymbol{P} = eN\boldsymbol{x} = eN\int \boldsymbol{v}dt = jeN\boldsymbol{v}/\omega\,, \tag{2}$$

where $\boldsymbol{x}$, $\boldsymbol{v}$ are displacement and velocity of respective electron and N is electron density, taking the time dependence as $\exp(-j\omega t)$. Substituting (2) into (1), the internal field is:

$$\boldsymbol{E}_i = \boldsymbol{E} - j\frac{LeN}{\varepsilon_0\omega}\boldsymbol{v}\,. \tag{3}$$

The equation of motion for an electron in unbounded infinite plasma under the influence of constant magnetic field and rf electric field is given by:

$$m(-j\omega+\nu)\,\boldsymbol{v} = e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\,, \tag{4}$$

where ν is the effective collision frequency. On the other hand, for the bounded plasma the equation of motion is:

$$m\left\{-j\omega\left(1-L\frac{\omega_p^2}{\omega^2}\right)+\nu\right\}\boldsymbol{v} = e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\,, \tag{5}$$

where ω_p is the plasma frequency. Thus, the effect of depolarization is to replace ω by ω', which is:

$$\omega' = \omega\left\{1-L(\omega_p/\omega)^2\right\} = \omega(1-L\eta)\,. \tag{6}$$

It is noted that the uniformity of the depolarizing electric field in the transversal plane is necessary for the above conclusion, as it is satisfied in this case.

3. Plasma resonance in absence of magnetic field

By substituting ω' for ω in the specific inductive capacity K for the unbounded plasma, K for the bounded plasma is given by:

$$K = 1-\frac{1}{\{1-L(\omega_p/\omega)^2\}}\frac{(\omega_p/\omega)^2}{\{1-L(\omega_p/\omega)^2\}+j\nu/\omega}\,, \tag{7}$$

or using Brown's notations (**5**),

$$K = 1-\frac{1}{1-L\eta}\frac{\eta}{(1-L\eta)+j\beta}\,. \tag{8}$$

(*a*) *Refractive index n*

From (8), n at zero collision limit is given by:

$$n^2 = 1-\frac{\eta}{(1-L\eta)^2}\,. \tag{9}$$

In Fig. 2 n^2 is plotted against η for $L=1/2$. In this figure, the dotted curve presents the dispersion curve for unbounded plasma ($L=0$).

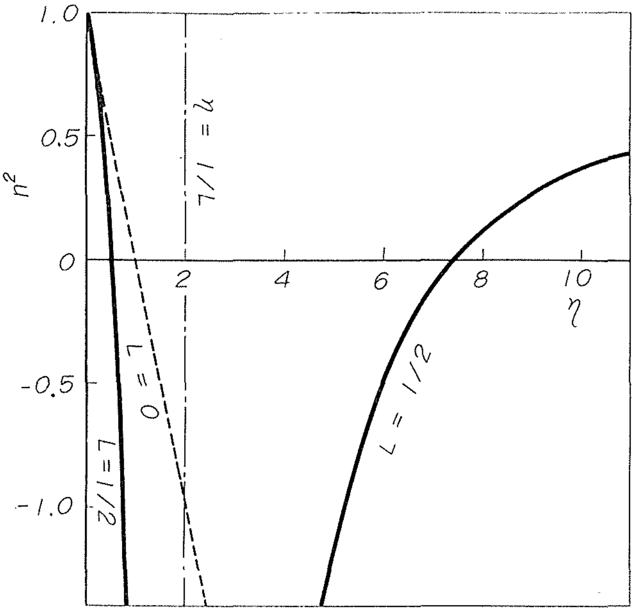

Fig. 2. Dispersion curves for bounded plasma.

(*b*) *Conductivity*

From (8), the real conductivity (σ_R) is given by:

$$\frac{\sigma_R}{\sigma_0} = \frac{\beta^2}{(1-L\eta)^2+\beta^2}, \tag{10}$$

where $\sigma_0 = e^2N/m\nu$ is the static conductivity. According to (10), plasma resonance occurs at

$$1-L\eta = 0 \quad \text{or} \quad \omega = \sqrt{L}\,\omega_p, \tag{11}$$

as shown in Fig. 3.

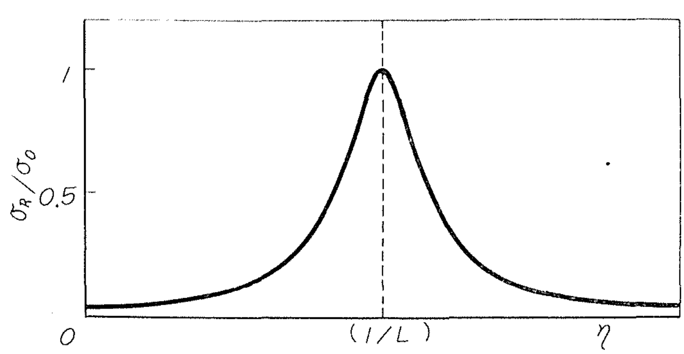

Fig. 3. Resonance absorption curve for bounded plasma.

Especially, the plasma resonance of circular cylindrical plasma ($L=1/2$) occurs at:

$$\omega = \omega_p/\sqrt{2}\ , \tag{11a}$$

which was called the transverse resonance of plasma column by Gould (**14**). This plasma resonance has been studied by various workers (**6**~**11**). Solving the electric fields inside and outside the circular cylindrical plasma, Herlofson obtained the result (11a) (**11**).

4. Magnetoplasma resonance for transversal wave ($k_0 /\!/ B$)

A linearly polarized wave incident on an infinite slab of plasma in a parallel magnetic field is split up into two circularly polarized waves, the right-handed and the left-handed waves with different phase velocities (**15, 16**). We consider a wave propagating through a circular cylindrical plasma, along the magnetic field, parallel to the cylinder axis.

(*a*) *Right-handed circularly polarized wave*

The specific inductive capacity K_- for the bounded plasma is:

$$K_- = 1 - \frac{1}{(1-L\eta)}\frac{\eta}{(1-L\eta-\gamma)+j\beta}\,. \tag{12}$$

Accordingly, n_- at zero collision limit is given by:

$$n_-^2 = 1 - \frac{\eta}{(1-L\eta)(1-L\eta-\gamma)}\,. \tag{13}$$

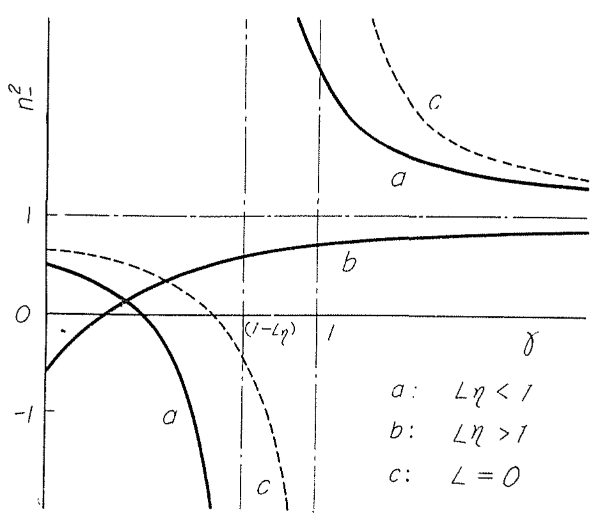

Fig. 4. Dispersion curves for right-handed wave.

n_-^2 is plotted against γ with η=const. in Fig. 4, which shows that n_-^2 increases to infinity at $\gamma=1-L\eta$ for the case of $L\eta<1$. As n_-^2 increases to infinity at $\gamma=1$ for infinite plasma ($L=0$), as is represented by a dotted curve, the effect of depolarization results in the displacement of n_-^2-axis by $(-L\eta)$.

On the other hand, σ_R is given by:

$$\frac{\sigma_R}{\sigma_0}=\frac{\beta^2}{(1-L\eta-\gamma)^2+\beta^2}. \tag{14}$$

From (14), the cyclotron resonance for the bounded plasma ($\sigma_{R\max}$), which is called magnetoplasma resonance, occurs at

$$\gamma_0=1-L\eta. \tag{15}$$

The resonance position γ_0 *versus* η is shown in Fig. 5, which implies that the magnetic field γ_0 at resonance decreases with increasing electron density η. For $\eta>1/L$, the resonance does not occur, as γ_0 becomes negative. The resonance absorption curve represented by (14) has the half-width $\Delta\gamma=2\beta$. It is to be noted that for the unbounded plasma, $\gamma_0=1$ independently of η.

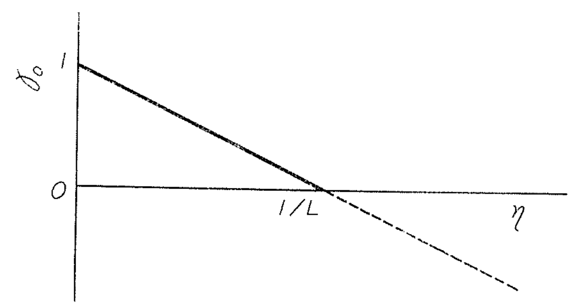

Fig. 5. γ_0 *versus* η for right-handed wave.

(*b*) *Left-handed circularly polarized wave*

Similarly K_+ of the left-handed wave is:

$$K_+=1-\frac{1}{(1-L\eta)}\frac{\eta}{(1-L\eta+\gamma)+j\beta}. \tag{16}$$

Accordingly, n_+ at zero collision limit is given by:

$$n_+^2=1-\frac{\eta}{(L\eta-1)(L\eta-1-\gamma)}. \tag{17}$$

n_+^2 *versus* γ with η=const. is plotted in Fig. 6, which shows that n_+^2 increases to infinity at $\gamma=L\eta-1$ for the case of $L\eta>1$, but it monotonously increases to unity for the case of $L\eta<1$, in contrast to the infinite plasma ($L=0$), in which n_+^2

monotonously increases to unity as showh by a dotted curve in Fig. 6.

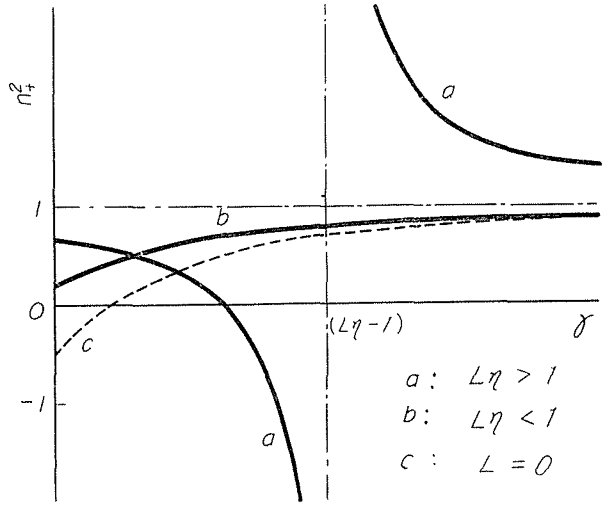

Fig. 6. Dispersion curves for left-handed wave.

Next, σ_R is given by:

$$\frac{\sigma_R}{\sigma_0}=\frac{\beta^2}{(L\eta-1-\gamma)^2+\beta^2}, \tag{18}$$

which implies that the magnetoplasma resonance occurs at

$$\gamma_0 = L\eta-1. \tag{19}$$

Therefore, although the cyclotron resonance does not occur for the infinite plasma, the magnetoplasma resonance occurs at γ_0 for the bounded plasma in the range $L\eta>1$. In Fig. 7 the resonance position γ_0 is plotted against η.

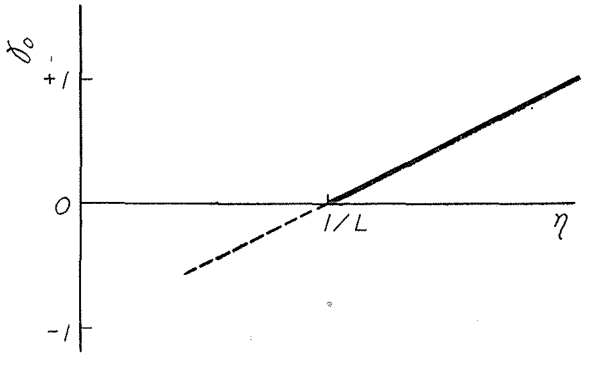

Fig. 7. γ_0 *versus* η for left-handed wave.

(*c*) *Linearly polarized wave*

Combining the right-handed and the left-handed waves, σ_R for the linearly polarized wave is given by:

$$\frac{\sigma_R}{\sigma_0} = \beta^2 \frac{(1-L\eta)^2+\gamma^2+\beta^2}{\{(1-L\eta)^2-\gamma^2-\beta^2\}^2+4\beta^2(1-L\eta)^2} . \tag{20}$$

Accordingly, the magnetoplasma resonance occurs approximately at γ_0 given by:

$$\gamma_0^2 = (1-L\eta)^2-\beta^2 . \tag{21}$$

In Fig. 8 the resonance position γ_0 is plotted against η. It is evident from this figure that the resonance occurs at two positions of η, when the electron density η is varied with the magnetic field fixed. This is characteristic of the bounded plasma. It is to be noted that the resonance at the smaller η is due to the right-handed wave and that at the larger η to the left-handed.

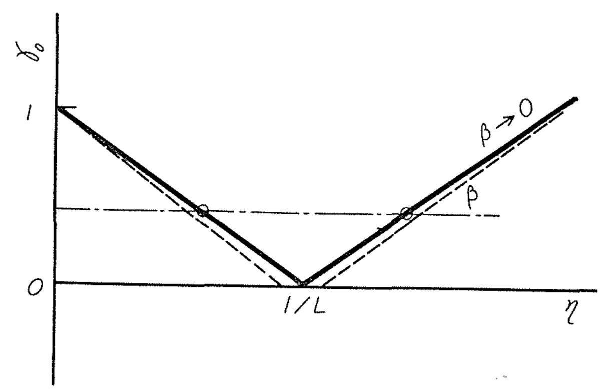

Fig. 8. γ_0 *versus* η for transversal wave ($\boldsymbol{k}_0 /\!/ \boldsymbol{B}$).

The absorption curve(σ_R *versus* γ) is asymmetric and its half-width ($\Delta\gamma$) is:

$$\frac{1}{\gamma_0}\frac{\Delta\gamma_\mp}{2} = \left| -1+\sqrt{1\mp 2\beta\frac{1-L\eta}{\gamma_0^2}} \right| . \tag{22}$$

For $\beta^2 \ll 1$, $\Delta\gamma$ is approximated by:

$$\Delta\gamma \simeq 2\beta , \tag{22a}$$

with the exception of $\gamma_0^2 \simeq 0$.

McGrath *et al.* (**17**) derived (20) in a similar way, but they made error in taking $L=1$. From the experimental studies on the cyclotron resonance of the afterglow, they reported that there may be two resonance positions for each magnetic field actually.

5. Magnetoplasma resonance for hybrid wave ($k_0 \perp B$)

We consider a wave propagating perpendicularly to the cylinder axis. $K_\perp$ for the hybrid wave is given by:

$$K_\perp = 1 - \frac{\eta}{1-L\eta} \frac{(1-L\eta)^2 - \eta + j\beta(1-L\eta)}{(1-L\eta)\{(1-L\eta)^2 - \eta - \gamma^2 - \beta^2\} + j\beta\{2(1-L\eta)^2 - \eta\}} . \tag{23}$$

Accordingly, $n_\perp$ at zero collision limit is given by:

$$n_\perp^2 = 1 - \eta \frac{(1-L\eta)^2 - \eta}{(1-L\eta)^2\{(1-L\eta)^2 - \eta - \gamma^2\}} . \tag{24}$$

The dispersion occurs as follows:

[A-I] $1-L\eta \neq 0$

(a) $\gamma_0^2 = (1-L\eta)^2 - \eta > 0$

$n_\perp^2$ increases to infinity at $\gamma^2 = \gamma_0^2$, as shown in Fig. 9, in which $n_\perp^2$ is plotted against γ^2 with η=const..

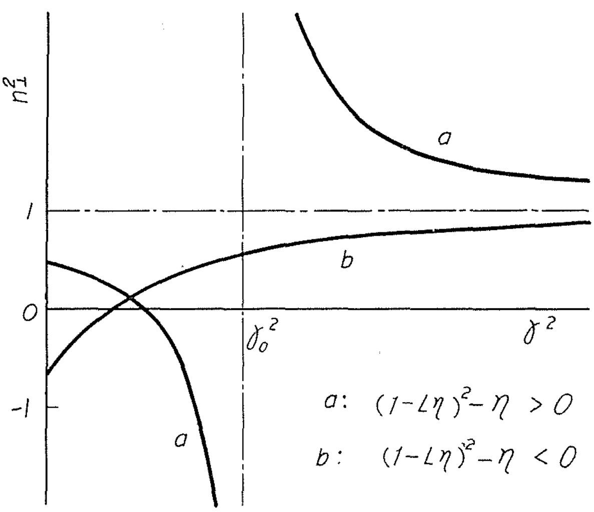

Fig. 9. Dispersion curves for hybrid wave ($k_0 \perp B$).

(b) $\gamma_0^2 = (1-L\eta)^2 - \eta < 0$

$n_\perp^2$ increases monotonously to unity as γ^2 increases. The dispersion is similar to that for the infinite plasma, for which $\gamma_0^2 = 1-\eta$ is taken. It is to be noted, however, that there are two η-regions where $\gamma_0^2 > 0$ for the bounded plasma, while there is one η-region where $\gamma_0^2 > 0$ for the unbounded plasma (as discussed later).

[A-II] $1-L\eta = 0$

According to (24), $n_\perp^2 = -\infty$ for any γ, and this is the same condition that was discussed in §3.

Next, σ_R is given by:

$$\frac{\sigma_R}{\sigma_0} = \beta^2 \frac{\{(1-L\eta)^2-\eta\}^2+(1-L\eta)^2(\gamma^2+\beta^2)}{(1-L\eta)^2\{(1-L\eta)^2-\eta-\gamma^2-\beta^2\}^2+\beta^2\{2(1-L\eta)^2-\eta\}^2}. \tag{25}$$

Therefore, the magnetoplasma resonance occurs approximately at γ_0 obtained from the following equation:

$$(1-L\eta)^2\left\{(1-L\eta)^2-\eta-\gamma_0^2-\beta^2\right\}^2 = 0. \tag{26}$$

[B-I] $1-L\eta \neq 0$

The resonance position γ_0 is given approximately by

$$\gamma_0^2 = (1-L\eta)^2-\eta-\beta^2 = \left\{L\eta-\left(1+\frac{1}{2L}\right)\right\}^2-\left(\frac{1}{L}+\frac{1}{4L^2}\right)-\beta^2, \tag{27}$$

which is plotted against η in Fig. 10.

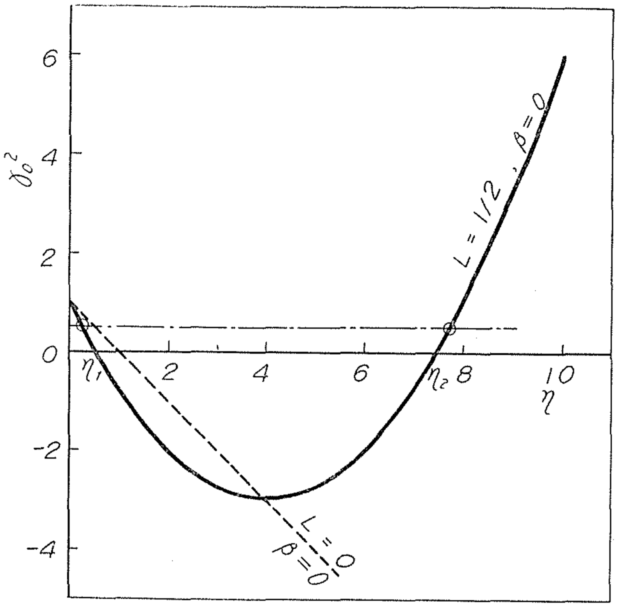

Fig. 10. γ_0^2 *versus* η for hybrid wave ($\boldsymbol{k}_0 \perp \boldsymbol{B}$).

Since the inequality $\gamma_0^2 > 0$ must be satisfied, the resonance occurs only in the following two η-regions:

$$0 < \eta < \eta_1 \quad \text{and} \quad \eta > \eta_2, \tag{28}$$

where

$$\left.\begin{matrix}\eta_1\\ \eta_2\end{matrix}\right\} = \frac{(2L+1)\mp\sqrt{4L(1+L\beta^2)+1}}{2L^2}, \quad (\eta_1 < \eta_2). \tag{29}$$

On the contrary, for the unbounded plasma the resonance position is given by:

$$\gamma_0^2 = 1-\eta-\beta^2,$$

which implies that the resonance occurs in the region $\eta < 1-\beta^2$ only, as shown in Fig. 10. Therefore, it is characteristic to the bounded plasmas that the resonance occurs at two positions of η at a fixed magnetic field γ.

The absorption curve (σ_R *versus* γ) is asymmetric and its half-width ($\Delta\gamma$) is:

$$\frac{1}{\gamma_0}\frac{\Delta\gamma_\mp}{2} = \left| -1+\sqrt{1\mp\frac{\beta}{\gamma_0^2}\left\{2(1-L\eta)-\frac{\eta}{1-L\eta}\right\}} \right|. \tag{30}$$

For $\beta^2 \ll 1$, $\Delta\gamma$ is approximated by:

$$\frac{\Delta\gamma}{2} = \frac{\beta}{\sqrt{(1-L\eta)^2-\eta}}\left\{(1-L\eta)-\frac{\eta}{2(1-L\eta)}\right\}, \tag{30a}$$

with the exception of $\gamma_0^2 \simeq 0$.

[B-II] $1-L\eta=0$

From (25), $\sigma_R/\sigma_0=1$ is obtained for any γ. The resonance is the same as discussed in § 3. The plasma resonance appears independently of a magnetic field. According to Dattner's experiments (**9**), this conclusion may be true, although the resonance absorption curve becomes complicated with increasing magnetic field.

6. Conclusion

Introducing the depolarization effect, the expressions for microwave conductivity for the bounded plasma in the absence of as well as in the presence of magnetic field were derived. It was shown that this effect is to replace ω in the specific inductive capacity for the unbounded plasma by $\omega' = \omega\{1-L(\omega_p/\omega)^2\}$.

The characteristic properties of the bounded plasma are summerized as follows:

(i) Plasma resonance occurs at $\omega = \sqrt{L}\,\omega_p$ both in the absence of and in the presence of magnetic field.

(ii) Magnetoplasma resonance occurs for both the left-handed wave and the right-handed and therefore, there are two resonance positions of electron density η at each magnetic field γ both for the transversal wave ($\boldsymbol{k}_0 /\!/ \boldsymbol{B}$) and for the hybrid wave ($\boldsymbol{k}_0 \perp \boldsymbol{B}$).

It has been assumed in these derivations that the electron density and the polarization induced are spatially uniform in the transversal plane. For the actual plasma, however, the electron density is not uniform in the plasma. Also, the surface charge induced, if the plasma is inserted within a waveguide, on the guide wall has been neglected above.

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