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# A STATISTICAL TREATMENT OF THE INTENSITY FLUCTUATION OF RANDOM NOISE CURRENT*

BY

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ABSTRACT

As a method of treating statistically many correlative physical quantities fluctuating only in positive region, the joint characteristic function in the form of Hankel transform was introduced. Then it was demonstrated that this function is more effective for the analyses of the noise current than the functions of Fourier or Laplace type which have been used so far. When the system of correlative random currents are led into an energy detector and the energy output fluctuation is considered, it can be reduced to a probability problem of distance in $N$-dimensional signal space, and further can correspond to a problem of random walks in $N$-dimensional space. From this point of view, the explicit expressions for the probability distribution of intensity fluctuation of noise current were derived by means of the Hankel type characteristic function, in connection with the random walk problem. These expressions are more general than the well-known expressions due to Rice, Rayleigh and Watson, because they include the latter ones as special cases.

## 1. Introduction

The probability variables defined over positive region in a functional space are fundamental quantities in physics, and the probability density distribution of distance in the multi-dimensional space (**1**, **2**) gives statistically the intensity fluctuation of random noise current, if we take the distance as the mean energy. We are well aware of the fact that the energy fluctuation comes to the front in almost all of noise measurements (**3**).

In this paper, the energy fluctuation of random noise in the multi-dimensional space is particularly considered, by treating the generalized problem of random walks in multi-dimensional vector space (**1**, **4**, **5**, **6**). First, we shall introduce a characteristic function in the form of the Hankel transform (**1**, **7**) applicable as a postern to probability problems of $K$ correlative physical quantities fluctuating only in positive region, and then, prove that the above characteristic function of Hankel type is more useful for calculating the joint-probability distribution of energy fluctuation of random noise than the usual characteristic function of

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Fourier (or Laplace) type. As applications, we shall calculate explicitly the probability density distribution of intensity fluctuation of random noise current: $P(R)$ or $P(E)$, and derive the relation between the resultant vector and respective element vectors affecting parameters of the resultant distribution.

Between the statistical theory of random noise and the information theory of message which are two main currents in general communication theory, this paper falls under the former, as S. O. Rice's way of treating (8).

## 2. The $N$-dimensional representation of the random intensity fluctuation

Now, we consider along time axis a set of the general stochastic processes:

$$G_h(t) \quad (h = 1, 2, \cdots, K),$$

which are respectively composed of the regular component $S_h(t)$ and the random component $N_h(t)$ ($\langle N_h(t)\rangle = 0$), and especially the integral energies:

$$E_h = \int_{t_1}^{t_2} G_h^2(t)\omega(t)dt\,,$$

such as the output from energy detector, where $G_h^2(t)$ corresponds to the instantaneous energy fluctuation and $\omega(t)$ is a weight function, since our observation coincides essentially with the weighted mean of energy, in that an information which we can obtain for the crude random phenomenon through our observation is a certain mean image. The form of the mean operation (especially the weight function) will be successively improved with the steady progress in science. However, at the stage where the effect of observation on the crude random fluctuation is unknown, a special form:

$$\frac{1}{T}\int_{t_1}^{t_2} G_h^2(t)dt$$

taking $\omega(t) = \frac{1}{T}$, $T = t_2 - t_1$, of the mean operation (**1, 2, 5, 9**) seems to be a most natural one to take.

Generally, in investigating properties of $G_h(t)$ $(= S_h(t) + N_h(t))$ in a finite time interval, say $(t_1, t_2)$, it is convenient to expand it in terms of an orthonormal set of functions, $\varphi_i(t)$ $(i = 1, 2, \cdots)$ with a weight function $\omega(t)$. That is, we can write:

$$\left.\begin{aligned} G_h(t) &= S_h(t) + N_h(t) = \sum_{i=0}^{\infty} g_{hi}\varphi_i(t)\,, \\ g_{hi} &= s_{hi} + n_{hi} = \int_{t_1}^{t_2} \omega(t)G_h(t)\varphi_i(t)dt\,, \\ s_{hi} &= \int_{t_1}^{t_2} \omega(t)S_h(t)\varphi_i(t)dt\,, \\ n_{hi} &= \int_{t_1}^{t_2} \omega(t)N_h(t)\varphi_i(t)dt\,, \end{aligned}\right\} \qquad (1)$$

where

$$\int_{t_1}^{t_2} \omega(t)\varphi_i(t)\varphi_j(t)dt = \delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \,. \end{cases}$$

On the other hand, by use of Khintchine-Wiener's theorem (**10**), auto- and cross-correlation functions of $G_h(t)$:

$$\rho_{hl}(t, t') = \langle (G_h(t) - \langle G_h(t) \rangle)(G_l(t') - \langle G_l(t') \rangle) \rangle ,$$

assuming

$$\rho_{hl}(t, t') = \rho_{hl}(\tau), \ \tau = |t - t'| ,$$

are derived from its auto- and cross-spectral densities *a priori* assigned.

In the above expression (1) of $G_h(t)$, it is very convenient to choose, as the orthonormal set $\varphi_i(t)$ $(i=1, 2, \cdots)$ with $\omega(t)$, the solutions (eigenfunctions) of the homogeneous Fredholm integral equation:

$$\underset{(j)}{\Lambda_{hl}} \sqrt{\omega(t)}\varphi_j(t) = \int_{t_1}^{t_2} K_{hl}(t, t')\sqrt{\omega(t')}\varphi_j(t')dt' \quad (t_1 \leqslant t \leqslant t_2) \tag{2}$$

with the symmetric kernel:

$$K_{hl}(t, t') = \rho_{hl}(t, t')\sqrt{\omega(t)\omega(t')} .$$

Then, from the Parseval equation, our weighted average power $E_h$ can be expressed by:

$$E_h \left( = \int_{t_1}^{t_2} \omega(t) G_h^2(t) dt \right) = \sum_{i=1}^{N} x_{hi}^2 , \ (x_{hi} = g_{hi}) . \tag{3}$$

By finding eigenvalues $\underset{(i)}{\Lambda_{hl}}$'s and eigenfunctions $\varphi_i(t)$'s of the above integral equation and letting $N$ be the effective number of eigenvalues (therefore, $i=1, 2, \cdots, N$), it can easily be shown that the mean value and the covariance for the expansion coefficients:

$$g_{hi}, \ (h = 1, 2, \cdots, K ; i = 1, 2, \cdots, N)$$

become:

$$\left.\begin{aligned} &\langle g_{hi} \rangle = \langle s_{hi} \rangle = \int_{t_1}^{t_2} \omega(t) \langle S_h(t) \rangle \varphi_i(t) dt (\equiv a_{hi}) , \\ \text{and} \quad &\langle (g_{hi} - \langle g_{hi} \rangle)(g_{lj} - \langle g_{lj} \rangle) \rangle = \underset{(j)}{\Lambda_{hl}} \cdot \delta_{ij} (\equiv \underset{(i)}{\sigma_{hl}}) , \\ &\sigma_{g_{hi}}^2 = \langle (g_{hi} - \langle g_{hi} \rangle)^2 \rangle = \underset{(i)}{\Lambda_{hh}} = \sigma_{s_{hi}}^2 + \sigma_{n_{hi}}^2 , \end{aligned}\right\}$$

where $s_{hi}$'s are independent of $n_{hi}$'s and we must notice that our eigenvalues and eigenfunctions are affected by $\omega(t)$.

Thus, a joint-distribution density of $x_{hi}$'s can be expressed in the form of product of $K$-dimensional normal distributions (**4, 6, 11**), that is,

$$P(x_{11}, x_{21}, \cdots, x_{K1}; x_{12}, x_{22}, \cdots, x_{K2}; \cdots, x_{1N}, x_{2N}, \cdots x_{KN})$$
$$= \prod_{i=1}^{N} \left(\frac{1}{\sqrt{2\pi}}\right)^K \sqrt{|\sigma_{(i)}^{hl}|} \exp\left\{-\frac{1}{2}\sum_{h}^{K}\sum_{l}^{K} \sigma_{(i)}^{hl}(x_{hi}-a_{hi})(x_{li}-a_{li})\right\}, \tag{4}$$

where $[\sigma_{(i)}^{hl}]$ is an inverse-symmetric matrix of the variance matrix $[\underset{(i)}{\sigma_{hl}}]$.

After all, the problem is: What is the joint-probability density of energies $P(E_1, E_2, \cdots, E_K)$ for $K$ correlative random output fluctuations, from Eqs. (3) and (4)?

On the other hand, we shall take up the generalized problem of $K$ correlative series of random walks $\boldsymbol{R}_h = \sum_{j=1}^{S_h} \boldsymbol{r}_{hj}$ $(h = 1, 2, \cdots, K)$ as shown in Fig. 1, where $\boldsymbol{r}_{hj} = \sum_{i=1}^{N_h} e_i x_{hji}$, $e_i$ being a unit vector along $x_i$-axis in the $N$-dimensional vector space. Now, in a special case where $N_1 = N_2 = \cdots = N_K = N$ $(= N')$,

$$R_h^2 = \sum_{i=1}^{N} x_{hi}^2 \,;\; x_{hi} = \sum_{j=1}^{S_h} x_{hji}\,. \tag{5}$$

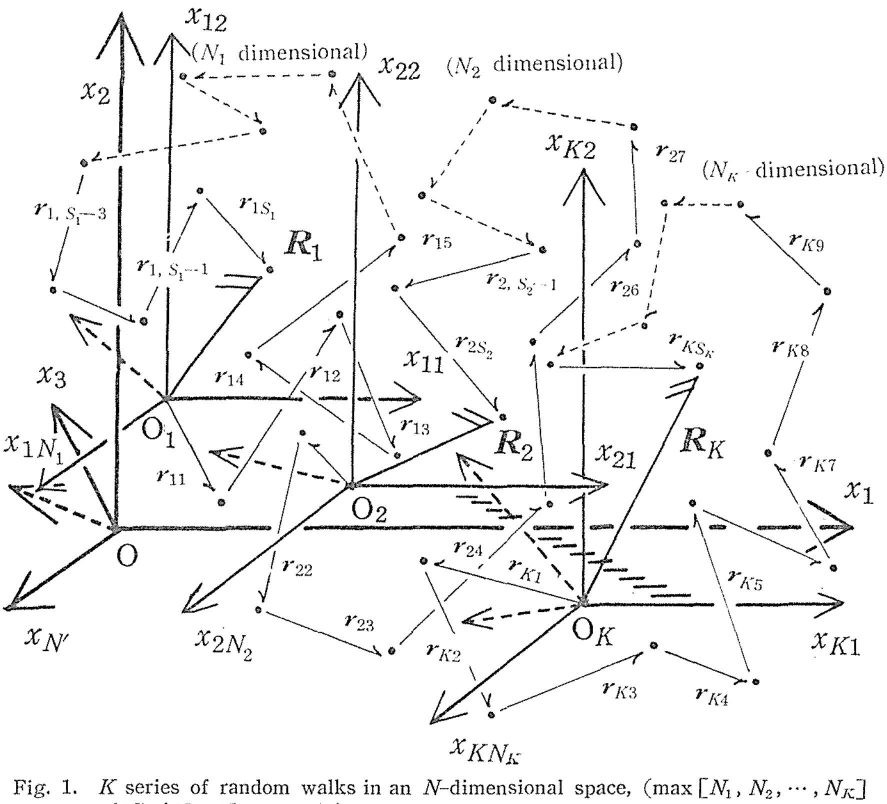


Fig. 1. $K$ series of random walks in an $N$-dimensional space, $(\max[N_1, N_2, \cdots, N_K] \leqslant N' \leqslant N_1 + N_2 + \cdots + N_K)$.

Thus, the problem can be transformed into the same form as Eq. (3) by letting $E_h = R_h^2$. Then, we are able to regard the above random process as a probability problem in multi-dimensional vector space (**1**, **7**).

We may mention several interesting cases.

( i ) As shown in Fig. 2, the problem of the resultant of several random waves such as the signal and the noise, corresponds to the random walk problem in the Gaussian plane with $N=2$, where we can consider the noise character as the amplitude fluctuation or the phase fluctuation (**1**, **12**).

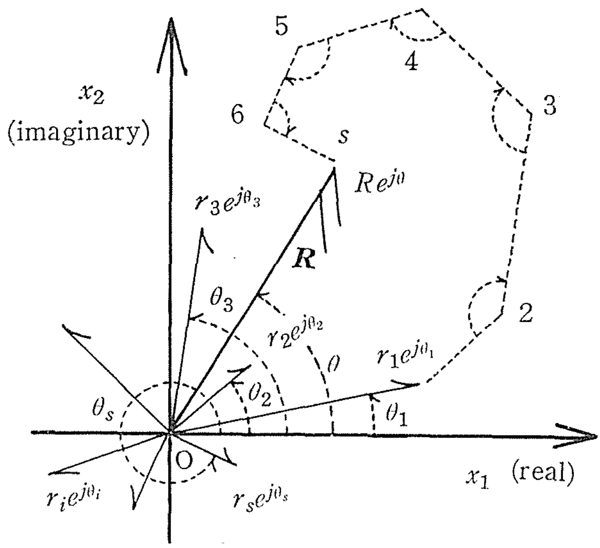


Fig. 2. The relation between random waves and random walks.

(ii) A probability calculation of average power of the random signals limited within the frequency interval $W$ and time interval $T$ can be regarded, following Shannon's sampling theorem (**13**), as a problem of the random process in the form of Eq. (3) in the functional space with $N=2TW$ (**1**, **7**, **11**).

## 3. The application of Hankel transform to random processes

In the preceding paragraph, it has been confirmed that we can regard the actual 3-dimensional random processes along time axis as a certain problem of random walks in an $N$-dimensional vector space, when the energy fluctuation is concerned. We are also well aware of the effectiveness of using the moment generating function of Laplace type: $m(t) = \langle e^{-tE} \rangle = \int_{-\infty}^{\infty} e^{-tE} P(E) dE$ or characteristic function of Fourier type as a postern for the probability calculation in a scalar field.

The problem is: What is the postern for the probability calculation in an $N$-dimensional vector space?

**[A] Derivation of the joint characteristic function of Hankel type**

First, we shall introduce a joint characteristic function $F(\lambda_1, \lambda_2, \cdots, \lambda_K)$ in the form of the Hankel transform applicable to respective probability problems of $K$ correlative physical quantities $R_h$'s $(h=1, 2, \cdots, K)$ fluctuating only in positive region, and express the joint-probability density $P(R_1, R_2, \cdots, R_K)$ (**1, 2, 7**).

In general, the joint-probability distribution function:

$$\mathcal{P}(R_1, R_2, \cdots, R_K) = \int_0^{R_1}\int_0^{R_2}\cdots\int_0^{R_K} P(R_1, R_2, \cdots, R_K)dR_1dR_2\cdots dR_K$$

can be expressed by

$$\left.\begin{aligned}&\mathcal{P}(R_{10}, R_{20}, \cdots, R_{K0}) = \int_0^{\infty}\cdots\int_0^{\infty} P(R_1, R_2, \cdots, R_K)\prod_{h=1}^{K} D_h(R_h)dR_h\,,\\ &D_h(R_h) = \begin{cases}1 & (R_h < R_{h0})\,,\\ 0 & (R_h > R_{h0})\,.\end{cases}\end{aligned}\right\} \tag{6}$$

By using, instead of $D_h(R_h)$, the discontinuous integral due to Weber-Schafheitlin (**14**):

$$R_{h0}^{m_h}\int_0^{\infty} J_{m_h}(R_{h0}\lambda_h)\frac{J_{m_h-1}(R_h\lambda_h)}{R_h^{m_h-1}}d\lambda_h = \begin{cases}1 & (R_h < R_{h0})\,,\\ 0 & (R_h > R_{h0})\,,\end{cases} \tag{7}$$

we obtain:

$$\left.\begin{aligned}\mathcal{P}(R_1, R_2, \cdots, R_K) &= \left(\frac{1}{2}\right)^{\sum^{K} m_h - K}\\ &\quad\times\prod_{h=1}^{K}\frac{R_h^{m_h}}{\Gamma(m_h)}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\{\prod_{h=1}^{K}\lambda_h^{m_h-1}J_{m_h}(\lambda_hR_h)\}F(\lambda_1, \lambda_2, \cdots, \lambda_K)d\lambda_1d\lambda_2\cdots d\lambda_K\,,\\ P(R_1, R_2, \cdots, R_K) &= \left(\frac{1}{2}\right)^{\sum^{K} m_h - K}\\ &\quad\times\prod_{h=1}^{K}\frac{R_h^{m_h}}{\Gamma(m_h)}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\{\prod_{h=1}^{K}\lambda_h^{m_h}J_{m_h-1}(\lambda_hR_h)\}F(\lambda_1, \lambda_2, \cdots, \lambda_K)d\lambda_1d\lambda_2\cdots d\lambda_K\,,\\ F(\lambda_1, \lambda_2, \cdots, \lambda_K) &= \left\langle\prod_{h=1}^{K} 2^{m_h-1}\Gamma(m_h)\frac{J_{m_h-1}(\lambda_hR_h)}{(\lambda_hR_h)^{m_h-1}}\right\rangle,\end{aligned}\right\} \tag{8}$$

with $m_h \geqq 1/2$ $(h=1, 2, \cdots, K)$, or, more explicitly,

$$\left.\begin{aligned}\mathcal{P}(R_1, R_2, \cdots, R_K) &= (\prod_{h=1}^{K} R_h^{m_h})\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\{\prod_{h=1}^{K} J_{m_h}(\lambda_hR_h)\}\\ &\qquad\times\langle\prod_{h=1}^{K} R_h^{1-m_h}J_{m_h-1}(\lambda_hR_h)\rangle d\lambda_1d\lambda_2\cdots d\lambda_K\,,\\ P(R_1, R_2, \cdots, R_K) &= (\prod_{h=1}^{K} R_h^{m_h})\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\{\prod_{h=1}^{K}\lambda_hJ_{m_h-1}(\lambda_hR_h)\}\\ &\qquad\times\langle\prod_{h=1}^{K} R_h^{1-m_h}J_{m_h-1}(\lambda_hR_h)\rangle d\lambda_1d\lambda_2\cdots d\lambda_K\,.\end{aligned}\right\} \tag{9}$$

But $F(\lambda_1, \lambda_2, \cdots, \lambda_K)=\prod_{h=1}^{K} F(\lambda_h)$ when $P(R_1, R_2, \cdots R_K)=\prod_{h=1}^{K} P(R_h)$, i.e., $R_1, R_2, \cdots, R_K$ are statistically independent.

Especially, when we consider the probability problem given in the form $R_h^2=\sum_{i=1}^{N_h} x_{hi}^2$ of Eq. (5), we can also derive the characteristic function $F(\lambda_1, \lambda_2, \cdots, \lambda_K)$ in Eq. (8) under the additional condition: $m_h=N_h/2$, as follows:

$$\left.\begin{aligned} F(\lambda_1, \lambda_2, \cdots, \lambda_K) &= \frac{1}{\prod_{h=1}^{K} S_{(N_h)}} \int_{S(N_1)} \cdots \int_{S(N_2)} \cdots \int \cdots \int_{S(N_K)} \cdots \int \prod_{h=1}^{K} ds_{(N_h)} \\ &\qquad \times [F(\mu_{11}, \mu_{12}, \cdots, \mu_{1N_1}; \cdots; \mu_{K1}, \mu_{K2}, \cdots, \mu_{KN_K})]\,|\,, \\ &(\mu_{h1}, \mu_{h2}, \cdots, \mu_{hN_h}) \rightarrow (\lambda_h, \varphi_{h1}, \cdots, \varphi_{h,N_h-1})\,(\forall h) \\ F(\mu_{11}, \mu_{12}, \cdots, \mu_{1N_1}; \cdots; \mu_{K1}, \mu_{K2}, \cdots, \mu_{KN_K}) &= \langle \exp(i[\sum_{h=1}^{K}\sum_{j=1}^{N_h} \mu_{hj} x_{hj}]) \rangle, \end{aligned}\right\} \quad (10)$$

where

$$S_{(N_h)} = \frac{(\sqrt{\pi})^{N_h} N_h}{\Gamma\left(\frac{N_h}{2}+1\right)} \quad \text{and} \quad ds_{(N_h)} = \prod_{j=1}^{N_h-1} (\sin\varphi_{hj})^{N_h-1-j} d\varphi_{hj} \qquad (h=1, 2, \cdots, K)$$

mean respectively a surface area and a surface element of an $N_h$-dimensional unit hypersphere, and $(\mu_{h1}, \mu_{h2}, \cdots, \mu_{hN_h}) \rightarrow (\lambda_h, \varphi_{h1}, \cdots, \varphi_{h,N_h-1})$ denotes the transformation to the polar coordinates. After a somewhat troublesome calculation, we can find that Eq. (10) agrees with $F(\lambda_1, \lambda_2, \cdots, \lambda_K)$ in Eq. (8), by use of an inversion formula of the Hankel transformation, and also $P(R_1, R_2, \cdots, R_K)$ in Eq. (8) can be derived from Eq. (10) by application of the transformation to an $N_h$-dimensional polar coordinates, $(x_{h1}, x_{h2}, \cdots, x_{hN_h}) \rightarrow (R_h, \theta_{h1}, \theta_{h2}, \cdots, \theta_{h,N_h-1})$.

If a special case $K=1$ is taken up, we have

$$\left.\begin{aligned} P(R) &= \frac{R^m}{2^{m-1}\Gamma(m)} \int_0^\infty F(\lambda)\lambda^m J_{m-1}(\lambda R)d\lambda, \\ F(\lambda) &= \left\langle 2^{m-1}\Gamma(m) \frac{J_{m-1}(\lambda R)}{(\lambda R)^{m-1}} \right\rangle \\ &= 1+\sum_n \frac{(-1)^n \Gamma(m)\langle R^{2n}\rangle}{2^{2n} n! \Gamma(m+n)} \lambda^{2n}, \end{aligned}\right\} \quad (11)$$

or,

$$\left.\begin{aligned} F(\lambda) &= \frac{1}{S_{(N)}} \int_0^{2\pi}\int_0^{\pi} \cdots \int_0^{\pi} [F(\mu_1, \mu_2, \cdots, \mu_N)]\,|\,ds_{(N)}, \\ &\qquad (\mu_1, \mu_2, \cdots, \mu_N) \rightarrow (\lambda, \varphi_1, \cdots, \varphi_{N-1}) \\ F(\mu_1, \mu_2, \cdots, \mu_N) &= \langle \exp(i\sum_{K=1}^{N} x_K \mu_K) \rangle, \end{aligned}\right\} \quad (12)$$

corresponding to Eq. (8) or Eq. (10).

**[B] The effectiveness of the characteristic function of Hankel type**

The method using the moment generating function $m(t)$ of Laplace type is indeed useful for calculating the probability density $P(E)$ for the sum: $E=\sum_{i=1}^{s} e_i$ of independent $s$ scalar functions $e_i(t)$, since we have simply

$$m(t) = \prod_{i=1}^{S} m_i(t), \quad m_i(t) = \langle e^{-te_i} \rangle .$$

The method using the characteristic function of Hankel type: $F(\lambda)=\left\langle 2^{m-1}\Gamma(m) \times \frac{J_{m-1}(\lambda R)}{(\lambda R)^{m-1}} \right\rangle$ (see Eq. (11)) is, however, generally effective in calculating the probability density $P(R)$ for the sum $\boldsymbol{R}=\sum_{i=1}^{S} \boldsymbol{r}_i$ $(R=|\boldsymbol{R}|)$ of independent $S$ vector functions $\boldsymbol{r}_i(t)$ in an $N$-dimensional vector space, when the magnitude $r_i$ $(=|\boldsymbol{r}_i|)$ is independent of the direction of $\boldsymbol{r}_i$.

Consequently, we shall prove Eq. (11) under the special condition that $F(\lambda)=\prod_{i=1}^{S} F_i(\lambda)$ and $m=N/2$, where $F_i(\lambda)=\left\langle 2^{m-1}\Gamma(m)\frac{J_{m-1}(\lambda r_i)}{(\lambda r_i)^{m-1}} \right\rangle$, more explicitly,

$$\left.\begin{aligned} \mathcal{P}(R) &= \Gamma\left(\frac{N}{2}\right)^{S-1}\left(\frac{1}{2}\right)^{(N/2)-1} R\int_0^\infty (\lambda R)^{(N/2)-1} J_{N/2}(\lambda R) \\ &\qquad \times \prod_{j=1}^{S}\left\langle \frac{J_{(N/2)-1}(\lambda r_j)}{\left(\frac{1}{2}\lambda r_j\right)^{(N/2)-1}} \right\rangle d\lambda , \\ P(R) &= \Gamma\left(\frac{N}{2}\right)^{S-1}\left(\frac{1}{2}\right)^{(N/2)-1} \int_0^\infty (\lambda R)^{N/2} J_{(N/2)-1}(\lambda R) \\ &\qquad \times \prod_{j=1}^{S}\left\langle \frac{J_{(N/2)-1}(\lambda r_j)}{\left(\frac{1}{2}\lambda r_j\right)^{(N/2)-1}} \right\rangle d\lambda . \end{aligned}\right\} \tag{13}$$

The proof consists in answering our question given at the beginning of this paragraph. It is sufficient to verify that $F(\lambda)=F_1(\lambda)F_2(\lambda)$ holds when $\boldsymbol{R}=\boldsymbol{r}_1+\boldsymbol{r}_2$ (i.e., $S=2$) in an $N$-dimensional vector space. But the way of verification differs between the next two cases: $N\neq 2$ and $N=2$.

( i ) Case when $N\neq 2$ $(m\neq 1)$

We shall take up the problem of random walks with two steps in an $N$-dimensional vector space as shown in Fig. 3.

Since $R^2=r_1^2+r_2^2-2r_1r_2\cos\Phi$, we shall apply to $F(\lambda)$ an addition-formula for Bessel function of order $\nu(=m-1)$ (14):

$$\left.\begin{aligned} \rho^{-\nu}J_\nu(\rho) &= 2^\nu\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)\frac{J_{\nu+k}(Z)}{Z^\nu}\,\frac{J_{\nu+k}(X)}{X^\nu}C_k^\nu(\cos\Phi) , \\ \rho &= \sqrt{Z^2+X^2-2ZX\cos\Phi} \quad (\nu\neq 0, -1, -2, \cdots) . \end{aligned}\right\} \tag{14}$$

Then, our characteristic function $F(\lambda)$ in Eq. (11) can be expressed by

$$F(\lambda) = 4^{m-1}\Gamma(m)\Gamma(m-1) \times \sum_{k=0}^{\infty}(m-1+k)\left\langle\frac{J_{m-1+k}(\lambda r_1)}{(\lambda r_1)^{m-1}}\right\rangle\left\langle\frac{J_{m-1+k}(\lambda r_2)}{(\lambda r_2)^{m-1}}\right\rangle\left\langle C_k^{m-1}(\cos\Phi)\right\rangle. \quad (15)$$

Now, we take, along the first vector $\boldsymbol{r}_1$, the negative direction of $x_1$-axis of an $N$-dimensional rectangular coordinate-system and express the second vector $\boldsymbol{r}_2$ by $N$-dimensional polar coordinates, say $\boldsymbol{r}_2=[r_2, \theta_1, \theta_2, \cdots, \theta_{N-1}]$ as in Fig. 3, taking $\theta_1=\Phi$. Under the assumption that the magnitude of each vector is independent of its direction, we have $P(r_2, \theta_1, \theta_2, \cdots, \theta_{N-1})=P(r_2)P(\theta_1, \theta_2, \cdots, \theta_{N-1})$. Further, from the randomness, i.e., the isotropic property of each vector, the probability that $\boldsymbol{r}_2$ should face to any surface element $dS_{(N)}$ of the $N$-dimensional hypersphere (radius $r_2$, surface area $S_{(N)}$) is constant.

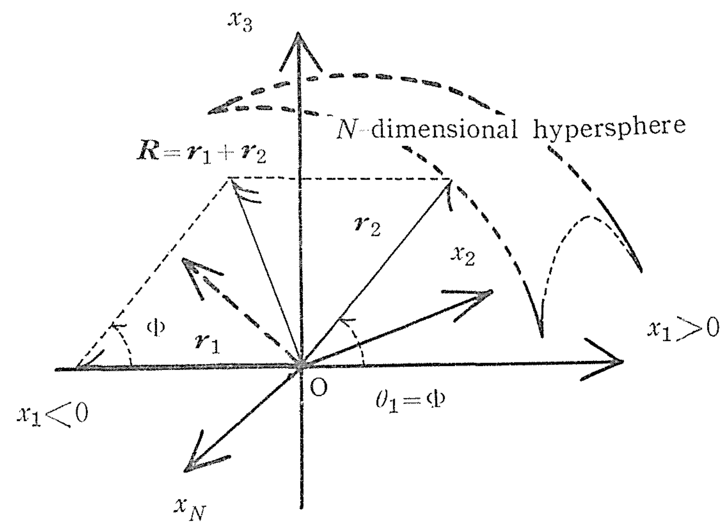


Fig. 3. Random walks with two steps in an $N$-dimensional space.

Thus, we have the following relation:

$$P(\theta_1, \theta_2, \cdots, \theta_{N-1})d\theta_1 d\theta_2\cdots d\theta_{N-1} = \frac{1}{S_{(N)}}ds_{(N)} = \frac{\Gamma(m+1)}{2m(\sqrt{\pi})^{2m}}\prod_{i=1}^{N-1}(\sin\theta_i)^{N-1-i}d\theta_1 d\theta_2\cdots d\theta_{N-1}, \quad (2m=N), \quad (16)$$

where

$$S_{(N)} = r_2^{N-1}\frac{\Gamma\left(\frac{1}{2}\right)^N 2m}{\Gamma(m+1)} \quad \text{and} \quad ds_{(N)} = r_2^{N-1}\prod_{i=1}^{N-1}(\sin\theta_i)^{N-1-i}d\theta_i.$$

Accordingly, from Eq. (16), the mean value of the Gegenbauer function (**14**): $\langle C_k^{m-1}(\cos\Phi)\rangle$ in Eq. (15) becomes:

$$\langle C_k^{m-1}(\cos \Phi)\rangle = \int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi} C_k^{m-1}(\cos\theta_1)P(\theta_1,\theta_2,\cdots,\theta_{N-1})d\theta_1 d\theta_2\cdots d\theta_{N-1}$$

$$= \int_0^{\pi} C_k^{m-1}(\cos\theta_1)(\sin\theta_1)^{N-2}d\theta_1\int_0^{\pi}(\sin\theta_2)^{N-3}d\theta_2\int_0^{\pi}\cdots\int_0^{\pi}\cdots\int_0^{2\pi}d\theta_{N-1}\frac{\Gamma(m+1)}{2m(\sqrt{\pi})^{2m}}. \quad (17)$$

Hereupon, if we use an orthogonality relation for the Gegenbauer function (14):

$$\int_0^{\pi} C_k^{m-1}(\cos\theta_1)C_0^{m-1}(\cos\theta_1)\sin^{2(m-1)}\theta_1 d\theta_1$$

$$= \delta_{k0}\frac{\pi\Gamma(2m-2)}{2^{2m-3}(m-1)\Gamma(m-1)^2}, \quad (18)$$

under the particular condition $m=N/2$, we can derive:

$$F(\lambda) = \left\langle 2^{m-1}\Gamma(m)\frac{J_{m-1}(\lambda r_1)}{(\lambda r_1)^{m-1}}\right\rangle\left\langle 2^{m-1}\Gamma(m)\frac{J_{m-1}(\lambda r_2)}{(\lambda r_2)^{m-1}}\right\rangle$$

$$= F_1(\lambda)F_2(\lambda). \quad (19)$$

(ii) Case when $N=2$ $(m=1)$

In this case, since $m=1$, we cannot use the same addition-formula for Bessel functions as in the preceding case. Therefore, we had better use instead the following addition-formula for Bessel funcitons (14):

$$J_\nu(\rho)_{\sin}^{\cos}\nu\Psi = \sum_{k=-\infty}^{\infty} J_{\nu+k}(Z)J_k(X)_{\sin}^{\cos}k\Phi, \quad (20)$$

as illustrated in Fig. 4.

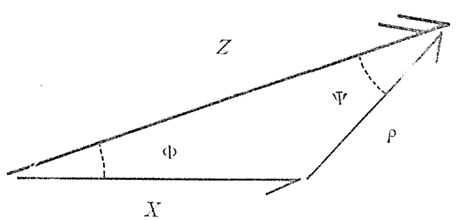


Fig. 4. Addition-formula for Bessel functions.

Thus, $F(\lambda)$ in the case $m=1$ becomes:

$$F(\lambda) = \langle J_0(\lambda R)\rangle = \langle J_0(\lambda r_1)\rangle\langle J_0(\lambda r_2)\rangle + 2\sum_{k=1}^{\infty}\langle J_k(\lambda r_1)J_k(\lambda r_2)\rangle\langle\cos k\Phi\rangle. \quad (21)$$

Since $P(\Phi)=\frac{1}{2\pi}$ corresponding to the same isotropic property as in the case when $N\neq 2$, we find

$$\langle\cos k\Phi\rangle = 0 \quad (k\neq 0).$$

Finally, we have

$$F(\lambda) = \langle J_0(\lambda r_1)\rangle\langle J_0(\lambda r_2)\rangle = F_1(\lambda)F_2(\lambda). \quad (22)$$

Thus, the proof has been completed.

**[C] The transformation between Hankel and Laplace type characteristic functions.**

As stated in [B], it goes without saying that our joint-characteristic function of Hankel type: $F(\lambda_1, \lambda_2, \cdots, \lambda_K)$ in Eq. (8) is also effective in calculating the joint-probability: $P(R_1, R_2, \cdots, R_K)$ in the vector field, in contrast to the usual joint moment generating function: $m(t_1, t_2, \cdots, t_K) = \langle \exp(-\sum_{h=1}^{K} t_h E_h) \rangle$ being effective in calculating the joint-probability: $P(E_1, E_2, \cdots, E_K)$ in the scalar field.

We must, however, point out that $F(\lambda_1, \lambda_2, \cdots, \lambda_K)$ and $m(t_1, t_2, \cdots, t_K)$ are closely connected with each other (**11**). When setting $E_h = R_h^2$, by use of the integral formula (**15**):

$$\int_0^\infty e^{-tR^2} R^m J_{m-1}(\lambda R) dR = \frac{\lambda^{m-1}}{(2t)^m} e^{-(\lambda^2/4t)}, \tag{23}$$

we can find that the transformability between the above two characteristic functions reduces itself to a $K$-dimensional Laplace transform as follows:

$$M(S_1, S_2, \cdots, S_K) = \int_0^\infty \cdots \int_0^\infty e^{-(S_1\Lambda_1 + S_2\Lambda_2 + \cdots + S_K\Lambda_K)} \times G(\Lambda_1, \Lambda_2, \cdots, \Lambda_K) d\Lambda_1 d\Lambda_2 \cdots d\Lambda_K, \tag{24}$$

where

$$\left.\begin{aligned} M(S_1, S_2, \cdots, S_K) &= m(t_1, t_2, \cdots, t_K) \prod_{h=1}^{K} 2^{2m_h} \Gamma(m_h) t_h^{m_h}, \\ &\qquad \left(S_h = \frac{1}{4t_h}\right), \\ G(\Lambda_1, \Lambda_2, \cdots, \Lambda_K) &= F(\lambda_1, \lambda_2, \cdots, \lambda_K) \prod_{h=1}^{K} \lambda_h^{2(m_h - 1)} \\ &\qquad (\Lambda_h = \lambda). \end{aligned}\right\} \tag{25}$$

Similarly, our $F(\lambda_1, \lambda_2, \cdots, \lambda_K)$ is essentially connected with a usual characteristic function $\varphi(t_1, t_2, \cdots, t_K) = \langle \exp(i \sum_{h=1}^{K} t_h E_h) \rangle$ by a $K$-dimensional Fourier transform.

## 4. The integral representation of probability density of $K$ correlative series of random walks.

We have indicated that the probability density $P(R)$ for one series (i.e., $\boldsymbol{R} = \sum_{i=1}^{S} \boldsymbol{r}_i$) of the random walks in an $N$-dimensional space can be given by Eq. (11) in a form of integral expression. Of course, when the magnitude and the direction of each vector are independent of all the preceding ones, we have $F(\lambda) = \prod_{i=1}^{S} F_i(\lambda)$ (cf. Eq. (13)). A case of special interest when $P(r_i) = \delta(r_i - r_{i0})$ arises when the magnitude of each vector is respectively constant, say $|\boldsymbol{r}_i| = r_{i0}$.

On this occassion, a probability $\mathcal{P}(R)(=\int_0^R P(R)dR)$ in Eq. (13) agrees with Watson's results (**15**), and also, using an integral formula (**15**):

$$\int_0^\infty t^{1+2\nu-\nu m} \prod_{n=1}^{S+1} J_\nu(a_n t)dt = 0 \qquad (a_1 > \sum_{n=2}^{S+1} a_n,\ \mathrm{Re}(\nu) > -1)\,, \tag{26}$$

we can find that $P(R)=0$ if $R > \sum_{i=1}^{S} r_{i0}$.

Generalizing the above, we shall now consider $K$ correlative series (i.e, $h=1, 2, \cdots, K$) of random walks of this kind, and assume that the $h$th series of random walks $\boldsymbol{R}_h = \sum_{j=1}^{S_h} \boldsymbol{r}_{hj}$ exists in an $N_h$-dimensional vector space, as shown in Fig. 1. Then, corresponding to Eq. (11), we can here use the same integral expression as in Eq. (8) or (9), under the special additional condition: $m_h = N_h/2$, by altering the interpretation (**1, 4, 6**).

## 5. The explicit repsesentation of intensity distribution of random noise current

[ i ] First, one series of random walks is considered, consisting of the regular components $\boldsymbol{I}_j(j=1, 2, \cdots, q)$ and the random components $\boldsymbol{r}_j(j=q+1, q+2, \cdots, S)$ in an $N$-dimensional vector space as shown in Fig. 5. That is,

$$R = \sum_{j=1}^{q} \boldsymbol{I}_j + \sum_{j=q+1}^{S} \boldsymbol{r}_j\,.$$

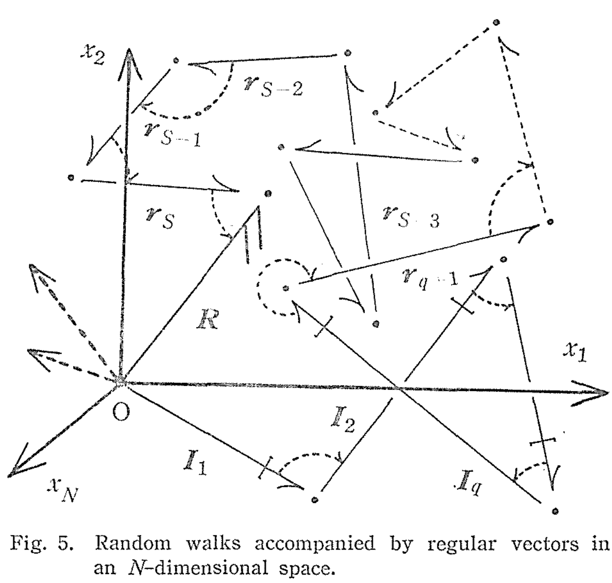


Fig. 5. Random walks accompanied by regular vectors in an $N$-dimensional space.

We shall now obtain the general expression of probability density $P(R)$ for very large values of $(S-q)$ under no special assumption concerning the distribution of the different walks except that all the $P(r_j)$'s $(r_j=|\boldsymbol{r}_j|)$ are nearly of the same functional form.

Since $\prod_{j=q+1}^{S} F_j(\lambda) \simeq e^{-(\Omega_0/4m)\lambda^2}$ $(\Omega_0=\sum_{j=q+1}^{S}\langle r_j^2\rangle)$ in Eq. (11) by use of Laplace's process or the method of steepest descent (**14, 15**) we can rewrite Eq. (13) in a form as follows:

$$P(R) = (2^{m-1}\Gamma(m))^{q-1}R^m \int_0^\infty \lambda^m J_{m-1}(\lambda R) e^{-(\Omega_0/4m)\lambda^2} \times \prod_{j=1}^{q} \frac{1}{(\lambda I_j)^{m-1}} J_{m-1}(\lambda I_j) d\lambda, \quad (I_j = |\boldsymbol{I}_j|). \tag{27}$$

In the special case of interest when $N=2$ $(m=1)$ corresponding to the random wave problem in the example (i) of §1 [B], i.e.,

$$I = \sum_{j=1}^{q} I_j \cos \omega_j t \text{ (signal)} + I_N \text{ (noise)},$$

the probability density for an envelope amplitude of $I$ agrees with S. O. Rice's expression (8) as follows:

$$P(R) = R\int_0^\infty \lambda J_0(\lambda R) e^{-(\Psi_0/4)\lambda^2} \prod_{j=1}^{q} J_0(\lambda I_j) d\lambda, \quad (\Psi_0 = \langle I_N^2\rangle). \tag{28}$$

Further, when $\boldsymbol{R}=\boldsymbol{I}_1+\sum_{j=2}^{S}\boldsymbol{r}_j$ (i.e., $q=1$) in the above, Eq. (27) can be expressed explicitly by

$$P(R) = \frac{NR^{N/2}}{\Omega_0}\left(\frac{1}{I_1}\right)^{(N/2)-1} e^{-\{N(R^2+I_1^2)/2\Omega_0\}} I_{(N/2)-1}\left(\frac{NI_1}{\Omega_0}R\right), \quad \text{(Bessel-distribution)}, \tag{29}$$

where $\Omega_0=\sum_{j=2}^{S}\langle \boldsymbol{r}_j^2\rangle$ (**1, 7, 13, 14, 16**). And if $q$ regular vectors $\boldsymbol{I}_j (j=1, 2, \cdots, q)$ are given instead of one regular vector $\boldsymbol{I}_1$, it can be proved that Eq. (29) approximately expresses $P(R)$ in this case, under the substitutions $I_1=\sqrt{\sum_{j=1}^{q} I_j^2}$ and $\Omega_0=\sum_{j=q+1}^{S}\langle \boldsymbol{r}_j^2\rangle$ by applying to Eq. (13) the property in the vicinity of $\lambda=0$ for the following addition-formula for Bessel functions (**14**):

$$\frac{J_\nu(\sqrt{Z^2+Y^2})}{(\sqrt{Z^2+Y^2})^\nu} = 2^\nu \sum_{m=0}^{\infty} (-1)^m(\nu+2m)\frac{\Gamma(\nu+m)}{m!} \times \frac{J_{\nu+2m}(Z)}{Z^\nu}\frac{J_{\nu+2m}(Y)}{Y^\nu}, \quad (Z=\lambda I_i,\ Y=\lambda I_j,\ i \neq j), \tag{30}$$

using the method of steepest descent.

Also, in the case of no regular vectors $(q=0)$, we have:

$$P(R) = \frac{2\left(\frac{N}{2}\right)^{N/2} R^{N-1}}{\Gamma\left(\frac{N}{2}\right)\Omega^{N/2}} e^{-(NR^2/2\Omega_0)}, \quad \left(P(E) = P(R)\left|\frac{dR}{dE}\right| : \Gamma\text{-distribution}\right). \tag{31}$$

Of course, in the above-mentioned expressions, we must take $\Omega_0=(S-q)r_0^2$ when all the $r_j$'s have equal lengths, say $r_j=r_0$ $(j=q+1, q+2, \cdots, S)$. In order to improve the approximation, we had better use the following asymptotic expression **(1, 2, 16)**:

$$P(R) = \frac{2\left(\frac{N}{2}\right)^{N/2} R^{N-1}}{\Gamma\left(\frac{N}{2}\right)\Omega_0^{N/2}} e^{-(NR^2/2\Omega_0)}\left\{1-\frac{2}{(N+2)S}L_2^{(N/2)-1}\left(\frac{N}{2Sr_0^2}R^2\right) + O\left(\frac{1}{S^2}\right)\right\} \quad (\Omega_0 = Sr_0^2), \quad (32)$$

instead of Eq. (31).

[ii] We shall again consider putting $p(r_j)=\delta(r_j-r_{j0})$ in Eq. (13) in the same way as we have done in §4, under a special condition $S=2$ corresponding to the special case $q=0$, $S=2$ in the above [i]. Then, we obtain the following explicit expression:

$$\left.\begin{aligned} P(R) &= \frac{1}{2^{N-3}}\left(\frac{1}{r_{10}r_{20}}\right)^{N-2}\frac{\Gamma(N/2)}{\sqrt{\pi}\,\Gamma\left(\frac{N-1}{2}\right)} \\ &\quad\times R[R^2-(r_{10}-r_{20})^2]^{(N-3)/2}[(r_{10}+r_{20})^2-R^2]^{(N-3)/2} \\ &\qquad\qquad \text{for } |r_{10}-r_{20}|<R<|r_{10}+r_{20}|, \\ &= 0 \qquad\qquad \text{for } 0<R<|r_{10}-r_{20}| \text{ or } r_{10}+r_{20}<R<\infty. \end{aligned}\right\} \quad (33)$$

When we especially fix $N=2$, this expression evidently means an interference effect between two random phase waves and becomes explicitly

$$\left.\begin{aligned} P(R) &= \frac{2}{\pi}\frac{R}{\sqrt{[R^2-(r_{10}-r_{20})^2][(r_{10}+r_{20})^2-R^2]}}, \\ &\qquad (|r_{10}-r_{20}|<R<r_{10}+r_{20}) \\ \mathcal{P}(R) &= \int_0^R P(R)dR = 0, \qquad (0<R<|r_{10}-r_{20}|) \\ &= \frac{1}{\pi}\cos^{-1}\frac{r_{10}^2+r_{20}^2-R^2}{2r_{10}r_{20}}, \quad (|r_{10}-r_{10}|<R<r_{10}+r_{20}) \\ &= 1, \qquad (r_{10}+r_{20}<R<\infty). \end{aligned}\right\} \quad (34)$$

[iii] In the next place, we consider the case of very large $S_h$ in two correlative series of such random walks: $\boldsymbol{R}_h=\sum_{j=1}^{S_h}\boldsymbol{r}_{hj}$ $(h=1, 2)$ having no regular vectors (i.e., $q=0$) in an $N$-dimensional space as shown in Fig. 6. Then, we can derive the joint-probability density of Eq. (31) from Eq. (8) under the particular conditions: $\Omega'_{h0}=\frac{1}{m}\Omega_{h0}=\frac{2}{N}\sum_{j=1}^{S_h}\langle r_{hj}^2\rangle$ $(h=1, 2)$ and $N_1=N_2=N=2m$ as follows:

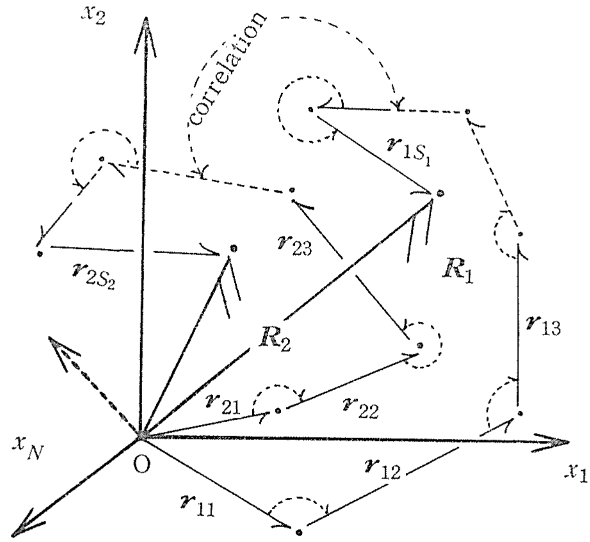


Fig. 6. Two correlative series of random walks in an $N$-dimensional space.

$$\left.\begin{aligned} P(R_1, R_2) &= \frac{4(R_1R_2)^{N/2}}{\Gamma\left(\frac{N}{2}\right)\Omega'_{10}\Omega'_{20}(1-\rho_E)}\left\{\frac{1}{\Omega'_{10}\Omega'_{20}\rho_E}\right\}^{(N/2)-1} \\ &\quad \times e^{-\frac{1}{1-\rho_E}\left\{\frac{R_1^2}{\Omega'_{10}}+\frac{R_2^2}{\Omega'_{20}}\right\}} I_{(N/2)-1}\left(\frac{2\sqrt{\rho_E}R_1R_2}{(1-\rho_E)\sqrt{\Omega'_{10}\Omega'_{20}}}\right) \\ &= P(R_1)P(R_2)\left\{1+\sum_{n=1}^{\infty} L_n^{(N/2)-1}\left(\frac{R_1^2}{\Omega'_{10}}\right)L_n^{(N/2)-1}\left(\frac{R_2^2}{\Omega'_{20}}\right)\rho_E^n \right. \\ &\qquad \left. \times nB\left(n, \frac{N}{2}\right)\right\} \quad (R_1, R_2 > 0)\,, \\ &= 0 \qquad\qquad\qquad\qquad (R_1, R_2 \leqslant 0)\,, \end{aligned}\right\} \tag{53}$$

$$F(\lambda_1, \lambda_2) = \frac{\Gamma\left(\frac{N}{2}\right)2^{N-2}(\lambda_1\lambda_2)^{1-N/2}}{(\sqrt{\rho_E\Omega'_{10}\Omega'_{20}})^{(N/2)-1}} e^{-\frac{1}{4}[\Omega'_{10}\lambda_1^2+\Omega'_{20}\lambda_2^2]} \times I_{(N/2)-1}\left(\frac{1}{2}\sqrt{\rho_E\Omega'_{10}\Omega'_{20}}\,\lambda_1\lambda_2\right), \tag{36}$$

where $P(R_h)$'s express respectively

$$P(R_h) = \frac{2\left(\frac{N}{2}\right)^{N/2}}{\Gamma\left(\frac{N}{2}\right)\Omega_{h0}^{N/2}} R_h^{N-1} e^{-(NR_h^2/2\Omega_{h0})} \quad (h = 1, 2)\,,$$

following Eq. (31), and $\rho_E$ denotes the correlation coefficient of $R_1^2$ and $R_2^2$. In the above derivation, we have used the following integrals (**14**):

$$\left.\begin{aligned}&\int_0^\infty \lambda_1 e^{-(\Omega/4)\lambda_1^2} I_{m-1}\Big(\frac{C}{2}\lambda_1\lambda_2\Big) J_{m-1}(\lambda R_1)d\lambda_1\\ &\qquad = \frac{2}{\Omega}\exp\Big\{\frac{1}{\Omega}\Big[\Big(\frac{C}{2}\lambda_2\Big)^2-R_1^2\Big]\Big\} J_{m-1}\Big(\frac{C}{\Omega}\lambda_2R_1\Big),\\ &\int_0^\infty xe^{-Ax^2}J_\nu(Bx)J_\nu(Rx)dx\\ &\qquad = \frac{1}{2A}\exp\Big\{-\frac{1}{4A}(B^2+R^2)\Big\} I_\nu\Big(\frac{1}{2A}BR\Big).\end{aligned}\right\} \tag{37}$$

Now, to apply Eq. (35) to the detector output such as considered in §2, Eqs. (3) and (5) are taken into account. Then, we can obtain the joint energy distribution $P(E_1, E_2)$ of detector output noise as follows:

$$P(E_1, E_2) = \frac{D_1D_2}{\Gamma(m)(1-\rho_E)}\Big(\sqrt{\frac{D_1D_2E_1E_2}{\rho_E}}\Big)^{m-1}\exp\Big\{-\frac{1}{1-\rho_E}(D_1E_1+D_2E_2)\Big\} \times I_{m-1}\Big(\frac{2\sqrt{\rho_E D_1D_2E_1E_2}}{1-\rho_E}\Big), \tag{38}$$

where

$$E_h = R_h^2,\ \sigma_{Eh}^2 = \langle(E_h-\langle E_h\rangle)^2\rangle,$$
$$D_h = m/\langle E_h\rangle = \langle E_h\rangle/\sigma_{Eh}^2,$$

and

$$\langle E_h\rangle = \Omega_{h0} \quad (h=1, 2).$$

By use of the integral formula (14):

$$\int_0^\infty e^{-p^2t^2}t^{\nu+1}J_\nu(at)dt = \frac{a^\nu}{(2p^2)^{\nu+1}}e^{-(a^2/4p^2)} \quad (\mathrm{Re}(\nu) > -1), \tag{39}$$

the moment generating function $m(t_1, t_2)$ of Eq. (38) can be derived as follows:

$$m(t_1, t_2) = \Big[\Big(1+\frac{t_1}{D_1}\Big)\Big(1+\frac{t_2}{D_2}\Big)-\rho_E\frac{t_1t_2}{D_1D_2}\Big]^{-m}. \tag{40}$$

When Eq. (39) is used again, we can easily confirm that the above $m(t_1, t_2)$ is essentially connected with $F(\lambda_1, \lambda_2)$ in Eq. (36) by the 2-dimensional Laplace transform from Eqs. (24) and (25) ($K=2$).

[iv] We shall again take up the generalized problem of $K$ correlative series of random walks, $\boldsymbol{R}_h=\sum_{j=1}^{S_h}\boldsymbol{r}_{hj}$ ($\boldsymbol{r}_{hj}=\sum_{i=1}^{N}e_ix_{hji}=[x_{hj1}, x_{hj2}, \cdots, x_{hjN}]$) which is the $h$th series of such random walks in the same $N$-dimensional vector space. When we are specially interested in the asymptotic expression of the joint-probability for a large number of steps corresponding to Eqs. (31) and (35), we had better fix our eyes upon the form given by Eq. (3), that is, treat the probability problem given in the same form as Eq. (5).

Since $S_h$'s are very large, by use of the central limit theorem in $NK$ dimensions, we can find that $x_{hi}$'s ($h=1, 2, \cdots, K$; $i=1, 2, \cdots, N$) are distributed according to the $NK$-dimensional Gaussian distribution. If the coordinate components of an $N$-dimensional Euclidean space are statistically independent of each other, the joint distribution density of $x_{hi}$'s can be expressed in the form of such product of $K$-dimensional Gaussian distribution as seen in Eq. (4).

This property completely agrees with the starting point of §IV, §V and §VI in a previous paper (**11**). Accordingly, by thinking each energy $E_h$ as the square of magnitude of each resultant vector $\boldsymbol{R}_h$, we can obtain many different explicit solutions to the generalized problem of random walks from separate energy distribution in the previous paper (**11**).

Moreover, if $N \to \infty \left( m = \frac{N}{2} = \frac{\langle E_h \rangle^2}{\sigma^2_{E_h}} \right)$ according to an estimation by the method of moment (**2, 7**), by using the central limit theorem in $N$ dimensions, we can also find that the joint-probability density of $E_h$ ($= R_h^2$, $h = 1, 2, \cdots, K$) is distributed asymptotically according to the $K$-dimensional Gaussian distribution.

## 6. Conclusion

In this paper, we have shown that the intensity fluctuation of random noise current can be investigated as a problem of many correlative series of random walks in multi-dimensional space, and from such a point of view, when we pay our attention to the energy fluctuation of white noise, the characteristic function of Hankel type is very useful for calculating its probability distribution. Then, for typical cases, we have derived several explicit expressions for the intensity distribution of white noise.

The following properties of random noise were derived:

(1) The number of dimensions for the used space is closely connected with the statistical characters of random noise and the characters of the observational device.

(2) The fluctuation in the output noise is reasonably described by one parameter $m$ involved in the energy distribution expressions.

(3) The above parameter $m$ is approximately equivalent to $TW$ ($W$: equivalent noise band-width, $T$: time-interval of observation) appearing in the sampling theorem.

The method described in this paper is also applicable to the other fields of measurement on the random phenomena, since the mean energy is a conservative physical quantity.

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## REFERENCES


1. M. ŌTA, Report to the Special Research Committee of Information Theory, I.E.C.E. of Japan, April, 1956 (in Japanese).
2. M. ŌTA, Report to the U.R.S.I. Commission VI in Japan, at the Scientific Convention, Dec. 1956 (in Japanese).
3. A. VAN DER ZIEL, "*Noise*" (Prentice-Hall, New York, 1954).
4. M. ŌTA, Proceedings of the 9th Japan National Congress for Applied Mechanics (1959), 473~481.
5. M. ŌTA, Reports to the Symposium at the Kansai Meeting of Electrical Society of Japan, Oct. 1958, p. 307 & Cct. 1959, p. 343 (in Japanese).
6. M. ŌTA, Report to the Symposium of Statistical Methods in the Theory of Automatic Control, Oct. 1959, pp. 6~22 (in Japanese).
7. M. ŌTA, Proceedings of the 6th Japan National Congress for Applied Mechanics (1956), 511~516.
8. S. O. RICE, Bell System Tech. J., **23** (1944), 282 & **24** (1945), 46.
9. D. SIEPIAN, Bell System Tech. J., **37** (1958), 163.
10. N. WIENER, "*Extrapolation, Interpolation and Smoothing of Stationary Time Series.*" (John Wiley & Sons, New York, 1949).
11. M. ŌTA, Proceedings of the 8th Japan National Congress for Applied Mechanics (1958), 425~430.
12. M. ŌTA, Proceedings of the 5th Japan National Congress for Applied Mechanics (1955), 483~492.
13. C. E. SHANNON, Bell System Tech. J., **27** (1949) p. 379 & 623.
14. YOSHIDA, AMEMIYA, ITŌ & MATSUSHIMA, "*Ōyōsūgaku-Binran*", (Maruzen, Tōkyo, 1954) (in Japanese).
15. G. N. WATSON, "*Theory of Bessel Functions*" (Cambridge Univ. Press, 1922).
16. M. ŌTA, Proceedings of the 7th Japan National Congress for Applied Mechanics (1957), 317~322.