<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>$&lt;q, r&gt;$-number systems and algebraic independence</td>
</tr>
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\( \langle q, r \rangle \)-number systems and algebraic independence

By

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This is an announcement of our results in [9].

Let \( q \) and \( r \) are integers with \( q \geq 2 \) and \( 0 \leq r \leq q - 1 \). In the \( \langle q, r \rangle \) number system, every integer \( n \in \mathbb{Z} \) is uniquely expressed with base \( q \) and digits \(-r, 1 - r, \ldots, 0, \ldots, q - 1 - r\); namely,

\[
n = \sum_{h=0}^{k} \delta_h q^h, \quad \delta_k \in \{-r, 1 - r, \ldots, q - 1 - r\}, \quad \delta_k \neq 0 \text{ if } n \neq 0,
\]

(1)

where \( \mathbb{Z} \) should be replaced by \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{< 0} \) if \( r = 0 \) and \( r = q - 1 \), respectively. The usual \( q \)-adic expansion is the \( \langle q, 0 \rangle \) number system. Symmetrically, in the \( \langle q, q - 1 - r \rangle \) number system \(-n\) is uniquely expressed as

\[
-n = \sum_{h=0}^{k} (-\delta_h) q^h,
\]

(2)

where \( \delta_h \) are as above (cf. [3], [5]).

Furthermore, taking the negative base \(-q\), we have the \( \langle -q, r \rangle \) number system, in which every \( n \in \mathbb{Z} \) is uniquely expressed as

\[
n = \sum_{h=0}^{l} \epsilon_h (-q)^h, \quad \epsilon_h \in \{-r, 1 - r, \ldots, q - 1 - r\}, \quad \epsilon_l \neq 0 \text{ if } n \neq 0
\]

(3)

(without exception on \( r \)). In the \( \langle -q, q - 1 - r \rangle \) number system, we have also an expansion of \(-n\) similar to (2).

An arithmetical function \( a_r(n) : \mathbb{Z} \rightarrow \mathbb{C} \) is called \( \langle q, r \rangle \)-linear, if there is an \( \alpha \in \mathbb{C}^\times \) such that

\[
a_r(nq + t) = \alpha a_r(n) + a_r(t)
\]

(4)
for any $n \in \mathbb{Z}$ and $t \in \mathbb{Z}$ with $-r \leq t \leq q - 1 - r$, where $\mathbb{Z}$ is replaced by $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{<0}$ if $r = 0$ and $r = q - 1$, respectively. By definition, $a_r(0) = 0$. Using the expansion (1), we have

$$a_r(n) = \sum_{h=0}^{k} a_r(\delta_h) \alpha^h,$$

and so $a_r(n)$ is determined by the coefficient $\alpha$ and the initial vector

$$a_r = (a_r(-r), a_r(1-r), \ldots, a_r(q-1-r)).$$

It follows from (2) and (5) that

$$a_{q-1-r}(-n) = \sum_{h=0}^{k} a_{q-1-r}(-\delta_h) \alpha^h.$$  

An arithmetical function $b_r(n) : \mathbb{Z} \to \mathbb{C}$ is called $(-q, r)$-linear, if there is a $\beta \in \mathbb{C}$ such that

$$b_r(n(-q) + t) = \beta b_r(n) + b_r(t)$$

for any $n \in \mathbb{Z}$ and $t \in \mathbb{Z}$ with $-r \leq t \leq q - 1 - r$. We have $b_r(0) = 0$ and

$$b_r(n) = \sum_{h=0}^{l} b_r(\epsilon_h) \beta^h,$$

using the expression (3), so that $b_r(n)$ is determined by the coefficient $\beta$ and the initial vector

$$b_r = (b_r(-r), b_r(1-r), \ldots, b_r(q-1-r)).$$

For $b_{q-1-r}(n)$, we have an expression similar to (7) 

**Examples.** We give some examples of $(q, r)$-linear functions using the expression (1) of $n \in \mathbb{Z}$, where $\mathbb{Z}$ should be replaced by $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{<0}$ if $r = 0$ and $r = q - 1$, respectively.

1. The sum of digits function in the $(q, r)$ number system defined by $s_{(q,r)}(n) = \sum_{h=0}^{k} \delta_h$ is $(q, r)$-linear with the coefficient 1 and the initial vector $(-r, 1-r, \ldots, q-1-r)$. Delange[1] proved for the ordinary $q$-adic sum of digits function $s_q(n) = s_{(q,0)}(n)$ that

$$\frac{1}{N} \sum_{n<N} s_q(n) = \frac{q-1}{2} \log_q N + F(\log_q N),$$
where $F(x)$ is a continuous, nowhere differentiable function of period 1, whose Fourier coefficients are given explicitly. Flajolet and Ramshaw[3] and Grabner and Thuswaldner[4] studied these phenomena in the $(q, r)$ number systems and in the $-q$ adic ones, respectively.

2. For any given $t = -r, 1 - r, \ldots, q - 1 - r$, $e_{tr}(n)$ denotes the number of the digits $t$ appearing in the $(q, r)$-expansion (1) of $n \in \mathbb{Z}$, which is $(q, r)$-linear with the coefficient 1 and the initial conditions $e_{tr}(s) = 1$ if $s = t$; 0 other wise. Flajolet and Ramshaw[3] proved Delange-type results for $e_{tr}(n)(-r \leq t \leq q - 1 - r)$ and applied them to the study of the summentary functions of $s_{(q, r)} = \sum_{t=r}^{q-1-r} te_{tr}(n)$.

3. The radical inverse function in the $(q, r)$ number system defined by $\phi_{(q,r)}(n) = \sum_{h=0}^{k} \delta_{h} q^{-h-1}$ is $(q, r)$-linear with the coefficient $q^{-1}$ and the initial vector $q^{-1}(-r, 1 - r, \ldots, q - 1 - r)$. Furthermore, for any given permutation $\sigma$ of $\{-r, 1 - r, \ldots, q - 1 - r\}$ with $0^\sigma = 0$, the generalized radical inverse function defined by $\phi_{(q,r)}^\sigma(n) = \sum_{h=0}^{k} \delta_{h} q^{-h-1}$ is $(q, r)$-linear with the coefficient $q^{-1}$ and the initial vector $q^{-1}((-r)^\sigma, (1 - r)^\sigma, \ldots, (q - 1 - r)^\sigma)$ (cf. [8] Chapter 3).

4. For any given $p \in \mathbb{Z}$ with $|p| \geq q$, the bases change function $\gamma_{pq}(n)$ is defined by $\gamma_{pq}(n) = \sum_{h=0}^{k} \delta_{h} p^h$, which is $(q, r)$-linear with the coefficient $p$ and the initial vector $(-r, 1 - r, \ldots, q - 1 - r)$ (cf. [2]).

5. The linear function $cn$ ($c \in \mathbb{C}^\times$) is $(q, r)$-linear with the coefficient $q$ and the initial vector $c((-r, 1 - r, \ldots, q - 1 - r)$.

Examples of $(-q, r)$-linear functions can be constructed similarly as above by using the expression (3).

Recently, Kuroswa and the second named author[6] gave a necessarily and sufficient condition for the generating functions of $(q, 0)$-linear functions and $(-q, 0)$-linear ones to be algebraically independent over $\mathbb{C}(z)$. We note that the generating function of $a(n) = cn$ given in Example 5 is

$$\frac{z}{(1-z)^2} \in \mathbb{C}(z).$$

We state our theorems. Let $\alpha_i, \beta_i \in \mathbb{C}^\times$ ($1 \leq i \leq I$) satisfy

$$\alpha_i \neq \alpha_j, \beta_i \neq \beta_j \quad (i \neq j, 1 \leq i, j \leq I).$$  \hspace{1cm} (11)

For any fixed $q$, let $a_{iir}(n)$ ($1 \leq l \leq m(i)$) and $b_{iir}(n)$ ($1 \leq l \leq n(i)$) be $(q, r)$-linear functions and $(-q, r)$-linear ones with coefficients $\alpha_i$ and $\beta_i$, respectively. We consider the generating functions

$$f_{iir}(z) = \sum_{n=0}^{\infty} a_{iir}(n) z^n, \quad f'_{iir}(z) = \sum_{n=0}^{\infty} a_{iir}(n) (z^n),$$
\[
g_{ilr}(z) = \sum_{n=0}^{\infty} b_{ilr}(n)z^n, \quad g_{ilr}^*(z) = \sum_{n=0}^{\infty} b_{ilr}(-n)z^n,
\]

which converge in \(|z| < 1\) by (4) and (8). We put

\[
a_{ilr} = (a_{ilr}(-r), a_{ilr}(1 - r), \ldots, a_{ilr}(q - 1 - r)),
\]

\[
b_{ilr} = (b_{ilr}(-r), b_{ilr}(1 - r), \ldots, b_{ilr}(q - 1 - r)).
\]

For any vector \(c = (c_1, c_2, \ldots, c_q)\), we write \(\overline{c} = (c_q, c_{q-1}, \ldots, c_1)\).

**Theorem 1.1.** The functions \(f_{ilr}(z) (1 \leq i \leq I, 1 \leq l \leq m(i), 0 \leq r < q - 1)\), \(f_{ilr}^*(z) (1 \leq i \leq I, 1 \leq l \leq m(i), 0 < r \leq q - 1)\), \(g_{ilr}(z)\) and \(g_{ilr}^*(z) (1 \leq i \leq I, 1 \leq l \leq n(i), 0 \leq r \leq q - 1, 2r \neq q - 1)\) are algebraically independent over \(\mathbb{C}(z)\) if and only if the following conditions (i) and (ii) hold;

(i) each one of the sets of vectors \(\{a_{ilr}, \overline{a}_{ilq-1-r} ; 1 \leq l \leq m(i)\} (1 \leq i \leq I, 0 \leq r < q - 1)\) and \(\{b_{ilr}, \overline{b}_{ilq-1-r} ; 1 \leq l \leq n(i)\} (1 \leq i \leq I, 0 \leq r \leq q - 1, 2r \neq q - 1)\) is linearly independent over \(\mathbb{C}\),

(ii) if \(\alpha_i = q\), then for any \(r\) with \(0 \leq r < q - 1\)

\((-r, 1 - r, \ldots, q - 1 - r) \notin \text{Span}_\mathbb{C}\{a_{ilr}, \overline{a}_{ilq-1-r} ; 1 \leq l \leq m(i)\},\)

and if \(\beta_i = -q\), then for any \(r\) with \(0 \leq r \leq q - 1, 2r \neq q - 1\)

\((-r, 1 - r, \ldots, q - 1 - r) \notin \text{Span}_\mathbb{C}\{b_{ilr}, \overline{b}_{ilq-1-r} ; 1 \leq j \leq n(i)\}.

**Remark 1.1** To prove the theorem, we use a criterion of algebraic independence over \(\mathbb{C}(z)\) of functions satisfying certain functional equations (cf. [7] Corollary of Theorem 3.2.1), which enable us to reduce the algebraic dependency over \(\mathbb{C}(z)\) of our functions to the linear dependency of them over \(\mathbb{C}\)mod \(\mathbb{C}(z)\). So we actually prove that the functions in the theorem are algebraically dependent over \(\mathbb{C}(z)\) if and only if, for some \(i\) and \(r\), \(f_{ilr}(z), f_{ilq-1-r}(z) (1 \leq l \leq m(i))\) are linearly dependent over \(\mathbb{C}\), \(g_{ilr}(z), g_{ilr}^*(z) (1 \leq l \leq n(i))\) are linearly dependent over \(\mathbb{C}\), \(\alpha_i = q\) and \(z/(1 - z)^2 \in \text{Span}_\mathbb{C}\{f_{ilr}(z), f_{ilq-1-r}(z) ; 1 \leq l \leq m(i)\}\), or \(\beta_i = -q\) and \(z/(1 - z)^2 \in \text{Span}_\mathbb{C}\{g_{ilr}(z), g_{ilq-1-r}(z) ; 1 \leq l \leq n(i)\}\).

**Remark 1.2** The conditions (i) and (ii) in Theorem 1.1 imply that \(m(i), n(i) \leq q\) for any \(i\), \(\alpha_i \neq q\) if \(m(i) = q\), and \(\beta_i \neq -q\) if \(n(i) = q\).

**Theorem 1.2.** Let the functions \(f_{ilr}(z), f_{ilq-1-r}(z), g_{ilr}(z),\) and \(g_{ilr}^*(z)\) satisfy the conditions (i) and (ii) in Theorem 1.1. Assume that \(\alpha_i, \beta_i, a_{ilr}(n),\) and \(b_{ilr}(n)\) are algebraic for all \(i, l, r\) and \(n\). Then, for any algebraic number \(\alpha\) with \(0 < |\alpha| < 1\), the numbers \(f_{ilr}(\alpha) (1 \leq i \leq I, 1 \leq l \leq m(i), 0 \leq r < q - 1)\), \(f_{ilq-1-r}(\alpha) (1 \leq i \leq I, 1 \leq l \leq m(i), 0 < r \leq q - 1)\), \(g_{ilr}(\alpha)\) and \(g_{ilr}^*(\alpha) (1 \leq i \leq I, 1 \leq l \leq n(i), 0 \leq r \leq q - 1, 2r \neq q - 1)\) are algebraically independent.
If we fix $r = 0$ in Theorem 1.1 and Theorem 1.2, we have the results of Kurosawa and the second named author[6] mentioned above. In their proof, they used another criterion ([7] Theorem 3.5) of algebraic independence of functions over $\mathbb{C}(z)$.

References


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