$\langle q, r \rangle$-number systems and algebraic independence

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This is an announcement of our results in [9].

Let \( q \) and \( r \) are integers with \( q \geq 2 \) and \( 0 \leq r \leq q - 1 \). In the \(\langle q, r\rangle\) number system, every integer \( n \in \mathbb{Z} \) is uniquely expressed with base \( q \) and digits \(-r, 1-r, \cdots, 0, \cdots, q-1-r\); namely,

\[
n = \sum_{h=0}^{k} \delta_h q^h, \quad \delta_h \in \{-r, 1-r, \cdots, q-1-r\}, \quad \delta_h \neq 0 \text{ if } n \neq 0,
\]

(1)

where \( \mathbb{Z} \) should be replaced by \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{< 0} \) if \( r = 0 \) and \( r = q - 1 \), respectively. The usual \( q \)-adic expansion is the \(\langle q, 0\rangle\) number system. Symmetrically, in the \(\langle q, q-1-r\rangle\) number system \(-n\) is uniquely expressed as

\[
-n = \sum_{h=0}^{k} (-\delta_h) q^h,
\]

(2)

where \( \delta_h \) are as above (cf. [3], [5]).

Furthermore, taking the negative base \(-q\), we have the \(\langle -q, r\rangle\) number system, in which every \( n \in \mathbb{Z} \) is uniquely expressed as

\[
n = \sum_{h=0}^{l} \epsilon_h (-q)^h, \quad \epsilon_h \in \{-r, 1-r, \cdots, q-1-r\}, \quad \epsilon_l \neq 0 \text{ if } n \neq 0
\]

(3)

(without exception on \( r \)). In the \(\langle -q, q-1-r\rangle\) number system, we have also an expansion of \(-n\) similar to (2).

An arithmetical function \( a_r(n) : \mathbb{Z} \rightarrow \mathbb{C} \) is called \(\langle q, r\rangle\)-linear, if there is an \( \alpha \in \mathbb{C}^\times \) such that

\[
a_r(nq + t) = \alpha a_r(n) + a_r(t)
\]

(4)
for any \( n \in \mathbb{Z} \) and \( t \in \mathbb{Z} \) with \( -r \leq t \leq q - 1 - r \), where \( \mathbb{Z} \) is replaced by \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{\leq 0} \) if \( r = 0 \) and \( r = q - 1 \), respectively. By definition, \( a_r(0) = 0 \). Using the expansion (1), we have

\[
a_r(n) = \sum_{h=0}^{k} a_r(\delta_h) \alpha^h, \tag{5}
\]

and so \( a_r(n) \) is determined by the coefficient \( \alpha \) and the initial vector

\[
a_r = (a_r(-r), a_r(1-r), \ldots, a_r(0), \ldots, a_r(q-1-r)). \tag{6}
\]

It follows from (2) and (5) that

\[
a_{q-1-r}(-n) = \sum_{h=0}^{k} a_{q-1-r}(-\delta_h) \alpha^h. \tag{7}
\]

An arithmetical function \( b_r(n) : \mathbb{Z} \to \mathbb{C} \) is called \((-q, r)\)-linear, if there is a \( \beta \in \mathbb{C}^\times \) such that

\[
b_r(n(-q) + t) = \beta b_r(n) + b_r(t) \tag{8}
\]

for any \( n \in \mathbb{Z} \) and \( t \in \mathbb{Z} \) with \( -r \leq t \leq q - 1 - r \). We have \( b_r(0) = 0 \) and

\[
b_r(n) = \sum_{h=0}^{l} b_r(\epsilon_h) \beta^h, \tag{9}
\]

using the expression (3), so that \( b_r(n) \) is determined by the coefficient \( \beta \) and the initial vector

\[
b_r = (b_r(-r), b_r(1-r), \ldots, b_r(0), \ldots, b_r(q-1-r)).
\]

For \( b_{q-1-r}(n) \), we have an expression similar to (7)

**Examples.** We give some examples of \((q, r)\)-linear functions using the expression (1) of \( n \in \mathbb{Z} \), where \( \mathbb{Z} \) should be replaced by \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{\leq 0} \) if \( r = 0 \) and \( r = q - 1 \), respectively.

1. The sum of digits function in the \((q, r)\) number system defined by \( s_{(q,r)}(n) = \sum_{h=0}^{k} \delta_h \) is \((q, r)\)-linear with the coefficient 1 and the initial vector \((-r, 1-r, \ldots, q-1-r)\). Delange[1] proved for the ordinary \( q \)-adic sum of digits function \( s_q(n) = s_{(q,0)}(n) \) that

\[
\frac{1}{N} \sum_{n < N} s_q(n) = \frac{q-1}{2} \log_q N + F(\log_q N), \tag{10}
\]
where $F(x)$ is a continuous, nowhere differentiable function of period 1, whose Fourier coefficients are given explicitly. Flajolet and Ramshaw[3] and Grabner and Thuswaldner[4] studied these phenomena in the $(q, r)$ number systems and in the $-q$ adic ones, respectively.

2. For any given $t = -r, 1 - r, \ldots, q - 1 - r$, $e_{tr}(n)$ denotes the number of the digits $t$ appearing in the $(q, r)$-expansion (1) of $n \in \mathbb{Z}$, which is $(q, r)$-linear with the coefficient 1 and the initial vector $q^{-1}(-r, 1 - r, \ldots, q - 1 - r)$. Furthermore, for any given permutation $\sigma$ of $\{-r, 1 - r, \ldots, q - 1 - r\}$ with $0^\sigma = 0$, the generalized radical inverse function defined by $\phi_{(q, r)}(n) = \sum_{h=0}^{k} \delta_{h} q^{-h-1}$ is $(q, r)$-linear with the coefficient $q^{-1}$ and the initial vector $q^{-1}((-r)^{\sigma}, (1 - r)^{\sigma}, \ldots, (q - 1 - r)^{\sigma})$ (cf. [8] Chapter 3).

3. The radical inverse function in the $(q, r)$ number system defined by $\phi_{(q, r)}(n) = \sum_{h=0}^{k} \delta_{h} q^{-h-1}$ is $(q, r)$-linear with the coefficient $q^{-1}$ and the initial vector $q^{-1}(-r, 1 - r, \ldots, q - 1 - r)$. Furthermore, for any given permutation $\sigma$ of $\{-r, 1 - r, \ldots, q - 1 - r\}$ with $0^\sigma = 0$, the generalized radical inverse function defined by $\phi_{(q, r)}(n) = \sum_{h=0}^{k} \delta_{h} q^{-h-1}$ is $(q, r)$-linear with the coefficient $q^{-1}$ and the initial vector $q^{-1}((-r)^{\sigma}, (1 - r)^{\sigma}, \ldots, (q - 1 - r)^{\sigma})$ (cf. [8] Chapter 3).

4. For any given $p \in \mathbb{Z}$ with $|p| \geq q$, the bases change function $\gamma_{qr}(n)$ is defined by $\gamma_{qr}(n) = \sum_{h=0}^{k} \delta_{h} q^{-h-1}$, which is $(q, r)$-linear with the coefficient $p$ and the initial vector $(-r, 1 - r, \ldots, q - 1 - r)$ (cf. [2]).

5. The linear function $cn$ ($c \in \mathbb{C}$) is $(q, r)$-linear with the coefficient $q$ and the initial vector $c(-r, 1 - r, \ldots, q - 1 - r)$.

Examples of $(-q, r)$-linear functions can be constructed similarly as above by using the expression (3).

Recently, Kurosawa and the second named author[6] gave a necessarily and sufficient condition for the generating functions of $(q, 0)$-linear functions and $(-q, 0)$-linear ones to be algebraically independent over $\mathbb{C}(z)$. We note that the generating function of $a(n) = cn$ given in Example 5 is

$$\frac{z}{(1-z)^2} \in \mathbb{C}(z).$$

We state our theorems. Let $\alpha_i, \beta_i \in \mathbb{C}$ $(1 \leq i \leq I)$ satisfy

$$\alpha_i \neq \alpha_j, \beta_i \neq \beta_j \ (i \neq j, 1 \leq i, j \leq I). \tag{11}$$

For any fixed $q$, let $a_{ilr}(n)$ $(1 \leq l \leq m(i))$ and $b_{ilr}(n)$ $(1 \leq l \leq n(i))$ be $(q, r)$-linear functions and $(-q, r)$-linear ones with coefficients $\alpha_i$ and $\beta_i$, respectively. We consider the generating functions

$$f_{ilr}(z) = \sum_{n=0}^{\infty} a_{ilr}(n) z^n, \quad f_{ilr}^{*}(z) = \sum_{n=0}^{\infty} a_{ilr}(-n) z^n.$$
which converge in $|z| < 1$ by (4) and (8). We put

$$a_{ir} = (a_{ir}(-r), a_{ir}(1-r), \ldots, a_{ir}(q-1-r)),$$

$$b_{ir} = (b_{ir}(-r), b_{ir}(1-r), \ldots, b_{ir}(q-1-r)).$$

For any vector $c = (c_1, c_2, \ldots, c_q)$, we write $\langle c \rangle = (c_q, c_{q-1}, \ldots, c_1)$.

**Theorem 1.1.** The functions $f_{ir}(z)$ (1 ≤ $i$ ≤ $I$, 1 ≤ $l$ ≤ $m(i)$, 0 ≤ $r$ < $q-1$), $g_{ir}(z)$ (1 ≤ $i$ ≤ $I$, 1 ≤ $l$ ≤ $m(i)$, 0 < $r$ ≤ $q-1$), $g_{ir}^*(z)$ (1 ≤ $i$ ≤ $I$, 1 ≤ $l$ ≤ $n(i)$, 0 ≤ $r$ < $q-1$, 2$r$ ≠ $q-1$) are algebraically independent over $\mathbb{C}(z)$ if and only if the following conditions (i) and (ii) hold:

(i) each one of the sets of vectors $\{a_{ir}, \alpha_{il_{q-1-r}}; 1 \leq l \leq m(i)\}$ (1 ≤ $i$ ≤ $I$, 0 ≤ $r$ < $q-1$) and $\{b_{ir}, \beta_{il_{q-1-r}}; 1 \leq l \leq n(i)\}$ (1 ≤ $i$ ≤ $I$, 0 ≤ $r$ ≤ $q-1$, 2$r$ ≠ $q-1$) is linearly independent over $\mathbb{C}$,

(ii) if $\alpha_i = q$, then for any $r$ with 0 ≤ $r$ < $q-1$

$$(-r, 1-r, \ldots, q-1-r) \notin \text{Span}_{\mathbb{C}} \{a_{ir}, \alpha_{il_{q-1-r}}; 1 \leq l \leq m(i)\},$$

and if $\beta_i = -q$, then for any $r$ with 0 ≤ $r$ ≤ $q-1$, 2$r$ ≠ $q-1$

$$(-r, 1-r, \ldots, q-1-r) \notin \text{Span}_{\mathbb{C}} \{b_{ir}, \beta_{il_{q-1-r}}; 1 \leq j \leq n(i)\}.$$

**Remark 1.1** To prove the theorem, we use a criterion of algebraic independence over $\mathbb{C}(z)$ of functions satisfying certain functional equations (cf. [7] Corollary of Theorem 3.2.1), which enable us to reduce the algebraic dependency over $\mathbb{C}(z)$ of our functions to the linear dependency of them over $\mathbb{C}$ mod $\mathbb{C}(z)$. So we actually prove that the functions in the theorem are algebraically dependent over $\mathbb{C}(z)$ if and only if, for some $i$ and $r$, $f_{ir}(z), f_{il_{q-1-r}}(z)$ (1 ≤ $l$ ≤ $m(i)$) are linearly dependent over $\mathbb{C}$, $g_{ir}(z), g_{ir}^*(z)$ (1 ≤ $l$ ≤ $n(i)$) are linearly dependent over $\mathbb{C}$, $\alpha_i = q$ and $z/(1-z)^2 \in \text{Span}_{\mathbb{C}} \{f_{il_{q-1-r}}(z); 1 \leq l \leq m(i)\}$, or $\beta_i = -q$ and $z/(1-z)^2 \in \text{Span}_{\mathbb{C}} \{g_{il_{q-1-r}}(z); 1 \leq l \leq n(i)\}$.

**Remark 1.2** The conditions (i) and (ii) in Theorem 1.1 imply that $m(i), n(i) \leq q$ for any $i$, $\alpha_i \neq q$ if $m(i) = q$, and $\beta_i \neq -q$ if $n(i) = q$.

**Theorem 1.2.** Let the functions $f_{ir}(z), f_{il_{q-1-r}}(z), g_{ir}(z)$, and $g_{ir}^*(z)$ satisfy the conditions (i) and (ii) in Theorem 1.1. Assume that $\alpha_i, \beta_i, a_{ir}(n)$, and $b_{ir}(n)$ are algebraic for all $i, l, r$ and $n$. Then, for any algebraic number $\alpha$ with 0 < $|\alpha|$ < 1, the numbers $f_{ir}(\alpha)$ (1 ≤ $i$ ≤ $I$, 1 ≤ $l$ ≤ $m(i)$, 0 ≤ $r$ < $q-1$), $f_{il_{q-1-r}}(\alpha)$ (1 ≤ $i$ ≤ $I$, 1 ≤ $l$ ≤ $m(i)$, 0 < $r$ ≤ $q-1$), $g_{ir}(\alpha)$ and $g_{ir}^*(\alpha)$ (1 ≤ $i$ ≤ $I$, 1 ≤ $l$ ≤ $n(i)$, 0 ≤ $r$ ≤ $q-1$, 2$r$ ≠ $q-1$) are algebraically independent.
If we fix \( r = 0 \) in Theorem 1.1 and Theorem 1.2, we have the results of Kurosawa and the second named author[6] mentioned above. In their proof, they used another criterion ([7] Theorem 3.5) of algebraic independence of functions over \( \mathbb{C}(z) \).

References


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