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OPTICAL THICKNESS OF THE PLANETARY NEBULAE AND TEMPERATURE OF THE CENTRAL STARS

BY

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ABSTRACT

Physical discussions on optical thickness of the planetary nebulae and temperature of the central stars are given on the basis of various data obtained by both the present author and other workers. As a first step, several methods of temperature determination were examined, and then the dependency of the optical thickness of the planetary nebula on its temperature, dimension and dilution factor for the ultraviolet radiation has been studied. Using the result obtained a more satisfactory model of such nebulae is proposed. Further, it may be suggested that future lines of study should be concentrated on the following points: (1) The deviation of the stellar radiation from the Planckian energy distribution should be estimated; (2) In the calculation of temperature the dilution factor for the continuum should be taken into account; (3) It is necessary to clearify the cause of the burst of the nova; (4) The intensity ratio of He II 4686 to O III 3133 should be examined in more detail.

1. Introduction

The planetary nebula is one of the largest celestial bodies in our Galaxy together with the diffuse gaseous nebula. But the former is alien from the latter. The planetary nebula is formed of the gaseous nebulosity called "the envelope" and the associated star called "the nuclei or central star". Some of the diffuse nebulae have the nearby stars, which are not relative to them and happen to be there. The very existence of the nuclei draws the sharp line between the two nebulae different in kind. Historically, the earliest studies on planetary nebulae were started from those on the diffuse nebulae. But, in connection with the advances of optical instruments and observational techniques, their real physical states have been since revealed theoretically by excellent investigators.

The schematic model for the planetary nebula is shown in Fig. 1. The envelope is the spherical shell bounded by the two spherical surfaces: the inner surface of radius R_1 and the outer surface of radius R_2. The inner surface is illuminated by the radiation coming from the central star. A part of the radiation

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is reflected at the inner surface. The remainder is transmitted to the envelope, and there the photons of higher energy are reduced to lower energy by absorption or scattering and by inelastic collisions between elementary particles, and turned to the visual radiation.

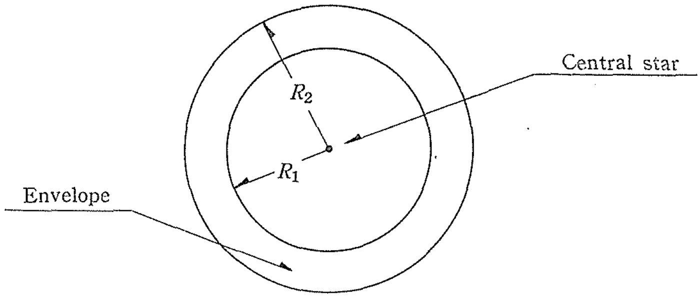

Fig. 1. The schematic model for the planetary nebula.
R_1 : Radius of the inner surface.
R_2 : Radius of the outer surface.

On the visual magnitudes m and apparent radii A of planetary nebulae, a relation between these quantities was derived by Hubble (1). So far, he had studied the illuminating mechanism in diffuse nebulae and got the well-known relation:

$$m+5 \log A = \text{constant}.$$

The assumptions employed by him are the inverse-square law of illumination, the scattering of light in all directions without change of colour, and the transparent nebula for the scattered light. In his paper, he mentioned that the intrinsic difference of the planetary nebula from the diffuse nebula might be the temporary existence of the central star. And he applied the same method to planetary nebulae, but did not obtain so obvious trace of correlation as in the case of diffuse nebulae. In the second paper (1), he suggested the both associations of bright stars with small nebulae and of faint stars with large nebulae.

Instead of the visual observation, the photographic images for some typical nebulae were given by Curtis (2) with Crossley reflector (3). On spectroscopic studies, Wright (4) in 1928 carried out the observations with slit and slitless spectrograms. On the basis of his data, Bowen (5) developed the interpretation of the spectra and structure of the nebulae. He suggested the excitation mechanism of the lines of the element "Nebulium" so far so called, and also the stratification of the envelope, which is formed by the succesive ionization due to the continuous radiation coming from the star. Afterwards, Strömgren (6) employed this principle of physical dilution to interpret the ionization of interstellar matters. Bowen and Wyse (7) performed the extensive research of spectroscopy with the image slicer, which is an ingenius device due to Bowen.

In 1927, Zanstra (8) attempted the application of the quantum theory to the luminosity of diffuse nebulae. In the same paper, he said that if the planetary nebula was sufficiently thick, each quantum of the Lyman continuous radiation emitted by the central star would be degraded into a quantum of the Balmer lines or continuum and a Lyman alpha quantum. In brief, it is mentioned as below:

$$L_{\alpha}+B_{\mathrm{ac}}+B_{\mathrm{a}} = N_{\mathrm{ul}}\,.$$

The each term on the left-hand side is in its order the number of photons of Lyman alpha, Balmer continuum and Balmer lines. The right-hand side is the number of photons of Lyman continuum. And further, he developed the principle determining the temperature of the central star. Some practical treatments of his principle were given in his subsequent works (9). The physical investigations of planetary nebulae owe much of their successive progresses to Zanstra's study as well as to Bowen's.

The spectrophotometric studies of planetary nebulae were made by Page (**10**), Aller (**11**), and Minkowski and Aller (**12**). Page performed a survey of the spectra of the planetary nebulae to reveal the continuous emission, and found the correlation of its intensity with the excitation and surface brightness. He classified the spectra of the nebulae and divided them into the nine classes. Aller measured the Balmer decrement in the spectra of the certain peculiar nebulae and classified them into the three groups according to the degree of their excitation.

As the theoretical treatment, there is the important Cillié's work (**13**) in which the theoretical studies of the hydrogen emission spectra of the planetary nebulae are given. As the next to his work, Menzel and his colleagues published a series of papers on the physical processes in gaseous nebulae (**14**). In these papers, the important subjects, namely the radiative processes, the excitation of forbidden lines, the density and temperature of electrons, and the chemical composition of planetary nebulae etc. are much discussed.

The problems of the radiative transfer and radiative equilibrium in planetary nebulae were treated by Ambarzumian (**15**) and Chandrasekhar (**16**). The latter solved the Lyman alpha radiation field with Eddington's approximation. According to their results, the flux of the Lyman alpha lines becomes of the same order of magnitude as that of the ultraviolet continuum. To avoid this difficulty, Zanstra (**17**), and Sobolev (**18**) and Zanstra (**19**) suggested respectively the assumption of the expanding nebula and the redistribution of the Lyman alpha quanta as referred to later.

Since the splitting-up of the emission lines was observed by Cambell and

Moore (**20**), the model of the nebula in outgoing motion has been current. But, the case of the nebula pushed back towards the nuclei on the inner surface was suggested by Chandrasekhar (**21**). Wilson (**22**), using the isophoto-contours of monochromatic images of the nebulae, discussed their structure and kinematic motion. The dynamical treatment is one of the matters of concern.

We have specially interested in the temperature of the central star and the optical thickness for the ultraviolet continuous radiation of gaseous nebulosity. Since the brightness of the nebulosity is caused by the central star, it is true that the relations between these physical quantities will be a clue to explain further the physical states of planetary nebulae.

The optical thickness for monochromatic radiation is directly connected with the number density of absorbing atoms or ions and their absorption coefficients for the existing radiation. If the thickness for the radiation of frequency ν is expressed by τ_ν, it will be defined as follows:

$$\tau_\nu = \int_{R_1}^{R_2} \kappa_\nu(x)dx ,$$

where $\kappa_\nu(x)$ is the absorption coefficient per unit length for the radiation concerned. Then, the value of optical thickness will be influenced by the energy distribution of the continuous radiation coming from the central star and the transfer procedure of its radiation through the nebulosity. The optical thickness τ_c for the ultraviolet continuous radiation of planetary nebulae is usually considered to be of the order of unity. But, a variety of physical states over many nebulae indicates the diversity of these thicknesses. In this paper, we shall investigate this diversity over nebulae.

In the next section, we shall mention the outline of excitation mechanisms of the planetary nebulae and explain in succession how the temperature of central star be determined. The relations of optical thickness to other physical quantities will be described in the third section. In the final section, some views on further researches will be given.

2. The excitation mechanisms of planetary nebulae and the temperature of central star

In this section we shall run over the emission mechanisms for bright lines present in the spectra of planetary nebulae and then describe the procedures determining the nuclear temperature.

In the following, we consider the emission mechanisms from the point of view of the energy valancing of electrons. The ultraviolet continuous radiation coming from the central star is absorbed by the most abundant matter, i.e. hydrogen atoms, and the liberated electrons lose their kinetic energy in three ways as follows: (1) by capture of protons and by free-free transitions; (2) by inelastic collisions with hydrogen atoms; and (3) by inelastic collisions with oxygen ions following to forbidden emission.

The recombination is the principal process accounting for nebular luminosity. This process was proposed by Zanstra (8) to explain the hydrogen spectrum, and its full account was given in the famous Cillié's work (**13**). The recombination to the higher level than the second level results in the cascade transitions to the second level and the subsequent emissions of the subordinate and Lyman alpha lines. The former passes through the nebulosity to give the nebular luminosity owing to their small excitation ability. If the complete absorption of the ultraviolet radiation is assumed, the one ultraviolet quantum will match to the one Balmer quantum. The resulting large abundance of the atoms in the second state impels the strong Lyman alpha radiation field. The investigation of its field is one of the important matters in the study on planetary nebulae, in particular in connection with the transfer of ultraviolet radiation. We shall give the energy losed in this process as $N_eN^+\sum_1^\infty C_ih\nu_i$, where N_e and N^+ are the densities of electrons and ionized hydrogen, respectively, and C_i is the probabitity of recombination when the photon of energy $h\nu_i$ is emitted. The summation is taken over all levels.

The free-free emission may be able to explain the spectrum of the solar corona, but in the planetary nebulae its process takes little part to compensate the Balmer continuous radiation of the nebulosity (**23**), owing to the low electron density. We shall denote this energy by N_eN^+F.

The influence of inelastic collision of electrons with hydrogen atoms upon the Balmer decrement was given by Miyamoto (**24**). According to him, this process acts to enhance the Balmer decrement and give the weaker Balmer continuous emission. The energy transformed in this process can be given by $N_eN^+(\sum_2^\infty D_ih\nu_i+D_ch\nu_{1c})$ where D_i and D_c are respectively the probabilities of collisional excitation of hydrogen atoms to the second or higher levels and to the continuous level from the ground level.

According to Bowen (**5**), the inelastic collision between electrons and oxygen ions O^{++} can explain the excitation of these ions and the following spontaneous emissions. The excited states are the metastable states $^1D_2$ and the emission lines are the forbidden lines, λ 5007 Å (N_1) and λ 4959 Å (N_2), being called the "nebulium lines". The intensities of these lines are the largest of the spectral lines of nebulae and are ten times stronger than that of H_β. The nebular image owes its brightness to these green lines. Since the strength of these lines depends on the kinetic energy and density of electrons, many important studies on the electron temperature and density of planetary nebulae were developed from the observed intensity of these lines. This process operates to reduce the electron temperature of nebula. We shall denote this loss of the kinetic energy of electrons as $N_e N^{++} \sum_i Q_i A_i h\nu_i$, where N^{++} is the density of oxygen ions O^{++} and Q_i is the probability of collisional excitation of these ions to the metastable levels from which the forbidden line ν_i is spontaneously emitted with the probability A_i. The summation is taken over all the forbidden transitions.

Therefore, the amount of the energy of electrons transformed to that of the visual radiation is given by

$$N_e N^{+}((\sum_1^\infty C_i h\nu_i + F) + N_e N^{+}(\sum_2^\infty D_i h\nu_i + D_c h\nu_{1c}) + N_e N^{++} \sum_i Q_i A_i h\nu_i . \qquad (1)$$

This energy is equal to the total kinetic energy of the electrons liberated by photo-ionization.

Measuring the visible radiation, each of which corresponds to one of the terms in the expression (1), we can obtain the temperature of central star. We shall now mention some methods of determining the temperature on the basis of the foregoing excitation mechanisms.

(1) *The hydrogen method*

Zanstra (9) obtained, for the first time, the temperature of the central stars of planetary nebulae, employing the theories on excitation. Under the assumptions of the hydrogen nebula and the complete absorption of the ultraviolet continuous radiation emitted from the central star, he showed that each absorbed ultraviolet quantum was transformed to a Balmer radiation quantum. The schematicexpression of the relation is

$$N_{\mathrm{ul}} = Q_{\mathrm{Ba}} , \qquad (2)$$

where N_{ul} is the quanta of the ultraviolet radiation of energies higher than the ionization limit of hydrogen atom and Q_{Ba} is the quanta of the Balmer lines emitted per unit time from the nebula. Zanstra obtained the mathematical expression for this relation as follows. If the total energy emitted from the central star in the frequency region ranging from zero to ∞ is denoted by L_s, the energy emitted into the frequency interval $d\nu$ per second is

$$\frac{\partial L_s}{\partial \nu} d\nu = \pi R^2 c \rho_\nu d\nu , \qquad (3)$$

where R is the radius of the central star and c is the light velocity. ρ_ν is the Planck function and given by

$$\rho_\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT}-1}. \tag{4}$$

From the right-hand side of (3) together with (4), the quanta N_{ul} of the ultraviolet continuous radiation become

$$N_{ul} = \frac{8\pi^2 R^3}{c^2} \int_{\nu_0}^{\infty} \frac{\nu^2}{e^{h\nu/kT}-1} d\nu . \tag{5}$$

The lower limit of integration ν_0 is the frequency of the ionization limit of hydrogen atom. Zanstra defined a quantity A_ν to be determined observationally as:

$$A_\nu = \frac{L_P}{\nu \frac{\partial L_s}{\partial \nu}}. \tag{6}$$

L_P is the total energy emitted per second in the monochromatic nebula image of frequency ν. A_ν is a non-dimensional quantity divided by ν. On the other hand, as the quanta of the Balmer lines are equal to $L_P/h\nu$, Q_ν is given by the following expression:

$$Q_\nu = \sum \frac{\nu^3}{e^{h\nu/kT}-1} \frac{8\pi^2 R^3}{c^2}. \tag{7}$$

The summation is taken over all the members of Balmer series appearing in the nebular spectrum. Writing $\chi = h\nu/kT$ in Eqs. (3), (5), (6) and (7), the relation (2) is reduced to

$$\int_{\chi_0}^{\infty} \frac{\chi^2}{e^{\chi}-1} d\chi = \sum \frac{\chi^3}{e^{\chi}-1} A_\nu . \tag{8}$$

T is the temperature of the central star and χ_0 corresponds to χ for the ionization frequency ν_0. To determine the temperature from this equation, we must calculate (8) by trial and error. The temperatures thus obtained by Zanstra are given in Table 1.

Table 1. The temperatures obtained by the hydrogen, helium and nebulium methods.

Nebula	Hydrogen method	Helium method	Nebulium method
NGC 6572	41,000	34,000～41,000	38,000
NGC 6543	37,000		35,000

The basic assumption that the energy distribution of stellar continuum is Planckian is adopted in this method. It will add an essential feature to the optical thickness τ_c of nebulae. The situation will be discussed later.

(2) *The helium method*

The foregoing method, which estimates the nuclear temperature on the basis of the observed intensity of the Balmer lines of hydrogen atoms, can be extended to the case of the helium ion He^{+}. By this method, Zanstra (9) obtained the temperature of central star. The application of this procedure is limited to the high excitation nebulae that show the helium II spectra.

This procedure contains the same assumption as the hydrogen method. The temperature determined by the helium method is usually higher than by the hydrogen one. That temperature gained by Zanstra is given for the nebula NGC 6572 in Table 1.

(3) *The nebulium method*

Zanstra (9) estimated the temperature of the central stars using the emission mechanism proposed by Bowen (5) on the forbidden lines N_1 and N_2. From Bowen's explanation, the energy of free electrons is used for the excitation of the O^{++} ions to the metastable states by inelastic collisions, and is equal to the energy of the ultraviolet quanta minus the ionization energy of hydrogen atoms. This relation is given by

$$E_{u1}-h\nu_0 N_{u1} = \sum L_P \,. \tag{9}$$

In the above equation, N_{u1} and E_{u1} are respectively the total number and its total energy of the ultraviolet radiation coming from the central star. $\sum L_P$ is the total energy of the both nebular lines. By the same treatment as before, the mathematical expression for this relation is given by

$$\int_{\chi_0}^{\infty} \frac{\chi^3}{e^{\chi}-1} d\chi - \chi_0 \int_{\chi_0}^{\infty} \frac{\chi^2}{e^{\chi}-1} d\chi = \sum \frac{\chi^4}{e^{\chi}-1} A_\nu \,, \tag{10}$$

or

$$J_1 - \chi_0 J_0 = \sum \frac{\chi^4}{e^{\chi}-1} A_\nu \,, \tag{11}$$

where

$$J_1 \equiv \int_{\chi_0}^{\infty} \frac{\chi^3}{e^{\chi}-1} d\chi \,,$$

and

$$J_0 \equiv \int_{\chi_0}^{\infty} \frac{\chi^2}{e^{\chi}-1} d\chi \,.$$

Here χ_0 denotes the ionization frequency of hydrogen atoms, and χ on the right-hand side is expressed in terms of frequency in the same manner as that of the nebulium lines. A_ν is the observed intensity of the lines.

In this treatment, the Planckian energy distribution of stellar continuum and the total energy of electrons being consumed in the excitation of nebulium lines are both assumed. The temperatures in the fourth column in Table 1 are those given by Zanstra.

(4) *The Δm-method*

Developing the method of nebulium lines, Zanstra (**9**) further obtained the different procedure of determining the temperature. He attributed the visual photographic brightness of the nebulosity to the nebulium emission lines N_1, N_2, and assumed a proportional relation between them as:

$$B_n = C_2 L_P \,, \tag{12}$$

where B_n is the visual photographic brightness of the nebulosity. The similar brightness B_* of the central star was given to be

$$B_* = C_1 \frac{\partial L_s}{\partial \nu_{\mathrm{ph}}} \,. \tag{13}$$

In the above equations (12) and (13), C_2 and C_1 are constants, and the other notations are the same as before. Using the expression introduced in the previous methods:

$$\frac{L_P}{\nu_{\mathrm{ph}} \dfrac{\partial L_s}{\partial \nu_{\mathrm{ph}}}} = A'_\nu = \frac{e^{\chi_{\mathrm{ph}}} - 1}{\chi_{\mathrm{ph}}^4} (J_1 - \chi_0 J_0) \,, \tag{14}$$

he got the relation between the visual magnitudes m_* and m_n of the star and nebulosity respectively,

$$\begin{aligned} m_* - m_n &= 2.5 \log \frac{B_n}{B_*} \\ &= 2.5 \log \left[\frac{1}{\chi_{\mathrm{ph}}^4} (J_1 - \chi_0 J_0)(e^{\chi_{\mathrm{ph}}} - 1) \right] + 20 + C_0 \,. \end{aligned} \tag{15}$$

The quantity on the right-hand side is a function of temperature except an unknown constant C_0. This constant was derived from the data of the planetary nebulae, NGC 6572 and 6543, with their temperatures determined by the nebulium method. For $\lambda = 4210$ Å, he obtained $C_0 = -10.9$ as the mean value. The temperatures, which were determined by Eq. (15) with the observed $m_* - m_n$, are given for the twenty-two planetary nebulae in Table 2.

This procedure adopted the nebulium temperature of Zanstra for the calculation of constant. Therefore, the defect of that method is held as it is.

(5) *The δ-method*

Berman (**25**) assumed that a part of the nebular luminosity arises from the recombination of protons and free electrons and the remainder from the inelastic collisions of oxygen ions with free electrons. Applying a method analogous to the one originally employed by Zanstra, he estimated the ratio of the brightness of the nebula, l_n, to that of the star, l_*, as follows;

Table 2. The temperatures determined by the Δm-method.

Nebula	m_*	m_n	Δm	$T\times10^{-3}$
NGC 6445	19	10.4	8.6	140
1952	15.9	8.4	7.5	100
2438	16.6	9.8	6.8	85
650-1	16.6	9.9	6.7	85
6853	13.6	7.3	6.3	75
6818	14.9	8.8	6.1	70
6720	14.7	8.8	5.9	70
3587	14.3	9.4	4.9	55
3242	11.7	7.2	4.6	55
7009	11.7	7.2	4.5	50
7662	12.7	8.4	4.3	50
6905	14.5	10.7	3.8	45
6309	14.1	10.8	3.3	40
6210	11.7	8.5	3.2	40
6543	11.3	8.1	3.2	40
1535	11.6	8.8	2.8	38
4361	12.8	10.1	2.7	37
6826	10.8	8.4	2.4	35
6572	10.5	8.4	2.1	34
6804	13.4	11.8	1.6	(31)
2392	10.0	8.4	1.6	(31)
NGC 7008	12.8	12.2	0.6	(31)

$$\frac{l_n}{l_*}=\frac{\alpha\int_{\chi_0}^{\infty}\frac{\chi^2}{e^{\chi}-1}d\chi}{\int_{\chi_1}^{\chi_2}\frac{\chi^2}{e^{\chi}-1}d\chi}+\frac{\beta\int_{\chi_0}^{\infty}\frac{\chi^2(\chi-\chi_0)}{e^{\chi}-1}d\chi}{\int_{\chi_1}^{\chi_2}\frac{\chi^3}{e^{\chi}-1}d\chi}. \tag{16}$$

The first term on the right-hand side represents the ratio of the number of observed Balmer radiation quanta to the number of photographic radiation quanta in the stellar light. The integration on the stellar radiation is calculated between the limits λ 3000 Å (χ_2) and λ 5050 Å (χ_1). The second term is the ratio of energy in the observed nebulium lines to that in the stellar spectrum. The correction factors $\alpha=0.5$ and $\beta=0.8$ allow for the omission of H_α line on the ordinary photographic plates and the percentage of the total intensity of nebulium lines observable over this limited wavelength region, respectively.

The schematic relation of this procedure is

$$m_*-m_n\equiv\delta=f(T)\,. \tag{16'}$$

Berman calculated δ_c's for some nebulae with their hydrogen temperature and compared with the observed δ_0. The results are given in Table 3.

Table 3. The calculated δ_c and observed δ_0.

Nebula	Temperature	δ_0	δ_c	$\delta_0-\delta_c$
NGC 6720	70,000	+5.4 m	+4.1 m	+1.3 m
NGC 7662	55,000	4.0	3.1	0.9
NGC 7009	50,000	4.0	2.7	1.3
IC 5217	50,000	3.5	2.7	0.8
NGC 6572	45,000	2.8	2.2	0.6
NGC 6543	33,000	2.5	+0.5	2.0
IC 4997	30,000	0.1	−0.1	0.2
NGC 6826	30,000	1.5	−0.1	1.6
IC 3568	30,000	1.2	−0.1	1.3
NGC 6210	30,000	+1.7	−0.1	+1.8
NGC 6891	30,000	−0.5	−0.1	−0.4
IC 4593	25,000	−0.9	−1.2	+0.3
BD 30 : 3639	25,000	+0.7	−1.2	+1.9

(6) *The continuum method*

Würm (**26**) suggested the utilization of the Balmer continuous emission instead of the subordinate lines employed in the preceding methods, and proposed the relation:

$$Q_{\mathrm{Ly}} \propto Q_{\mathrm{Bc}}\,. \tag{17}$$

If the intensity ratio of Balmer continuous emission to the stellar continuous emission is given by P_χ, we have

$$P_\chi = \frac{\zeta_\chi \int_{\chi_0}^{\infty} \frac{\chi^2}{e^\chi - 1} d\chi}{\int_{\Delta\chi} \frac{\chi^2}{e^\chi - 1} d\chi}\,. \tag{18}$$

In Eq. (18) χ_0 is the ionization frequency of hydrogen atom, χ in the denominator is the frequency to be compared, and $\Delta\chi$ is its width. Further, ζ_χ in the numerator is the rate of the ultraviolet quanta transformed to the Balmer continuous quanta. This factor is the product of two factors ε_χ and δ_χ. The former is the rate of the continuous quanta lower than the Balmer limit falling in the Balmer continuous quanta and is given as:

$$\varepsilon_\chi = \frac{\int_{\nu_2}^{\infty} F_{\kappa 2} d\nu}{\sum_{n'=2}^{\infty} \int_{\nu_{n'}}^{\infty} F_{\kappa n'} d\nu}\,, \tag{19}$$

and the latter is that of the Balmer continuous quanta falling in the frequency band considered and is given as:

$$\delta_\chi = \frac{F_{\kappa 2} d\nu}{\int_{\nu_2}^{\infty} F_{\kappa 2} d\nu}, \tag{20}$$

where F_{ij} is the number of transitions. Würm obtained, after some calculations, a value of $10^{0.15}$ as the factor ζ_χ for the electron temperature 10^4. The temperature determined by this method is given as a function of P_χ in Table 4.

Table 4. The temperature of central star with P_χ.

P_χ	Temperature	P_χ	Temperature
0.30	30,000	10	80,000
1.0	40,000	16	100,000
2.5	50,000	44	150,000
3.1	60,000	75	200,000

(7) *The helium-hydrogen method*

All the above-mentioned methods are not applicable to the central stars in which the continuous spectra are too weak to be measured in the visual region, since the observed intensity of the line emission considered is given in units of those of the continuous radiation of the star at the same energy region. This problem was solved by Ambarzumian's device (27) using the intensity ratio of HeII λ 4686 Å to HI λ 4861 Å at the same wavelength. The relation suggested by him is

$$\frac{\int_{\chi_0'}^{\infty} \frac{\chi^2}{e^\chi - 1} d\chi}{\int_{\chi_0}^{\infty} \frac{\chi^2}{e^\chi - 1} d\chi} = \frac{\alpha' Q'}{\alpha Q}, \tag{21}$$

where χ_0' and χ_0 correspond to $h\nu/kT$ at the ionization energy of HeII and HI, respectively. The left-hand side of Eq. (21) is a function of temperature. In the other side, α' and α are the percentage of the ultraviolet quanta transformed to the Balmer quanta. Q' and Q are the number of these quanta concerned of each ion. The values with dashes express those for the helium ions. Under the assumption of $\alpha' \approx \alpha \approx 1$, that is, of the inevitable transformation of an ultraviolet quantum into a Balmer one, the above equation (21) is reduced approximately to

$$\frac{\int_{\chi_0'}^{\infty} \frac{\chi^2}{e^\chi - 1} d\chi}{\int_{\chi_0}^{\infty} \frac{\chi^2}{e^\chi - 1} d\chi} = \Pi, \tag{22}$$

where Π is defined as

$$\Pi = \frac{I_{4686}}{I_\beta} = \frac{A'Q'}{A\,Q}. \tag{23}$$

Since the transition probabilities A' and A are nearly equal to each other, the above reduction may be satisfactory.

Though this procedure dodges the previous question for the continuous radiation of the central star, the Planckian distribution of the ultraviolet radiation is assumed for both ions. This assumption introduces the unreliability into the temperature of the central star.

The author and others (**28**) obtained the temperature of some nebulae by this method, considering the absorption of the ultraviolet radiation in the envelope.

(8) *The hydrogen nebulium method*

Another method which does not use the stellar continuum was proposed by Stoy (**29**), who combined both the hydrogen and the helium methods. Its formulation is

$$\frac{\int_{\chi_0}^{\infty} \frac{\chi^3}{e^{\chi}-1} d\chi - \chi_0 \int_{\chi_0}^{\infty} \frac{\chi^2}{e^{\chi}-1} d\chi}{\chi' \int_{\chi_0}^{\infty} \frac{\chi^2}{e^{\chi}-1} d\chi} = \frac{\sum\limits_{\text{neb}} I_P}{\sum\limits_{\text{hyd}} I_P \frac{\nu'}{\nu_P}}. \tag{24}$$

The notations used in this relation are similar to those in the preceding methods. Employing the line intensities measured by Plasket (**30**) and Berman (**31**), Stoy obtained the nuclear temperature of some nebulae. He asserts that this method seems to be simpler and of wide application. The results are shown in Table 5.

Table 5. The temperatures given by the hydrogen nebulium method.

Nebula	Temperature	
	$H_\alpha : H_\beta = 4.0 : 1$	$H_\alpha : H_\beta = 5.8 : 1$
NGC 6572	34,000	27,000
NGC 6543	24,000	18,500
IC 4593	20,000	15,500
NGC 6826	27,000	22,000
NGC 7009	51,000	41,000
NGC 7662	35,000	29,000
NGC 7027		43,000

(9) *The excitation temperature of central star*

The temperatures so far mentioned are the effective temperatures. The excitation temperature of a star can be estimated from the spectral type. But, for the early-type stars, the spectral classification is very difficult to be attacked owing to the complication of their spectra, and much more for the obscured nuclei of planetary nebulae. Adopting Petrie's temperature-spectral class calibration for the normal 0-type stars, Aller (**32**) and Wilson and Aller (**33**) estimated the excitation temperature of nuclei. Their results are tabulated in Table 6.

Table 6. The spectral class and excitation temperature.

Nebula	Spectral class	Excitation temperature
IC 418	0 7	33.2×10^3
IC 2149	0 7.5	32.5
NGC 2392	0 6	34.5
IC 4593	0 7	33.4
NGC 6210	0 7	32.9
NGC 6543	0 7	33.0
NGC 6826	0 6	34.6
NGC 6891	0 7	32.9

These temperatures are not necessarily consistent with the effective temperatures. And still more, the procedure of Petrie's calibration adopts the abundance ratio of helium to hydrogen in the Type I population stars. On the other hand, the planetary nebulae are said to be the Type II population stars. If the higher abundance ratio common to this Type is assumed, these excitation temperatures will be lower. In this paper, these temperatures are quoted only as a reference.

Each method of determination of temperature is based on the same assumption of the energy distribution of stellar radiation. The difference between their methods is the choice of the visible radiation with which the ultraviolet radiation is matched. Severely speaking from the viewpoint of energy balancing, all the visible radiations enumerated in the second section are necessary to be considered. Only Berman's δ-method reflects upon this situation, taking into consideration the first two terms. Therefore, the above temperatures will give the lower values of nuclear temperatures. Sobolev (**34**) allowed for the three terms given in Eq. (1)

Table 7. The temperatures of central stars obtained by the various methods (in units of 10^3 degrees).

Method		NGC 6543	NGC 6572	NGC 6826	NGC 7009	NGC 7662	NGC 7027	NGC 4593
Hydrogen method	z*	39	41		55			
	b			26		43	52	
Helium method	z		34∼41		70			
	b				70	81	86	
Nebulium method	z	35	38					
	b	33	45	30	50	55		25
Δm-method	z	40	34	35	50	50		
	b	35	45	30	50	55	80	25
Correct method	so	41	48	29	45			25
Excitation method	a	33		34.6				33.4
Hydrogen-helium method	s	24	34	27	51	35		20
		18.5	27	22	41	29	43.5	15.5

* z : Zanstra, b : Berman, so : Sobolev, a : Aller, s : Stoy.

and calculated the nuclear temperature (Table 7). The resulting temperatures are higher than the preceding ones, and then would give the upper limit of the temperatures.

We shall now compare the various temperatures obtained by the preceding methods. Table 7 represents their temperatures. The values based on the helium method are higher than the others. This tendency can be found much more in Würm's work. The high helium temperature in the high excitation nebula will be discussed in connection with the optical thickness of the nebulosity. We have investigated the temperatures of central stars on the basis of the observed phenomena. On the other hand, the assembly of the nebulous matters ought to be steady to some extent, radiatively due to the radiation coming from the star and thermally due to the thermal motions of most abundant elementary particles. The electrons gain their kinetic energies by photo-ionization, and lose them by some processes before mentioned. In such a situation, the electron temperature will have a correlation with the effective temperature for a star. Baker and others (35) studied this problem about the optically thin nebula. To formulate this treatment, they adopted the following four assumptions: (1) the electrons being produced by photo-ionization; (2) a Maxwell velocity distribution of electrons; (3) a steady distribution of atoms over various levels; and (4) the radiative equilibrium. Their starting equations are the four fundamental equations of equilibrium as follows:

(1) The statistical equilibrium for the ground states:—

$$\sum_{2}^{\infty} F_{n1} + \int_{\nu_1}^{\infty} F_{\kappa 1} d\nu = \sum_{2}^{\infty} F_{1n} + \int_{\nu_1}^{\infty} F_{1\kappa} d\nu , \tag{25}$$

where the left-hand side represents the number of atoms entering the ground state and the other side the number of atoms leaving it by all possible transitions. $F_{nn'}$ is the effective number of transitions from state n to state n' per unit time and per unit volume. The notation κ is referred to the continuous state.

(2) The statistical equilibrium for the nth state:—

$$\sum_{n+1}^{\infty} F_{n''n} + \int_{\nu_n}^{\infty} F_{\kappa n} d\nu + \sum_{1}^{n-1} F_{n'n} = \sum_{n+1}^{\infty} F_{nn''} + \int_{\nu_n}^{\infty} F_{n\kappa} d\nu + \sum_{1}^{n-1} F_{nn'} . \tag{26}$$

The physical meaning of this equation is as before.

(3) The statistical equilibrium for the continuous state:—

$$\sum_{1}^{\infty} \int_{\nu_{n'}}^{\infty} F_{n'\kappa} d\nu = \sum_{1}^{\infty} \int_{\nu_{n'}}^{\infty} F_{\kappa n'} d\nu . \tag{27}$$

(4) The radiative equilibrium:—

$$\sum_1^\infty \int_{\nu_n}^\infty F_{\kappa n} h\nu_{\kappa n} d\nu + \sum_2^\infty \sum_1^{n-1} F_{nn'} h\nu_{nn'} + \int_0^\infty \int_0^\infty F_{\kappa''\kappa} h\nu_{\kappa''\kappa} d\nu_{\kappa''} d\nu_\kappa$$
$$= \sum_1^\infty \int_{\nu_n}^\infty F_{n\kappa} h\nu_{n\kappa} d\nu + \sum_2^\infty \sum_1^{n-1} F_{n'n} h\nu_{n'n} + \int_0^\infty \int_0^\infty F_{\kappa\kappa''} h\nu_{\kappa\kappa''} d\nu_{\kappa''} d\nu_\kappa \,. \tag{28}$$

They adopted the relation of the radiation field to the transitions and the Planckian energy distribution with the dilution factor for the radiation from the star. After some simplifications, they obtained the final equation which combines the electron temperature with the temperature of central star,

$$\frac{\left(\frac{k}{2hRZ^2}\right) T_s \int_{\chi_0}^\infty \frac{d\chi}{e^\chi - 1}}{\int_{\chi_0}^\infty \frac{d\chi}{\chi(e^\chi - 1)}} = \frac{\left(\frac{k}{2hRZ^2}\right)^2 T_e^2 + \sum_1^\infty \frac{1}{n^3}\left(\frac{k}{2hRZ^2}\right) T_e + \frac{1}{2}\sum_1^\infty \frac{S_n}{n^3} - \frac{1}{2}\sum_1^\infty \frac{S_n}{n^5}}{\sum_1^\infty \frac{S_n}{n^3}}, \tag{29}$$

on briefly

$$f_1(T_s) = f_2(T_e)\,, \tag{29'}$$

where T_s is the effective temperature and T_e is the electron temperature. In Eq. (29), χ_0 corresponds to $h\nu/kT$ at the ionization limit of the hydrogen, and S_n is given as:

$$S_n = e^{X_n}\{-E_i(-X_n)\}\,, \quad X_n = \frac{hRZ^2}{n^2 k T_e}\,, \tag{30}$$

and the other notations are usual ones. From the relation (29), Baker *et al.* derived the result which is given in Table 8. According to them, for the higher nuclear temperature, the electron temperature of nebulosity deviates more from the effective temperature.

In this treatment, they assumed the dilution factor for the stellar radiation to be constant. But, this factor varies for the optical depth of nebulosity due to the diffuse radiation emitted there. We shall make reference to this situation in the next section. Furthermore, the model adopted by them is the hydrogen nebula. Besides the most abundant element, i. e. hydrogen, the envelope is formed

Table 8. The relation between the electron temperature and the effective temperature.

Effective temperature	Electron temperature
5,000	5,000
10,000	9,500
20,000	18,000
40,000	34,000
80,000	57,000
160,000	92,000
320,000	132,000

Table 9. The effective temperature for some nebulae.

Nebula	Effective temperature
NGC 7027	300,000
NGC 6572	70,000
NGC 6826	50,000
NGC 6543	30,000
NGC 7662	100,000
NGC 7009	50,000
NGC 1535	30,000
IC 418	20,000

of other elements. In particular, the oxygen ions operate to cool the nebulosity by the inelastic collisions with electrons as described in the preceding section. Allowing for this effect, the above equation (29′) is rewritten in the form:

$$f_1(T_s) = f_2(T_e) + C(T_e) = f_2'(T_e)\,. \qquad (29'')$$

This relation gives the nuclear temperature for a given electron temperature. Menzel and Aller (**36**) obtained the temperature for some nebulae using the data of T_e of their objects. The results are shown in Table 9.

3. The relations between the optical thickness and other physical quantities

In this section, we shall study some relations between the optical thickness and other physical quantities.

The optical thickness τ_c for the ultraviolet continuous radiation depends, as mentioned in the first section, on the temperature of central star. If the energy distribution of the Lyman continuous radiation is like the Planckian curve, the absorption by the hydrogen and helium method would be consistent to some extent. But the helium temperature is much higher than the hydrogen temperature as shown in Table 7. We are perplexed to face another question of hydrogen and helium. This question is the intensity ratio of HeII 4686 to H_β. The line 4686 is a recombination line of ionized helium. As the nebula is free from the self absorption for these lines, this intensity ratio is an indication of the ratio of the quanta of ionization of He^+ to those of H. Then, we can estimate the intensity ratio for any given effective temperature of the central star, with the complete absorption of the ionizing radiation. But the observed intensity ratio is much larger than that predicted theoretically. Würm and Singer (**37**) attributed this anomalous intensity ratio to the incomplete absorption of the Lyman continuous radiation resulting in the weakness of H_β.

The present author and others (**28**) calculated the intensity ratio, taking this incomplete absorption into account. The procedure employed is

$$\phi = \frac{\alpha(4686)}{\alpha(H_\beta)} \frac{\displaystyle\int_{\chi_0'}^{\infty} \frac{\chi^2}{e^\chi - 1} d\chi}{\displaystyle\int_{\chi_0}^{\infty} \frac{\chi^2}{e^\chi - 1}(1 - e^{-\tau}) d\chi}\,, \qquad (31)$$

where $\alpha(4686)$ and $\alpha(H_\beta)$ correspond to α' and α of Eq. (21), respectively. After some calculations, we obtained

$$\alpha(4686) = 0.138\,,$$
$$\alpha(H_\beta) = 0.118\,.$$

And τ is the optical thickness of the nebula for the frequency ν and is a function of χ. Our calculations were performed for $\tau_0(\mathrm{H})=0.1$, 0.5, 1.0, and 3.0 as a function of temperature T. Though the temperature used in this calculation is somewhat unreliable as shown in the foregoing section, it seems likely that there is the diversity of the optical thickness over nebulae.

The optical thickness depends on the number density of the nebular matters and the geometrical dimension of the envelope. Page and Greenstein (**38**) calculated the size of the ionized hydrogen regions by means of Strömgren's method (**6**), and said that the linear dimensions of the bright rings are approximately the same as the calculated radii of the HII region for some nebulae. Some question is left for their procedure, but here we let it away. Hattori and Yada (**39**) attacked this problem in another way. They performed an extensive survey for eighty-nine nebulae to reinvestigate their Hubble's diagram. To establish Hubble's relation, they allowed for the following three parameters: (1) the effective temperature of central star T; (2) the electron density of the nebulosity N_e; (3) the opitical thickness at the Lyman limit τ_0 of the envelope.

The basic relation is the fundamental equation:

$$\log a+\frac{m-\Delta m}{5}=\log\left(\frac{r}{L^{1/2}}\right)+\text{const}\,, \tag{32}$$

where m and Δm are the nebular magnitude and the space absorption in photographic region, and a and r are the angular radii in seconds of arc and the radii in persecs of the nebula, respectively. L is the true photographic brightness of the central star. They employed the Planckian form for the determination of L, and Strömgren's equation for r. With some assumptions, they obtained the final expression as follows:

$$\log a+\frac{m-\Delta m}{5}=\text{const}+f(T)+g(N_e)+h(\tau_0)\,, \tag{33}$$

where

$$f(T)=\log T-\frac{2280}{T}+\frac{1}{2}\log\left(10^{\frac{14700}{T}}-1\right), \tag{34}$$

$$g(N_e)=-\frac{2}{3}\log N_e\,, \tag{35}$$

$$h(\tau_0)=\frac{1}{3}\log\left(1-e^{-\tau_0}\right). \tag{36}$$

The unknown constant on the right-hand side is determined from the data of the planetary nebula IC 418, with the tentative assumption $\tau_0=\infty$.

Truly, the ionized hydrogen regions would agree with the theoretical radii. But, a fair deviation from the theoretical curve is perceived in their results. After the estimations of probable errors in their treatment, they gave the statistical discussions that not all of nebulae are HII regions and there is the diversity of the optical thickness τ_0, and further suggested that some extremely large nebulae will be in the advanced stage of expansion.

The dynamical study of the planetary nebulae was made by many investigators in connection with the Lyman alpha radiation field. The recombination theory of the excitation mechanism of nebula expects its strong field. Since the excitation ability of the Lyman alpha line is considerable, the radiation pressure due to radiation much acts on the stability of planetary nebulae. The problem has been first analyzed by Ambarzumian (**15**) and Chandrasekhar (**16**). Ambarzumian showed that the radiation pressure of Lyman alpha would be enormous and greatly excess of that of the ultraviolet radiation. But this conclusion is incompatible with observations. To avoid this defect, Zanstra (**17**) first proposed the model of an outergoing nebula and suggested the diminishing of the flux of Lyman alpha radiation due to a velocity gradient. Afterwards, Sobolev (**18**) and Zanstra (**19**) took the parabolic or Maxwell contour for the rectangular contour of the Lyman alpha line which had been adopted thus far, and proposed the change of frequency of its photons due to the Doppler effect in scattering process. The influence of this feature upon the transfer of Lyman alpha radiation is varied for the optical thickness τ_0 of nebulae. Miyamoto (**40**) and Unno (**41**) studied this effect over the peculiar stars and the planetary nebula, respectively. The present author (**42**) examined the variety of this effect over the optical thicknesses τ_0 of nebulae and arrived at the conclusion that the effect of the redistribution in frequency of photon energy operates for the nebulae with $\tau_0<1$ and becomes invalid for the nebulae with $\tau_0>1$. In this connection it may be interesting to recall the preceding results of the relation of intensity ratio Π to temperature and the survey of Hubble's diagram.

In the preceding section, we referred the relation of the nuclear temperature to the electron temperature. The constant dilution factor adopted by Baker *et al.* will be different from the real value owing to the diffused radiation. The dilution factor for the ultraviolet continuous radiation is given by

$$W_{1\kappa} = J_\nu + e^{-\tau_\nu}, \tag{37}$$

where J_ν is the diffused radiation of frequency ν and the τ_ν is the optical depth for the radiation of frequency ν. For J_ν, they employed the solution of the equation of transfer on J_ν:

$$\frac{d^2 J_\nu}{d\tau_\nu^2} = \lambda_\nu J_\nu - 3p_\nu e^{-\tau_\nu} - 3K_\nu, \tag{38}$$

where $\lambda_\nu^2 = 3(1-p_\nu)$ and K_ν is the interlocking term given by

$$K_\nu = \frac{e^{\chi_0}-1}{\sum_1^\infty \frac{S_n}{n^3}} e^{T_s(\chi_0-\chi)/T_e} \int_{\chi_0}^\infty (J_\nu + e^{-\tau_\nu}) \frac{d\chi}{\chi(e^\chi - 1)} - p_\nu (J_\nu + e^{-\tau_\nu}) . \tag{39}$$

From the solution of Eq. (38) as obtained by succesive approximations, they obtained the dilution factor $W_{1\kappa}$ as shown in Fig. 2. The results are given for the two cases: the nebulae of optical thickness $\tau_0=1$ and 10, respectively.

From Fig. 2 we can observe some piling-up of the continuous radiation near the inner boundary of the envelope. The effect of this piling-up is larger in the optically thick nebula than the other.

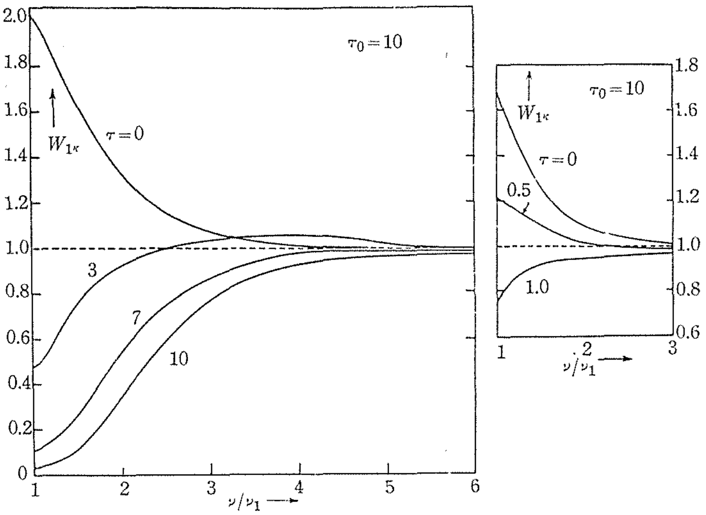

Fig. 2. The dilution factor for the optically thin and thick nebulae (in units of geometrical dilution factor). ν_1 is the frequency of the ionization. The curves are given by the parameter of optical depth. This is originally due to Aller's *"Gaseous Nebulae"* (1954).

We have discussed the relations of the optical thickness τ_c to the other physical quantities. The conclusion is that the star of higher effective temperature is associated with the optically thick nebulosity. The radiation emitted from the central star of higher excitation temperature is accumulated at the inner boundary and gives the higher ratio of helium to hydrogen by the succesive ionizations and recombinations. This nebula continues to expand by the large radiation pressure and results in the large optically thin nebula. In the later stage, the mechanism of expansion will be the thermal diffusion as well as the radiation pressure. The thermal diffusion is more effective in the atmosphere of high electron temperature. The negative effect of the condensation may be neglected in such a situation of high kinetic temperature.

4. Some views for further research

In the foregoing discussions, the energy distribution of the stellar radiation is frequently assumed to be of the Planckian. The effective energy distribution of the outergoing radiation of a star is deformed by the absorption and scattering in the outer layer of the star. The model of stars is calculated by many investigators. The spectral type and population of the central star are as yet uncertain. If these situations are cleared up, more reliable nuclear temperature will be obtained.

The methods of determining stellar temperature cited in the previous section are based on the constant dilution factor for the ultraviolet continuous radiation. We must take this effect into account in those calculations.

The temperature of the exciting star of the nova is obtained (**43**), using the hydrogen and helium method. The values are shown in Table 10. The spectrum

Table 10. The temperatures of exciting stars of novae.

	BF Cygni	Z Andromedae	CI Cygni
H-method	23,000-34,000	28,000	60,000
He-method		90,000	130,000

of nova varies over some stages. In the first stage the spectrum is of the continuous one and in the next stage the numerous broad emission lines appear with absorption component on their violet side. In the last stage, the faint continuous spectrum of a normal hot star is left behind. In the intermediate stage, the spectrum is like that of the planetary nebulae in which the strong forbidden lines appear. In such a stage, the physical treatment of planetary nebulae may be applied to novae. The cause of the burst of nova is unknown yet, so that the mechanisms of softening of ultraviolet radiation are not applicable to the novae. The study on the mechanism of burst will advance that of the higher helium temperature and the higher helium to hydrogen intensity ratio.

According to Bowen's suggestion (**44**), the excitation of oxygen ion O^{++} is due to resonance absorption of the helium radiation. An O^{++} ion, which is raised to the $3d\,^3P$ level by absorption of HeII 303.78 Å, either returns to the ground level with the emission of λ 303.8 Å or cascades to the ^3S level with the emission of λ 3133 Å. HeII 303.78 Å line is the "Lyman alpha" of helium ion. We can expect the strong field of its radiation. The emission line λ 3133 Å will owe its intensity to the helium radiation. A study of the intensity ratio of λ 3133 Å line HeII 4684 Å line may somewhat clear the ultraviolet continuous radiation field.

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