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NOTE ON THE FLOW OF AN INCOMPRESSIBLE VISCOUS FLUID PAST A CIRCULAR CYLINDER AT LOW REYNOLDS NUMBERS

BY

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ABSTRACT

The flow of an incompressible viscous fluid past a circular cylinder at low Reynolds numbers is discussed, with special reference to the formation of a vortex-pair behind the body. An approximate expression for the stream function given in a previous paper is revised and supplementary discussion is made. The flow pattern at Reynolds number equal to 4.0 is re-calculated and the existence of a vortex-pair at this value of the Reynolds number is confirmed. A brief consideration is also made on the critical Reynolds number for the first appearance of a pair of vortices.

1. Introduction

In a previous paper (**1**), the senior writer discussed, in conjunction with T. Aoi, the steady flow of an incompressible viscous fluid past a sphere and a circular cylinder at small Reynolds numbers, on the basis of Oseen's equations of motion. It was found out later, however, that the approximate expressions for the current function in the case of a sphere and the stream function in the case of a circular cylinder as given there (§7) in the forms of Reynolds number expansion are unfortunately rather incomplete. It is likely that such an incompleteness has considerable influences upon the shape and configuration of a vortex-pair which may appear behind the obstacle. Thus, it is necessary to revise some of the discussions in the previous paper (**1**, §§4 and 8).

One of the objects of the present note is to obtain additional terms to the previous expressions and to make supplementary discussions. It is shown that contrary to the previous result, no standing vortex-pair is given at all by revised approximate formulae for sufficiently small Reynolds number R.

On the other hand, the flow pattern at $R=4.0$ for the case of a circular cylinder was also computed in the previous paper (**1**) by making use of another approximation in which Lamb's potential is truncated by finite terms but not

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expanded in R. The resulting flow involved a fairly large standing vortex-pair behind the cylinder. Therefore, we repeat here the same analysis by taking more terms in the series for the stream function so that the result is almost convergent. Thus, we have confirmed the existence of a standing vortex-pair at $R=4.0$, though smaller in size than that found previously. In this connection it may be mentioned that the corresponding discussion for the case of a sphere was made recently by T. Pearcey and B. McHugh (2), who concluded that in the case of a sphere, no standing vortex-ring appears up to $R=10$.

In the present note, a brief discussion is also made on the critical Reynolds number for the first appearance of a standing pair of vortices behind a circular cylinder.

2. Solution of Oseen's equations of motion

We consider the steady two-dimensional flow of an incompressible viscous fluid past a circular cylinder of radius a. With the origin at the centre of the cylinder, we take the rectangular coordinates (x, y) in the plane of fluid motion in such a way that the x-axis is along the direction of the uniform stream of velocity U.

Let u and v be the x- and y-components of the fluid velocity at any point in the field of flow, p the pressure, ρ the density and ν the coefficient of kinematic viscosity of the fluid. Then, Oseen's equations of motion and the equation of continuity are given by

$$\left.\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta u\,, \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\Delta v\,; \end{aligned}\right\} \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{2}$$

where Δ stands for $\partial^2/\partial x^2+\partial^2/\partial y^2$. The boundary conditions at the surface of the cylinder are

$$u=v=0 \quad \text{at} \quad r=a\,. \tag{3}$$

In the previous paper (1) use was made of Lamb's method of solution. Here we shall use a different but more direct method. In conformity with the equation of continuity (2) we first introduce the stream function ψ such that

$$u=\frac{\partial\psi}{\partial y}\,, \qquad v=-\frac{\partial\psi}{\partial x}\,. \tag{4}$$

Then, elimination of p from Eqs. (1) gives the fundamental equation for ψ in the form:

$$\Delta\left(\Delta-2k\frac{\partial}{\partial x}\right)\psi=0\,, \tag{5}$$

where $k=U/(2\nu)$.

According to L. N. G. Filon (3), the general solution of this equation (5) can be expressed as follows:

$$\begin{aligned}\psi &= Ur\sin\theta+A_0\log\frac{r}{a}+B_0\theta+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{-n}(A_n\cos n\theta+B_n\sin n\theta)\\ &\quad+b_0\int_0^{\theta}kr\Big[K_1(kr)+K_0(kr)\cos\theta\Big]e^{kr\cos\theta}d\theta\\ &\quad+e^{kr\cos\theta}\sum_{n=0}^{\infty}K_n(kr)(a_n\cos n\theta+b_n\sin n\theta)\,,\end{aligned} \tag{6}$$

where r, θ are polar coordinates, $K_n(kr)$ is the modified Bessel function and A_n's, B_n's, a_n's and b_n's are arbitrary constants.

We further introduce the complex velocity $u-iv$. By use of (6) above, it can be written in the form:

$$\begin{aligned}u-iv &= e^{-i\theta}\left[\frac{1}{r}\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial r}\right]\psi\\ &= U+\sum_{n=0}^{\infty}C_n\left(\frac{r}{a}e^{i\theta}\right)^{-n-1}+e^{kr\cos\theta}\sum_{n=-\infty}^{\infty}c_nK_n(kr)\,e^{in\theta}\,,\end{aligned} \tag{7}$$

with

$$c_{-n}=\bar{c}_{n-1}\,, \tag{7a}$$

where bar implies the complex conjugate and the constants C_n's and c_n's are related to the A_n's, etc. in the forms:

$$\left.\begin{aligned}C_0 &= (B_0+iA_0)/a\,,\\ C_n &= n(B_n-iA_n)/a\,, && (n=1,2,3,\cdots)\,,\\ c_0 &= k\left\{b_0-\frac{b_1}{2}+i\left(a_0-\frac{a_1}{2}\right)\right\},\\ c_n &= \frac{k}{2}\left\{b_n-b_{n+1}+i(a_n-a_{n+1})\right\}, && (n=1,2,3,\cdots)\,.\end{aligned}\right\} \tag{8}$$

In the present case of a circular cylinder, $A_n=a_n=0$ for all positive integral values of n, zero inclusive, and therefore the C_n's and c_n's are all real.

Expanding (7) in a Fourier series in θ and then applying the boundary conditions (3), we get the following set of linear equations for determining the constants C_n's and c_n's, namely:

$$\left.\begin{aligned} &C_{s-1}+\sum_{n=-\infty}^{\infty} c_n I_{n+s}(ka)K_n(ka) = 0\,, && (s=1,2,3,\cdots)\,, \\ &\sum_{n=-\infty}^{\infty} c_n I_{n-s}(ka)K_n(ka) = \begin{cases} -U, & (s=0)\,, \\ 0\,, & (s=1,2,3,\cdots)\,. \end{cases} \end{aligned}\right\} \tag{9}$$

Also, if the values of the C_n's and c_n's are obtained by solving these equations, the values of the constants B_n's and b_n's can be obtained immediately by (8), namely:

$$\left.\begin{aligned} B_0 &= a\,C_0\,, \\ B_n &= \frac{a}{n}\,C_n\,, && (n=1,2,3,\cdots)\,, \\ b_0 &= \frac{1}{k}\sum_{j=0}^{\infty} c_j\,, \\ b_n &= \frac{2}{k}\sum_{j=n}^{\infty} c_j\,, && (n=1,2,3,\cdots)\,. \end{aligned}\right\} \tag{10}$$

3. Expansion formula for the stream function

For small ka Eqs. (9) can be solved by the method of successive approximations. The results are series expression in ka for the C_n's and c_n's, and it is thus found that $C_n = \mathrm{O}\,[(ka)^{n-1}]$ and $c_n = \mathrm{O}\,[(ka)^{2n}]$. Also, we get from (10) similar expressions for the B_n's and b_n's. Inserting these values of the B_n's and b_n's into (6) and expanding the modified Bessel functions for small kr and further rearranging the resulting terms with respect to the order in ka, we get the expansion formula for ψ in the form:

$$\psi = \psi_0 + R\psi_1 + \mathrm{O}(R^2)\,, \tag{11}$$

with

$$\left.\begin{aligned} \psi_0 &= c_0\Big[\frac{1}{2}\Big(r_1 - \frac{1}{r_1}\Big) - r_1 \log r_1\Big]\sin\theta\,, \\ \psi_1 &= -\frac{1}{16}\Big[r_1^2 - \frac{1}{r_1^2} + c_0 r_1^2 \log r_1 - \Big(\frac{c_0}{2}+2\Big)\Big(1-\frac{1}{r_1^2}\Big)\Big]\sin 2\theta\,, \end{aligned}\right\} \tag{11a}$$

where $R=4ka$, $r_1=r/a$, $c_0=-2/(1+2\Omega)$, $\Omega=\log(8/R)-\gamma$ and γ is Euler's constant. In deriving the formula (11) it is sufficient to take into account only the first three of the C_n's and c_n's respectively. It will readily be seen that the corresponding formula given in the previous paper (**1**, Eq. (44)) is lacking in the last term in the square brackets in ψ_1. The cause of such a discrepancy lies in the rather inadequate truncation of the system of equations, corresponding to the above equations (9), in the previous analysis.

It is expected that the present approximate expression (11) can represent fairly accurately the flow field around the cylinder for $kr \ll 1$ at sufficiently small values of R. In this expression, ψ_0 is a solution of the Stokes equation

$\Delta^2\psi=0$, which gives however logarithmic divergence in velocity at great distances, while the second term $R\psi_1$ represents the correction arising from the linearized inertia term in Eqs. (1). ψ vanishes obviously for $r_1=1$ (i.e., at the surface of the cylinder) and for $\theta=0$ (i.e., on the axial streamline). But the streamline $\psi=0$ may have one more branch given by

$$c_0\left[\frac{1}{2}\frac{r_1+1}{r_1}-\frac{r_1\log r_1}{r_1-1}\right]$$
$$-\frac{R}{8}\left[\frac{(r_1+1)(r_1^2+1)}{r_1^2}+c_0\frac{r_1^2\log r_1}{r_1-1}-\left(\frac{c_0}{2}+2\right)\frac{r_1+1}{r_1^2}\right]\cos\theta=0\,. \qquad (12)$$

In fact, if we drop the last term in the second square brackets, we certainly obtain the separated streamline $\psi=0$ embracing a vortex-pair even for very small R, as described in the previous paper (**1**). However, it can be shown that the full equation (12) does never have real root for all values of the Reynolds number

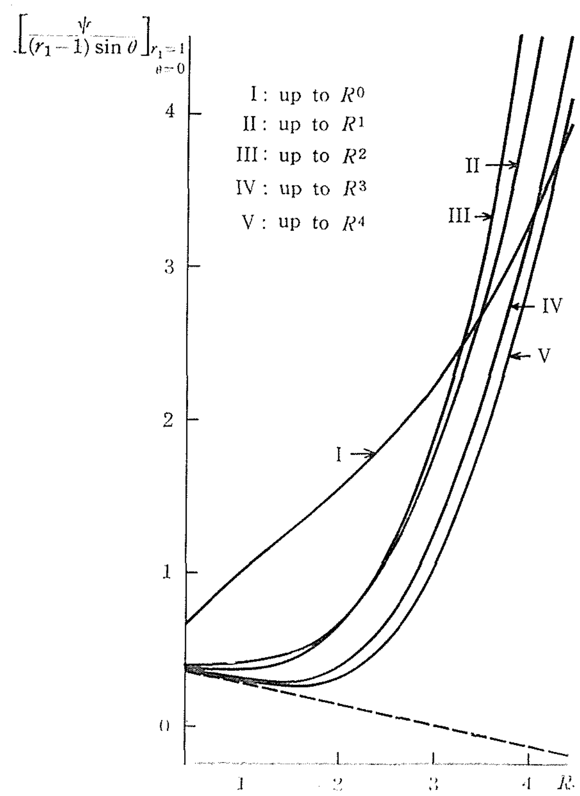

Fig. 1. Values of $\left[\frac{\psi}{(r_1-1)\sin\theta}\right]_{\substack{r_1=1\\ \theta=0}}$ *versus* R for each degree of approximation.

R. Thus, the lost term in the previous paper affects seriously the delicate flow pattern immediately behind the cylinder.

The present writers have also extended the expansion (11) up to the fourth order term (proportional to R^4). The result is fairly complicated and will not be given here. It may be added, however, that the separated streamline $\psi=0$ and hence a vortex-pair does not yet appear for small values of R as far as the fourth order approximation is concerned. In Fig. 1 the values of $[\psi/\{(r_1-1)\sin\theta\}]_{r_1=1,\,\theta=0}$ are plotted against R for each degree of approximation. The zero point of this function is expected to give the critical Reynolds number for the first appearance of a standing pair of vortices behind the cylinder. From the curves for successive approximations and from the nature of the function, it may be conjectured that the true values of $[\psi/\{(r_1-1)\sin\theta\}]_{r_1=1,\,\theta=0}$ lie near a dotted-line curve in Fig. 1, and thus the value of the critical Reynold number for the first appearance of a vortex-pair is supposed to be about 3. This point will be re-considered in the next section.

4. Flow pattern at $R=4.0$

In the previous paper (**1**) the flow field at $R=4.0$ was also computed by making use of another approximation in which the series for Lamb's potential is truncated by finite terms but not expanded in R. The resulting flow involved a fairly large vortex-pair behind the cylinder. In view of the non-existence of a standing vortex-pair at sufficiently small R as mentioned in the preceding section, however, it is desirable to refine the previous calculation for $R=4.0$.

We have started again from the simultaneous equations (9) for the C_n's and c_n's. We have first evaluated the functions $I_{n-s}(ka)\ K_n(ka)$, etc. for $ka=1$ (i.e., $R=4.0$) and then we have solved the resulting linear algebraic equations for the C_n's and c_n's. Inserting these values of the C_n's and c_n's in (10), we have also obtained the B_n's and b_n's. With various constants thus determined, Eqs. (6) and (7) give respectively the stream function and the complex velocity. The simultaneous equations (9) are doubly infinite, and therefore in actual computation we must truncate them adequately by finite terms. The calculation in the previous paper (**1**) corresponds to taking the c_n's up to c_4. Here we have taken the c_n's up to c_9. The values of the C_n's and c_n's are listed in Table I, in which are also given the values obtained when truncated c_8, c_7, c_6 and c_5 respectively. It can readily be seen that our c_9-approximation represents almost convergent result.

The vorticity changes its sign at the separation point of the streamline $\psi=0$ on the cylinder and this occurs at $\theta=20.072^\circ$, 20.691°, 20.658°, 20.659° and 20.659°

for the c_5-, c_6-, c_7-, c_8- and c_9-approximation, respectively. On the other hand, the point of intersection of the separated and the axial streamlines is found to lie at $x=1.2057\,a$, $1.2051\,a$, $1.2052\,a$ and $1.2052\,a$, and $y=0$ for the c_6-, c_7-, c_8- and c_9-approximation respectively. For the c_5-approximation we cannot obtain the value of the said point of intersection, because in this case the stream function is very near to zero inside and near the separated streamline. Thus it is seen that the flow pattern immediately behind the cylinder is in general rather delicate.

Table I. Values of the c_n's and C_n's for each degree of approximation.

	c_9-approx.	c_8-approx.	c_7-approx.	c_6-approx.	c_5-approx.
c_0	-1.5703051	-1.5703051	-1.5703051	-1.5703051	-1.5703051
$c_1\times10$	7.0777998	7.0777998	7.0777998	7.0777998	7.0777997
$c_2\times10^2$	-6.9880077	-6.9880077	-6.9880077	-6.9880077	-6.9880044
$c_3\times10^3$	2.7411856	2.7411856	2.7411856	2.7411851	2.7411258
$c_4\times10^5$	-5.5879335	-5.5879335	-5.5879330	-5.5878643	-5.5821547
$c_5\times10^7$	6.9156370	6.9156367	6.9155837	6.9105374	6.6306723
$c_6\times10^9$	-5.7320823	-5.7320529	-5.7289149	-5.5295436	
$c_7\times10^{11}$	3.4008551	3.3993972	3.2956557		
$c_8\times10^{13}$	-1.5147279	-1.4733203			
$c_9\times10^{16}$	5.1530823				
C_0	9.2971925	9.2971925	9.2971925	9.2971925	9.2971924
$C_1\times10$	-5.7679289	-5.7679289	-5.7679289	-5.7679288	-5.7679288
$C_2\times10^2$	5.4303441	5.4303441	5.4303440	5.4303440	5.4303439
$C_3\times10^3$	4.541038	4.541038	4.541038	4.541038	4.541044
$C_4\times10^4$	2.43383	2.43383	2.43383	2.43383	2.43355
$C_5\times10^6$	5.0856	5.0856	5.0856	5.0873	-14.894
$C_6\times10^7$	-5.238	-5.238	-5.238	9.220	
$C_7\times10^8$	-7.568	-7.568	-16.69		
$C_8\times10^8$	-1.111	-0.5977			
$C_9\times10^9$	-1.225				

The flow pattern at $R=4.0$ to the c_9-approximation is shown in Fig. 2.

Comparing Fig. 2 with Fig. 3 in the previous paper (**1**), we find that a standing pair of vortices given previously is too large in size, perhaps because of slight inaccuracy of the previous calculation. However, these two figures are quite in good agreement with each other in other domains of the flow field and this suggests that so far as the pressure distribution on the surface of the cylinder, the total drag and etc. are concerned, the previous results are sufficiently accurate.

Now, observing the definite asymmetry with respect to the y-axis of the over-all flow pattern as shown in Fig. 2 and reflecting upon the accuracy of our

c_9-approximation, it may be concluded that a standing vortex-pair is certainly present at $R=4.0$ on the basis of the Oseen approximation.

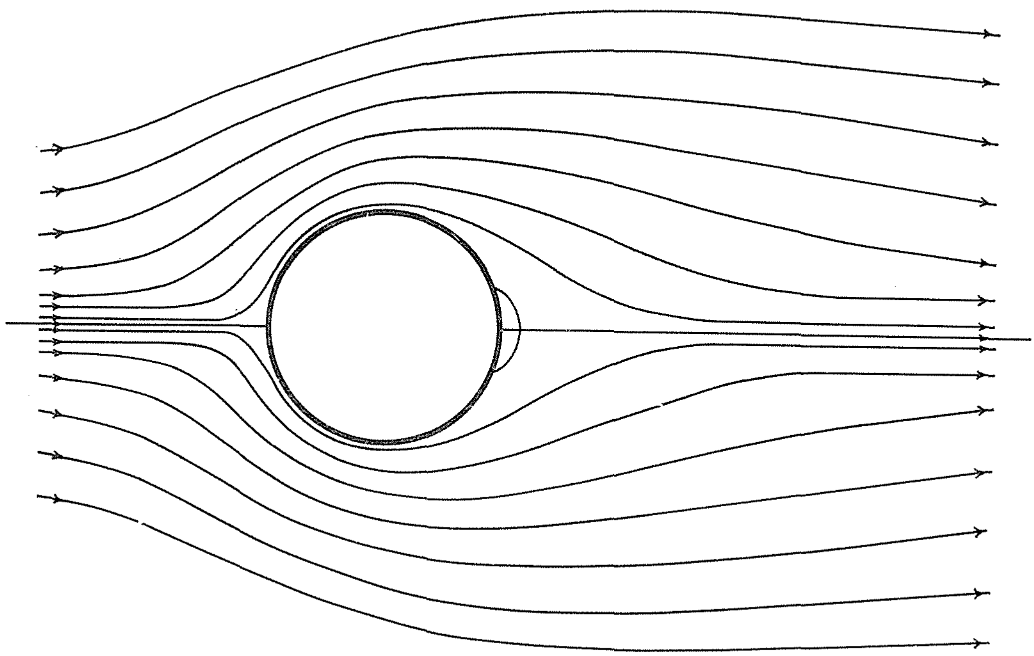

Fig. 2. Calculated streamlines past a circular cylinder at $R=4.0$.

Since, however, the size of the vortex-pair in this case is fairly small and the fluid is almost at rest there, it is expected that a slight reduction of the Reynolds number below 4.0 will make the vortex-pair vanish immediately. Based upon this expectation and in view of the result shown in Fig. 1, we have further computed the flow pattern at $R=3.2$ to the c_9-approximation, obtaining the value $\theta=9.732°$ for the point of separation of the streamline $\psi=0$ on the cylinder and $x=1.044\,a$, $y=0$ for the point of intersection of the separated and the axial streamlines. In this case we find a standing vortex-pair with extremely small size. Thus, it may be concluded that so far as we are concerned with the Oseen approximation, the value of the critical Reynolds number for the first appearance of a standing pair of vortices behind a circular cylinder will not differ too much from 3. It is of interest to compare this value with an experimental value of 5 found recently by S. Taneda (4).

Lastly, we shall give here, as an addendum, a revised R-expansion of the current function for the case of the flow past a sphere, which should take place of Eq. (20) in the previous paper (**1**). It is given by

$$\psi = \frac{3}{4}\Bigg[\left(r_1-\frac{1}{r_1}\right)-\frac{2}{3}\left(\frac{1}{r_1}-r_1^2\right) + \frac{R}{16}\left\{3\left(\frac{1}{r_1}-r_1\right)-2\left(\frac{1}{r_1}-r_1^2\right)+\left[2\left(\frac{1}{r_1^2}-r_1^2\right)-4\left(\frac{1}{r_1^2}-1\right)\right]\cos\theta\right\}\Bigg]\sin^2\theta + O(R^2)\,. \tag{13}$$

This result is due to H. Yosinobu (**5**). The flow field around a sphere has also been discussed later by T. Pearcey and B. McHugh in a paper cited before (**2**) and they have confirmed that a stationary vortex-ring behind the sphere does not make its appearance at least up to $R=10$.

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