Memoirs of the College of Science, University of Kyoto, Series A,
Vol. XXX, No. 2, Article 6, 1962.

ON THE ATMOSPHERE OF THE LATE-TYPE COMPONENT OF ZETA AURIGAE-TYPE STARS

BY

Shusaku KAWABATA*

(*Received May 9, 1962*)

ABSTRACT

Atmospheres of late-type supergiant component of Zeta Aurigae-type stars are discussed, especially with respect to the state of ionization and to the condensed structure. After deriving the ionization formulae for metallic elements in a locally condensed atmosphere, the reciprocal of the packing fraction of condensation, α, is, using Wilson and Abt's model, shown to have very high values and to increase towards upper layer of the atmosphere in order to suffice the observed values of degree of metallic ionization, even though the ratio of number of atoms contained in condensed region to that of smoothed case, β, is nearly equal to unity. To avoid these difficulties, high electron temperature of the atmosphere is assumed, and we extend the ionization formula by taking account of the effects of electron collision due to the high electron temperature. Temperature distribution of the atmosphere that can explain the observed degree of ionization is given with constant α as parameter for Zeta Aurigae and 31 Cygni. As for calcium, calculations are done for both cases, namely that of $\tau_{\mathrm{CaII}} \simeq 0$ and $\tau_{\mathrm{CaII}} \gg 1$. Our discussion leads to results that in the lower layer of the atmosphere $T_{\varepsilon}=4000° \sim 8000°$K for both stars and in the upper layer $T_{\varepsilon}=12000° \sim 16000°$K for Zeta Aurigae and $T_{\varepsilon}=10000° \sim 12000°$K for 31 Cygni, and that $\log \alpha = 2.5 \sim 5$ for both stars. Treatment of r^{-m}-distribution of density is also discussed, but results are shown to be about the same with those of exponential form. It is shown that at least for 31 Cygni those condensations cannot be directly identified as prominence-like clouds detected in the outer layer of the primary.

1. Introduction

When an eclipsing binary system consists of a giant or supergiant star of low surface temperature and a much smaller and hotter star, observations of the eclipses of the small star by its primary star are quite interesting, because we have opportunity to study the atmospheric structure of the large and cool star, though we cannot apply directly to the atmosphere of a single late-type supergiant star. Four systems with this eclipsing nature, namely, Zeta Aurigae, 31 Cygni, 32 Cygni and VV Cephei, were ascertained, and a number of observations and analyses are carried out. Some figures about the eclipses and physical dimensions of these four binaries are given in Table 1.

* Present address: 4th Division, Meguro Technical Research Laboratories, National Defence Agency, Tokyo, Japan.

Table 1.

	Zeta Aurigae	31 Cygni	32 Cygni	VV Cephei
Period (day)	973	3802	1141	7400
Duration of total elipse (day)	38	61	12	450
Radius of primary star (10^6 km)	140	120	150	720
Height of chromosphere (10^6 km)	350	350	280	2100
Radius of secondary star (10^6 km)	3.5	7	2.1	3.5

In the discussions of the atmosphere of primary component of system, condensation structure is considered by several authors (**1**, **2**, **3**, **4**). There are two reasons to assume the presence of condensations: The first reason is that the state of metallic ionization in the atmosphere could not be explained with smooth distribution of matter in the atmosphere, but locally condensed cloud would be sufficient for the observational values, because higher electron density in the clouds could reduce the degree of ionization. This hypothesis is proposed by Wilson and Abt (**1**) for Zeta Aurigae, and later by Wright (**2**) for 31 Cygni. Deutsch (**5**) also tried to apply this hypothesis to discuss the circumstellar envelope of Alpha Herculis.

The other reason is the prominence-like motions of calcium clouds at the pre- and post-total eclipses which are confirmed from complex structure of CaII lines. Such motions of calcium clouds are detected by McKellar *et al.* (**3**, **6**) for 31 Cygni, by McKellar and Petri (**7**) for Zeta Aurigae, and by McLaughlin (**8**) for VV Cephei. In the case of Wilson and Abt's model, however, the packing fraction of condensations becomes very small, and the existence of condensation in the atmosphere makes the treatment of the problem complicated. For Zeta Aurigae, the present writer (**9**) proposed a model which has smoothed distribution of matter and high electron temperature. But it is not sufficient to explain the degree of ionization especially with regard to the supply of electrons.

In this paper we discuss atmospheric models which have increasing electron temperature with height and constant or decreasing α which is defined as reciprocal of packing fraction. But we do not refer to 32 Cygni because of the lack of quantitative data.

In § 2, geometry and the penetration of radiation from the hotter star into the cool atmosphere during atmospheric eclipse are presented. In § 3, atmosphere of Zeta Aurigae is discussed. Considering the ionization formulae in the locally condensed atmosphere, variation of α through the atmosphere in the case of Wilson and Abt's model is derived. In this case, α seems to have very high values and increases rapidly towards the upper atmosphere.

To avoid these characters of α, we assume high electron temperature in the atmosphere and derive the ionization formulae for these conditions and the distribution of electron temperature with heights. In §4, similar treatment will be applied to 31 Cygni. In §5, treatment of VV Cephei is discussed. In §6, calculation is carried out under the assumption of r^{-m}-distribution of matter instead of exponential form. In the last section, discussions of our results are given.

2. Treatment of atmospheric eclipse

2.1 Before we enter into the discussions of Zeta Aurigae-type stars, it is necessary to prepare the geometry of atmospheric eclipse and the pass of radiation from the secondary star to the extended atmosphere of the primary star. Although it has been already formulated by several authors, we summarize and present in a rather generalized way. Now the secondary star is assumed to be far distant from the primary compared with the effective dimensions of the atmosphere of primary. Spherical symmetry of the primary star is also assumed. These assumptions would be reasonable in our case. At first, we discuss the number of atoms along the line of sight (Fig. 1). Let us consider a column of unit cross section in the primary atmosphere along the line of sight. Here r_c is minimum distance from the center of the primary star to the column, and r_0 the radius of the primary star. In Fig. 1 the dotted line represents a section plane whose inter-

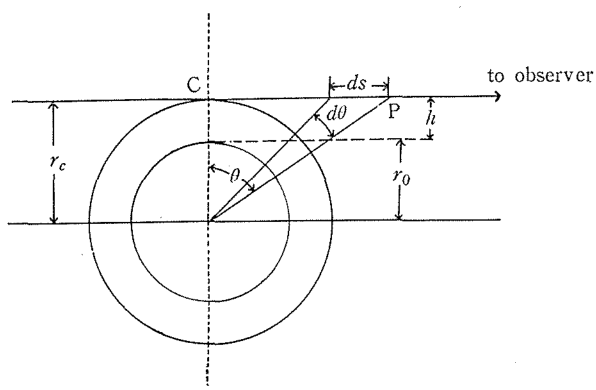


Fig. 1.

section with the star presents the disc of this star to the observer. s is measured from this plane along the directed line. Let θ be the angle between the vertical line from the center to the line of sight and a line joining the center and a point P whose distance from C is s. h means the atmospheric height of point C. Then we have

$$ds = r_c \sec^2\theta\, d\theta . \tag{2.1}$$

Let $n(r)$ be particle density in the atmosphere at distance r. Then the total number of atoms $N(s)$ per unit cross section from the secondary star, which is considered to be at infinity from the atmosphere of primary, to the observer, is given by

$$\begin{aligned} N(s) &= \int_{-\infty}^{s} n(r)\, ds = \int_{-\pi/2}^{\theta} n(r)\, r_c \sec^2\theta\, d\theta \\ &= \int_{-\pi/2}^{\theta} n(r_c \sec\theta)\, r_c \sec^2\theta\, d\theta , \end{aligned} \tag{2.2}$$

which may be rewritten as:

$$N(s) = \int_{-\pi/2}^{0} n(r_c \sec\theta)\, r_c \sec^2\theta\, d\theta + \int_{0}^{\theta} n(r_c \sec\theta)\, r_c \sec^2\theta\, d\theta . \tag{2.3}$$

Usually the hydrostatic distribution of density is assumed, but the Gaposchkins (**10**), Linnel (**11**) and others considered the form proportional to r^{-m}. We shall consider these two cases.

Case A. Exponential distribution of density

Let n_0 be the density at atmospheric base, and γ be the density gradient. Then

$$\left.\begin{aligned} n(r) &= n_0 e^{-\gamma(r-r_0)} , \\ n_c &= n_0 e^{-\gamma(r_c-r_0)} , \end{aligned}\right\} \tag{2.4}$$

and we have

$$n(r) = n_c e^{-\gamma(r-r_c)} . \tag{2.5}$$

Thus, we obtain from (2.2),

$$\begin{aligned} N(s) \equiv N(\theta) &= \int_{-\pi/2}^{\theta} n_c e^{-\gamma(r_c \sec\theta - r_c)} r_c \sec^2\theta\, d\theta \\ &= n_c r_c \int_{-\pi/2}^{\theta} e^{-\gamma r_c(\sec\theta - 1)} \sec^2\theta\, d\theta . \end{aligned} \tag{2.6}$$

Putting $\tan\theta = t$ and changing variables in (2.6), we have

$$\begin{aligned} N(s) \equiv N(t) &= n_c r_c \int_{-\infty}^{t} e^{-\gamma r_c(\sqrt{1+t^2}-1)} dt \\ &= n_c r_c \left\{ \int_{-\infty}^{0} e^{-\gamma r_c(\sqrt{1+t^2}-1)} dt + \int_{0}^{t} e^{-\gamma r_c(\sqrt{1+t^2}-1)} dt \right\} \\ &= n_c r_c \{ F(\infty, \gamma r_c) + F(t, \gamma r_c) \} , \end{aligned} \tag{2.7}$$

where we have put

$$F(t, k) = \int_{0}^{t} e^{-k(\sqrt{1+t^2}-1)} dt . \tag{2.8}$$

The total number of particle in the column is easily obtained as:

$$N_{\text{total}} = N(s=\infty) = N\left(\theta = \frac{\pi}{2}\right) = N(t=\infty)$$
$$= n_c r_c \int_{-\infty}^{\infty} e^{-\gamma r_c(\sqrt{1+t^2}-1)} dt = 2n_c r_c F(\infty, \gamma r_c) . \tag{2.9}$$

Now, we assume

$$\gamma r_c \gg 1 . \tag{2.10}$$

Then, only the region $t \ll 1$ is effective for integrations and the function F becomes the error function E:

$$E(t, k) \equiv \int_0^t e^{-(k/2)t^2} dt , \qquad k > 0 , \tag{2.11}$$

and we have

$$N(s) = n_c r_c \int_{-\infty}^{t} e^{-\gamma r_c(t^2/2)} dt$$
$$= n_c r_c \left(\int_{-\infty}^{0} e^{-\gamma r_c(t^2/2)} dt + \int_0^t e^{-\gamma r_c(t^2/2)} dt \right)$$
$$= n_c r_c [E(\infty, \gamma r_c) + E(t, \gamma r_c)] , \tag{2.12}$$

and

$$N_{\text{total}} = 2n_c r_c E(\infty, \gamma r_c)$$
$$= n_c \sqrt{\frac{2\pi r_c}{\gamma}} , \tag{2.13}$$

or equivalently

$$n_c = N_{\text{total}} \sqrt{\frac{\gamma}{2\pi r_c}} . \tag{2.14}$$

Expressions (2.12) and (2.13) are approximate formulae valid only in the case $\gamma r_c \gg 1$, and exact values of $N(s)$ and N_{total} can be obtained from (2.7) and (2.9) respectively by performing numerical integrations. But the difference of approximate value of N_{total} from the corresponding exact value is roughly estimated with the ratio $F(\infty, \gamma r_c)/E(\infty, \gamma r_c)$. Fig. 2 shows this estimation.

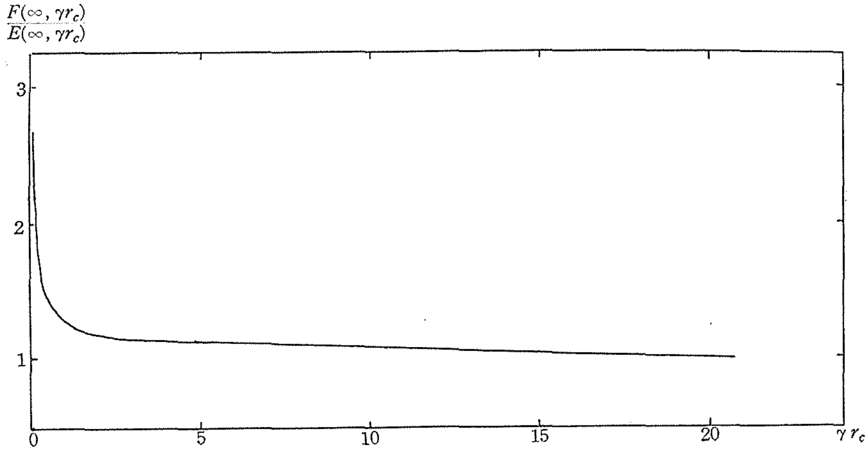


Fig. 2.

Case B. r^{-m}-distribution

Recently, the ejection of the matter from the atmosphere of late-type super-giant star has been confirmed (5). If we take these circumstances, together with hydrostatic case, into account, it will be important to consider the case of r^{-m}-distribution in the atmosphere of Zeta Aurigae-type stars.

We assume a distribution law of the form:

and

$$\left.\begin{aligned} n(r) &= n_0\left(\frac{r}{r_0}\right)^{-m}, \\ n_c &= n_0\left(\frac{r_c}{r_0}\right)^{-m}. \end{aligned}\right\} \tag{2.15}$$

Then we have

$$n(r) = n_c\left(\frac{r}{r_c}\right)^{-m}. \tag{2.16}$$

Using (2.16), Eq. (2.2) is written as:

$$\begin{aligned} N(s) \equiv N(\theta) &= \int_{-\pi/2}^{\theta} n_c\left(\frac{r_c \sec\theta}{r_c}\right)^{-m} r_c \sec^2\theta\, d\theta \\ &= n_c r_c \int_{-\pi/2}^{\theta} \cos^{m-2}\theta\, d\theta = n_c r_c \left\{\int_{-\pi/2}^{0} \cos^{m-2}\theta\, d\theta + \int_0^{\theta} \cos^{m-2}\theta\, d\theta\right\} \\ &\equiv n_c r_c\left[G\left(\frac{\pi}{2},\ m-2\right) + G(\theta,\ m-2)\right], \end{aligned} \tag{2.17}$$

where

$$G(\theta,\ m) = \int_0^{\theta} \cos^m\theta\, d\theta. \tag{2.18}$$

The total number is expressed by

$$\begin{aligned} N_{\text{total}} &= 2n_c r_c G\left(\frac{\pi}{2},\ m-2\right) \\ &= n_c r_c \sqrt{\pi}\,\frac{\Gamma([m-1]/2)}{\Gamma(m/2)}, \end{aligned} \tag{2.19}$$

which gives

$$n_c = \frac{N_{\text{total}}}{r_c\sqrt{\pi}}\,\frac{\Gamma(m/2)}{\Gamma([m-1]/2)}. \tag{2.20}$$

2.2 The problem of penetration of radiation from a high temperature star into the cool extended atmosphere was treated by Strömgren (**12**) for Epsilon Aurigae system. Again we consider Fig. 1. Let n_1, n_2 and n_e be the densities of neutral atom, singly ionized ion and free electron. Then, the formula of ionization in the column is given by

$$\frac{n_2}{n_1} n_e = 10^{15.38-\chi\theta} T_\star T_\varepsilon^{1/2} W e^{-\tau_1} \equiv C e^{-\tau_1}, \tag{2.21}$$

where

$$C = 10^{15.38-\chi\theta} T_\star T_\varepsilon^{1/2} W, \tag{2.22}$$

where $T_\star$ means the effective temperature of secondary star, T_ε the local electron temperature of the cool star, W the dilution factor, and χ and θ have usual meanings. τ_1 is the ultraviolet optical depth along the ray from the ionizing star to the point considered. Let x and n denote the degree of ionization and the total density of atom. Then, we have

$$\left.\begin{aligned} n_2 &= xn\,, \\ n_1 &= (1-x)n\,, \\ n_e &= xn\,. \end{aligned}\right\} \tag{2.23}$$

Using (2.23), Eq. (2.21) can be written as:

$$\frac{x^2}{1-x}n = Ce^{-\tau_1}\,. \tag{2.24}$$

τ_1 can be expressed in terms of the absorption coefficient a_1 in the ultraviolet per atom and the number of neutral atoms per unit volume $(1-x)n$ as:

$$d\tau_1 = (1-x)na_1 ds\,. \tag{2.25}$$

From (2.24) and (2.25) we have

$$d\tau_1 = \frac{a_1}{C}n^2 x^2 e^{-\tau_1} ds\,, \tag{2.26}$$

and then

$$\int_0^{\tau_1} \frac{1}{x^2} e^{-\tau_1} d\tau_1 = \frac{a_1}{C}\int_{-\infty}^{s} n^2 ds\,. \tag{2.27}$$

Discussing the nature of the integrations of (2.27), Strömgren shows that, along a given ray from the companion, considering atom will be almost completely ionized up to a certain point s_0 defined by the following formula (2.28), and then the ionization will drop very rapidly to a value close to zero.

$$1 = \frac{a_1}{C}\int_{-\infty}^{s_0} n^2 ds\,. \tag{2.28}$$

Consequently s_0 can be considered as the position of boundary between the ionized and neutral zones. We apply this result to the two cases of density distribution derived in §2.1.

Case A. Exponential distribution

Here, the right-hand side of (2.28) is written as:

$$\begin{aligned} 1 = \frac{a_1}{C}\int_{-\infty}^{s_0} n^2 ds &= \frac{a_1}{C}\int_{-\infty}^{t_0} n_c^2 e^{-2\gamma r_c(\sqrt{1+t^2}-1)} dt \\ &= \frac{a_1}{C} n_c^2 [F(\infty,\ 2\gamma r_c) + F(t_0,\ 2\gamma r_c)]\,. \end{aligned} \tag{2.29}$$

From this formula we can find the boundary of ionization for a given n_c. Of course, when approximation (2.10) is valid, F-functions in (2.29) are replaced by error function E. The ray for which the ionization just extends through the atmosphere, i.e. the critical ray, is the one for which $t_0 = +\infty$. Hence, the central density on the critical ray is expressed by

$$n_{c,\,\mathrm{crit}} = \left[\frac{C}{2a_1 F(\infty,\ 2\gamma r_c)}\right]^{1/2}. \tag{2.30}$$

If F is approximated by E, then (2.30) becomes

$$n_{c,\,\mathrm{crit}} = \left(\frac{C}{a_1}\right)^{1/2}\left(\frac{\gamma}{\pi r_c}\right)^{1/4}. \tag{2.31}$$

This is derived by Wilson and Abt (**1**).

Case B. r^{-m}*-distribution*

In this case, from (2.16) and (2.28), we have

$$\begin{aligned} 1 = \frac{a_1}{C}\int_{-\infty}^{s_0} n^2\, ds &= \frac{a_1}{C} n_c^2 \int_{-\infty}^{\theta_0} \cos^{2m-2}\theta\, d\theta \\ &= \frac{a_1}{C} n_c^2\left[G\left(\frac{\pi}{2},\ 2m-2\right) + G(\theta_0,\ 2m-2)\right]. \end{aligned} \tag{2.32}$$

The central density on the critical ray is given by

$$\begin{aligned} n_{c,\,\mathrm{crit}} &= \left[\frac{C}{2a_1 G(\pi/2,\ 2m-2)}\right]^{1/2} \\ &= \left[\frac{C}{a_1}\frac{\Gamma(m)}{\sqrt{\pi}\,\Gamma([2m-1]/2)}\cdots\right]^{1/2}. \end{aligned} \tag{2.33}$$

3. The atmosphere of the K-type component of Zeta Aurigae

3.1 The system of Zeta Aurigae has been known longer and investigated more extensively than the others, from both observational and theoretical aspects. From the observation of 1934 eclipse, Christie and Wilson (**13**) obtained the relative number of atoms of the various elements at various levels above the photosphere of K star, and discussed the structure of the atmosphere. Roach and Wood (**14**) discussed the photoelectric observations during the eclipses of 1939~1940 and 1947~1948. Their conclusion is that the probable mechanism of light extinction is due to the weakening of the radiation of B star by absorption lines produced in the K star's atmosphere, and the role of negative hydrogen ion is ineffective. Wilson and Abt tried detailed discussions on the K star's atmosphere with the spectroscopic data of 1939~1940 and 1947~1948 eclipses. They derived a number of important results for physical quantities; number of atoms along the line of sight, density gradient and excitation temperature, and so on. One of the most important conclusions of their discussions is that the atmosphere of K-type supergiant star should be consisted with discrete condensations of very high particle density. Groth (4) also mentioned the condensed structure in the atmosphere to explain metallic ionization in the discussion of 1950 and 1953 eclipses.

3.2 The condensation structure in atmosphere is convenient to interpret low degree of ionization, and it is necessary to estimate the packing fraction, or in other words, the degree of condensation, and to know its variation through the atmosphere qualitatively. In this section we consider the ionization in locally condensed atmosphere. As a first approximation we shall take rather simplified case. Our assumptions are as follows:

(i) Assuming a static atmosphere, we put the density distribution of matter as an exponential form. Treatment of r^{-m}-distribution will be discussed in a later section.

(ii) For the degree of ionization we take that of mean values obtained by dividing the total number of ionized atom in the column considered by that of neutral atom. In our case this approximation seems to be sufficient.

(iii) Normal cosmic abundance ratio is assumed.

As observational quantities to which our discussions are extended we take from data of 1939～1940 and 1947～1948 eclipses, which are shown in Figs. 3 and 4. Groth's data during 1950 and 1953 eclipses are also plotted in Fig. 4.

Total number of various atoms and degree of ionization greatly depend on the degree of ionization of CaII to CaIII. Following (2.31), the critical density of CaII in the atmosphere of Zeta Aurigae is estimated to be 1.06×10^{9}, with values of $r_c=1.4\times10^{13}$ cm, $\gamma=6\times10^{-12}$ cm^{-1}, $W=3\times10^{-6}$, $T_\star=T_B=15000°$K, $T_\varepsilon=4000°$K and $a_{\mathrm{CaII}}=2.4\times10^{-19}$. The central density of CaII at 8×10^{6} km derived from observation is found to be $10^{5.2}$, which is smaller than $n_{c,\mathrm{crit}}$, and then the ionization of CaII to CaIII proceeds by the radiation of B star. However, as is shown by Wilson and Abt, the presence of Lyman Beta absorption which lies 19 A shortward from the limit of CaII may affect the CaII ionization. The absorption coefficient in the wing of a line is given by

$$k = 16.5\times10^{-26}(\lambda/\Delta\lambda)^2 f\,. \tag{3.1}$$

With $\lambda=1026$ A, $\Delta\lambda=19$ A and $f=7.9\times10^{-2}$, and denoting the number of neutral hydrogen as $N(\mathrm{H})$, we find the optical depth for CaII due to the wing of Lyman Beta as:

$$\tau_{\mathrm{CaII}} = 3.84\times10^{-23}N(\mathrm{H})\,. \tag{3.2}$$

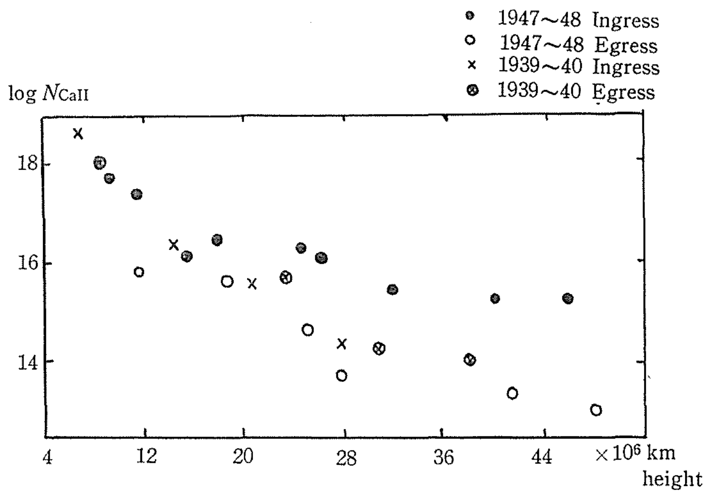


Fig. 3. Zeta Aurigae: log N *versus* height for CaII. (By Wilson and Abt)

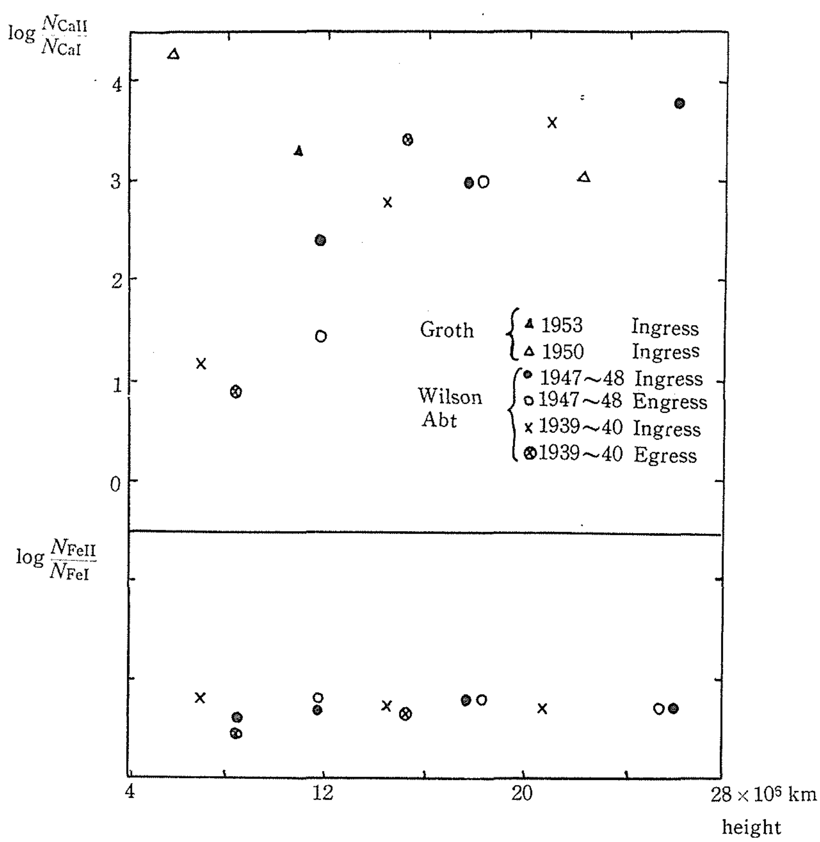


Fig. 4. Zeta Aurigae: Ionization of Ca and Fe as a function of height, (By Wilson and Abt)

Wilson and Abt estimate that the opacity due to Lyman Beta can suppress the ionization of CaII over about three-fourths of the light path at the height of 8×10^6 km. That is, the suppression due to Lyman Beta lies in critical region for Zeta Aurigae. By this reason we divide our problem into two extreme cases, namely, *Case I* in which $\tau_{\rm CaII}$ is nearly equal to zero and *Case II* in which $\tau_{\rm CaII}$ is much larger than unity.

3.3 As we assume the exponential distribution of matter, the particle density of any atom or ion at any height is expressed by (2.4) and the total number by (2.13). To make our treatment easy, let us define the "uniformly smoothed distance" s_s as:

$$s_s = \sqrt{\frac{2\pi r_c}{\gamma}}. \tag{3.3}$$

Here s_s has the meaning of length of column in which the all of atoms of an element on the line of sight are distributed with uniform density n_c.

Taking s_s and n_c as standards, we discuss the condensed region and the other region—we call it "ground region"—in the atmosphere. Of course, the condensation may have various values of density and dimension, but in our simplified case, we assume density of condensed parts in the column along the ray to be constant.

Now we proceed to calculate the degree of ionization of calcium in the locally condensed atmosphere. Let $N^{(c)}$, $n^{(c)}$, s_c, $N^{(g)}$, $n^{(g)}$ and s_g mean the total number of atoms, density, and length of condensed and ground region in the column. We assume further that electrons are supplied from metal.

(*a*) *Case I.* $\tau_{\rm CaII}\simeq 0$.

In this case, the ionization formulae for calcium are described as follows. For the condensed region

$$\frac{N^{(c)}_{\rm CaIII}}{N^{(c)}_{\rm CaII}} = \frac{f_{\rm CaII}}{n^{(c)}_{\rm e}}, \tag{3.4}$$

$$\frac{N^{(c)}_{\rm CaII}}{N^{(c)}_{\rm CaI}} = \frac{f_{\rm CaI}}{n^{(c)}_{\rm e}}, \tag{3.5}$$

$$n^{(c)}_{\rm e} = n^{(c)}_{\rm metal} = D_{\rm Ca}\, n^{(c)}_{\rm Ca}, \tag{3.6}$$

$$n^{(c)}_{\rm Ca} = \frac{N^{(c)}_{\rm Ca}}{s_c}, \tag{3.7}$$

and

$$N^{(c)}_{\rm Ca} = N^{(c)}_{\rm CaI} + N^{(c)}_{\rm CaII} + N^{(c)}_{\rm CaIII}. \tag{3.8}$$

Here $f_{\rm CaII}$ and $f_{\rm CaI}$ are usual ionization constants for CaII and CaI, namely:

$$f_{\rm CaII} = 10^{15.38-\chi_{\rm CaII}\theta}\, T_B\, T_e^{1/2} W, \tag{3.9}$$

and

$$f_{\mathrm{CaI}} = 10^{15.38-\chi_{\mathrm{CaI}}\theta} T_B T_\varepsilon^{1/2} W. \tag{3.10}$$

D_{Ca} is the normal cosmic abundance ratio of metal to calcium, $\log D_{\mathrm{Ca}}=1.8$. T_ε, the local electron temperature, is taken as the effective temperature of K star, namely 4000°K, $T_B=15000$°K and $W=3\times10^{-6}$. Remaining notations in the expressions of ionization constants f_{CaII} and f_{CaI} are as usual.

For the ground region:

$$\frac{N^{(g)}_{\mathrm{CaIII}}}{N^{(g)}_{\mathrm{CaII}}} = \frac{f_{\mathrm{CaII}}}{n^{(g)}_{\mathrm{e}}}, \tag{3.11}$$

$$\frac{N^{(g)}_{\mathrm{CaII}}}{N^{(g)}_{\mathrm{CaI}}} = \frac{f_{\mathrm{CaI}}}{n^{(g)}_{\mathrm{e}}}, \tag{3.12}$$

$$n^{(g)}_{\mathrm{e}} = n^{(g)}_{\mathrm{metal}} = D_{\mathrm{Ca}}\, n^{(g)}_{\mathrm{e}}, \tag{3.13}$$

$$n^{(g)}_{\mathrm{Ca}} = \frac{N^{(g)}_{\mathrm{Ca}}}{s_g}, \tag{3.14}$$

and

$$N^{(g)}_{\mathrm{Ca}} = N^{(g)}_{\mathrm{CaI}} + N^{(g)}_{\mathrm{CaII}} + N^{(g)}_{\mathrm{CaIII}}. \tag{3.15}$$

In *Case I*, in the exact sense, the number of electrons supplied by CaIII is twice the number of CaIII. But, with respect to n_{e}, these deviations can be practically neglected and we can put $n_{\mathrm{e}}=n_{\mathrm{metal}}$. Further, as is seen from Fig. 4, N_{CaII} is much larger than N_{CaI}, hence we can neglect N_{CaI} compared with $N_{\mathrm{CaII}}+N_{\mathrm{CaIII}}$ in the expressions (3.8) and (3.15).

Since the quantities to be compared with observations are mainly concerned with CaII and CaI, we obtain the two solutions of family of ionization formulae (3.4)~(3.8) and (3.11)~(3.12).

For the condensed region:

$$I^{(c)}_{\mathrm{CaI}} \equiv \frac{N^{(c)}_{\mathrm{CaII}}}{N^{(c)}_{\mathrm{CaI}}} = f_{\mathrm{CaI}} \Big/ \left(\frac{D_{\mathrm{Ca}}}{s_c} N^{(c)}_{\mathrm{CaII}} + \frac{1}{2}\left[-\frac{D_{\mathrm{Ca}}}{s_c} N^{(c)}_{\mathrm{CaII}} + \left\{ \left(\frac{D_{\mathrm{Ca}}}{s_c} N^{(c)}_{\mathrm{CaII}} \right)^2 + \frac{D_{\mathrm{Ca}}}{s_c} f_{\mathrm{CaII}} N^{(c)}_{\mathrm{CaII}} \right\}^{1/2} \right] \right), \tag{3.16}$$

and

$$n^{(c)}_{\mathrm{e}} = \frac{D_{\mathrm{Ca}}}{s_c} N^{(c)}_{\mathrm{CaII}} + \frac{1}{2}\left[-\frac{D_{\mathrm{Ca}}}{s_c} N^{(c)}_{\mathrm{CaII}} + \left\{ \left(\frac{D_{\mathrm{Ca}}}{s_c} N_{\mathrm{CaII}} \right)^2 + \frac{D_{\mathrm{Ca}}}{s_c} f_{\mathrm{CaII}} N^{(c)}_{\mathrm{CaII}} \right\}^{1/2} \right]; \tag{3.17}$$

and for the ground region:

$$I^{(g)}_{\mathrm{CaI}} \equiv \frac{N^{(g)}_{\mathrm{CaII}}}{N^{(g)}_{\mathrm{CaI}}} = f_{\mathrm{CaI}} \Big/ \left(\frac{D_{\mathrm{Ca}}}{s_g} N^{(g)}_{\mathrm{CaII}} + \frac{1}{2}\left[-\frac{D_{\mathrm{Ca}}}{s_g} N^{(g)}_{\mathrm{CaII}} + \left\{ \left(\frac{D_{\mathrm{Ca}}}{s_g} N^{(g)}_{\mathrm{CaII}} \right)^2 + \frac{D_{\mathrm{Ca}}}{s_g} f_{\mathrm{CaII}} N^{(g)}_{\mathrm{CaII}} \right\}^{1/2} \right] \right), \tag{3.18}$$

and

$$n_e^{(g)} = \frac{D_{\rm Ca}}{s_g} N_{\rm CaII}^{(g)} + \frac{1}{2}\left[-\frac{D_{\rm Ca}}{s_g} N_{\rm CaII}^{(g)} + \left\{ \left(\frac{D_{\rm Ca}}{s_g} N_{\rm CaII}^{(g)} \right)^2 + \frac{D_{\rm Ca}}{s_g} f_{\rm CaII} N_{\rm CaII}^{(g)} \right\}^{1/2} \right]. \quad (3.19)$$

These two regions are connected as follows:

$$\left.\begin{aligned} N_{\rm CaIII}^{(c)} + N_{\rm CaIII}^{(g)} &= N_{\rm CaIII}\,, \\ N_{\rm CaII}^{(c)} + N_{\rm CaII}^{(g)} &= N_{\rm CaII}\,, \\ N_{\rm CaI}^{(c)} + N_{\rm CaI}^{(g)} &= N_{\rm CaI}\,, \\ \text{and} \qquad s_c + s_g &= s_s\,. \end{aligned}\right\} \quad (3.20)$$

Let α be the ratio of the density in the condensed region to that of the uniformly smoothed case, and also β be the ratio of the total number in the condensed region to that of the uniformly smoothed case. Then we have

$$\left.\begin{aligned} \alpha &= \frac{n^{(c)}}{n_c}\,, \\ \text{and} \qquad \beta &= \frac{N^{(c)}}{N}\,. \end{aligned}\right\} \quad (3.21)$$

Then we get

$$n^{(c)} = \frac{N^{(c)}}{s_c} = \alpha \frac{N}{s_s} = \alpha n_c\,, \quad (3.22)$$

$$N^{(g)} = N - N^{(c)} = N(1-\beta)\,, \quad (3.23)$$

$$N\beta / s_c = \alpha N / s_s\,, \quad (3.24)$$

and

$$s_c = s_s \beta / \alpha\,. \quad (3.25)$$

Consequently, we obtain

$$s_g = s_s - s_c = s_s(1-\beta/\alpha)\,, \quad (3.26)$$

and

$$n^{(g)} = \frac{N^{(g)}}{s_g} = \frac{N(1-\beta)}{s_s(1-\beta/\alpha)} = n_c \frac{\alpha(1-\beta)}{\alpha-\beta}\,. \quad (3.27)$$

We must notice that following (3.22) and (3.27), the solutions (3.16) and (3.18) are functions of α. From the expressions (3.16)$\sim$(3.27), we get "the mean degree of ionization of calcium" $I_{\rm CaI,\,total}$:

$$\begin{aligned} I_{\rm CaI,\,total} &\equiv \{N_{\rm CaII}^{(c)} + N_{\rm CaII}^{(g)}\} / \{N_{\rm CaI}^{(c)} + N_{\rm CaI}^{(g)}\} \\ &= 1/\{\beta / I_{\rm CaI}^{(c)}(\alpha) + (1-\beta)/I_{\rm CaI}^{(g)}(\alpha)\}\,. \end{aligned} \quad (3.28)$$

Here, for convenience in further discussions, we shall define "packing fraction" as $1/\alpha$. Thus, the mean degree of ionization of calcium is calculated with (3.28) taking α and β as parameters.

(*b*) *Case II.* $\tau_{\mathrm{CaII}} \gg 1$

In this case, the expressions (3.4) and (3.11) can be dropped from the family of equations of *Case I*, and the terms N_{CaIII} in the expressions (3.8) and (3.15) are negligible. Then we have $N_{\mathrm{CaII}} = N_{\mathrm{Ca}}$. In *Case II*, $I^{(c)}_{\mathrm{CaI}}$, $I^{(g)}_{\mathrm{CaI}}$ and the mean degree of ionization $I_{\mathrm{CaI,\,total}}$ are easily calculated with

$$I^{(c)}_{\mathrm{CaI}} \equiv N^{(c)}_{\mathrm{CaII}}/N^{(c)}_{\mathrm{CaI}} = f_{\mathrm{CaI}}/n^{(c)}_{\mathrm{e}}\,, \tag{3.29}$$

$$n^{(c)}_{\mathrm{e}} = \frac{D_{\mathrm{Ca}}}{s_c} N^{(c)}_{\mathrm{CaII}}\,, \tag{3.30}$$

$$I^{(g)}_{\mathrm{CaI}} \equiv N^{(g)}_{\mathrm{CaII}}/N^{(g)}_{\mathrm{CaI}} = f_{\mathrm{CaI}}/n^{(g)}_{\mathrm{e}}\,, \tag{3.31}$$

$$n^{(g)}_{\mathrm{e}} = \frac{D_{\mathrm{Ca}}}{s_g} N^{(g)}_{\mathrm{CaII}}\,, \tag{3.32}$$

and

$$\begin{aligned} I_{\mathrm{CaI,\,total}} &= 1/\{\beta/I^{(c)}_{\mathrm{CaI}}(\alpha) + (1-\beta)/I^{(g)}_{\mathrm{CaI}}(\alpha)\} \\ &= \frac{K}{n_c[\alpha\beta + \alpha(1-\beta)^2/(\alpha-\beta)]}\,, \end{aligned} \tag{3.33}$$

where

$$K = f_{\mathrm{CaI}}/D_{\mathrm{Ca}}\,. \tag{3.34}$$

If $s_c/s_g \ll 1$, α is much greater than β, and then

$$I_{\mathrm{CaI,\,total}} = \frac{K}{n_c[\alpha\beta + (1-\beta)^2]}\,. \tag{3.35}$$

(*c*) *Ionization of iron*

We have also to discuss the degree of ionization of iron. In this case, too, formal difference of ionization formulae between *Case I* and *Case II* appears in the expression of electron density. Ionization formulae are;
for the condensed region

$$I^{(c)}_{\mathrm{FeI}} \equiv N^{(c)}_{\mathrm{FeII}}/N^{(c)}_{\mathrm{FeI}} = f_{\mathrm{FeI}}/n^{(c)}_{\mathrm{e}}\,, \tag{3.36}$$

and for the ground region

$$I^{(g)}_{\mathrm{FeI}} \equiv N^{(g)}_{\mathrm{FeII}}/N^{(g)}_{\mathrm{FeI}} = f_{\mathrm{FeI}}/n^{(g)}_{\mathrm{e}}\,, \tag{3.37}$$

where

$$f_{\mathrm{FeI}} = 10^{15.38 - \chi_{\mathrm{FeI}}\theta} T_B T_g^{1/2} W\,. \tag{3.38}$$

For *Case I*, we use (3.17) and (3.19) for $n^{(c)}_{\mathrm{e}}$ in (3.36) and $n^{(g)}_{\mathrm{e}}$ in (3.37), and for *Case II*, we use (3.30) and (3.32).

The mean degree of ionization is expressed by

$$\begin{aligned} I_{\mathrm{FeI,\,total}} &\equiv \{N^{(c)}_{\mathrm{FeII}} + N^{(g)}_{\mathrm{FeII}}\}/\{N^{(c)}_{\mathrm{FeI}} + N^{(g)}_{\mathrm{FeI}}\} \\ &= 1/\{\beta/I^{(c)}_{\mathrm{FeI}}(\alpha) + (1-\beta)/I^{(g)}_{\mathrm{FeI}}(\alpha)\}\,. \end{aligned} \tag{3.39}$$

Wilson and Abt showed that variations of the degree of ionization of calcium and iron are not explained with the single model of smooth distribution, and then assumed the model with condensation. But from their observations, an important trend of metallic ionization was confirmed; the degree of ionization is nearly constant throughout the atmosphere except for calcium. Although the calcium anomaly is not yet explained, we adopt the value of the degree of ionization of calcium at the upper atmosphere, i.e., $N_{CaII}/N_{CaI} \sim 5000$. With respect to iron, Wilson and Abt estimate the abundance of N_{FeII} from N_{Ca}, and N_{FeI} from λ 3720 of FeI. Consequently, N_{FeII} and N_{FeII}/N_{FeI} depend on the estimation of N_{Ca}. We get $N_{FeII}/N_{FeI} \sim 2.2 \times 10^6$ for *Case I* and 500 for *Case II*.

In the expressions (3.28), (3.33) and (3.39), setting values of $I_{CaI, total}$ and $I_{FeI, total}$ as the constant values described above, we can find relation between α and n_c. The central density n_c depends on the atmospheric height. Then we get α as a function of atmospheric height. Results are presented in Fig. 5 with β as a parameter. We do not estimate α of iron for *Case I*, because the observed degree of ionization exceeds the computed value and the assumption of condensation looses its validity. Further discussions are extended in § 7.

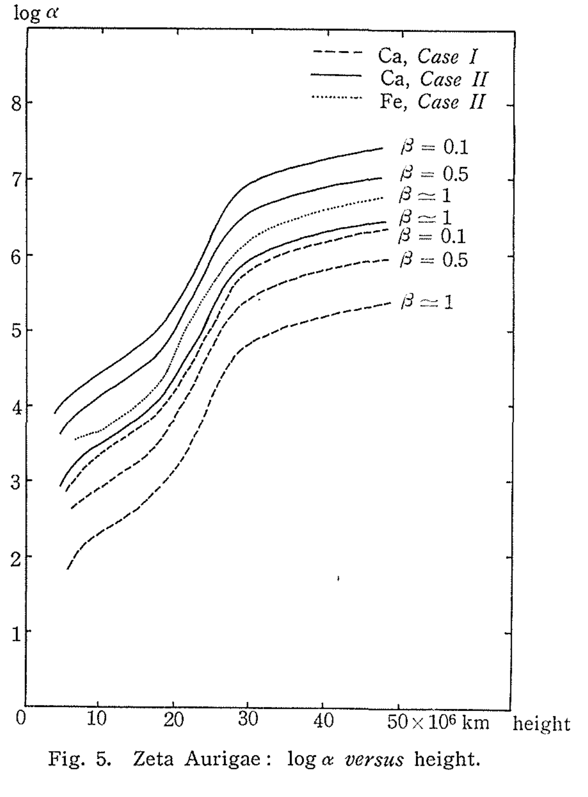


Fig. 5. Zeta Aurigae: log α *versus* height.

In Fig. 5 we can find various characteristics; namely, (i) $\log \alpha$ increases rapidly from lower to upper atmosphere by a factor of about 3.5; (ii) Decreasing of β makes α larger by an appreciable amount; (iii) The difference of $\log \alpha$ between *Case I* and *Case II* is about 1.3.

3.4 As is seen in Fig. 5, α takes a very high value. To explain the higher value of the degree of condensation, we need more force to make condensation. From the view of dynamical equilibrium, large value of the packing fraction, i.e., small value of α is desirable. Furthermore, rapid increase of α with height makes the problem more complicated, because it is natural that the condensation should be easily dissipated in the upper atmosphere than in the lower atmosphere. It should be desirable for α to decrease to the upper layer, or even not to increase. Here we try to get an atmospheric model in which electron temperature increases with height and α is smaller and non-increasing with height. Increasing of excitation temperature with height was already found by Wilson and Abt (**1**) and super-excitation in the upper atmosphere was suggested by Groth (**4**). Although there are ambiguities in the mechanism at the upper atmosphere, the increasing of temperature seems not so absurd.

When the electron temperature increases, the electron collision becomes more effective for ionization. In that case, new ionization formula should be used by taking account of collisional ionization.

Let Q_{12}, Q_{21}, S_{12} and S_{21} be the coefficients of photo-ionization, photo-recombination, collisional ionization and recombination by triple collision of atom or ion, respectively. Denoting the densities of atom and singly ionized ion by n_{x_0} and n_{x^+}, we have

$$\frac{n_{x^+}}{n_{x_0}} = \frac{Q_{12}(T_e)+Q_{21}(T_\varepsilon)}{Q_{21}(T_\varepsilon)+n_e S_{21}(T_\varepsilon)}. \tag{3.40}$$

Since it is a sufficient approximation that $Q_{21} \gg n_e S_{21}(T_\varepsilon)$, (3.40) becomes

$$\frac{n_{x^+}}{n_{x_0}} = \frac{1}{n_e} f_x(T_e,\ T_\varepsilon) + \phi_x(T_\varepsilon), \tag{3.41}$$

where

$$f_x(T_e,\ T_\varepsilon) = Q_{12}(T_e)/Q_{21}(T_\varepsilon), \tag{3.42}$$

and

$$\phi_x(T_\varepsilon) = S_{12}(T_\varepsilon)/Q_{21}(T_\varepsilon). \tag{3.43}$$

Suffix x of f and ϕ means the respective element.

f is the ratio of photo-recombination to photo-ionization and is given exactly by our previous ionization formulae as:

$$f_x = 10^{15.38-\chi_x\theta} T_e T_\varepsilon^{1/2} W. \tag{3.44}$$

Using Unsöld's expression (**15**), we have

$$\phi_x = \frac{3\sqrt{3}}{16\delta^3}\frac{\zeta_r}{\varepsilon n_0}\left(\frac{\chi_H}{\chi_x}\right)^2\frac{e^{-u_n}}{u_n}\left(1+\frac{0.3}{u_n}\right), \tag{3.45}$$

where

$$u_n = \frac{h\nu_n}{kT}, \tag{3.46}$$

and ζ_r is the number of outermost electron, δ the reciprocal of Sommerfeld's fine structure constant, $\delta=2\pi e^2/hc=1/137$, and ε the correction factor for the recombination in excited state due to Elwert (**16**).

(*a*) *Case I.* $\tau_{\mathrm{CaII}}\simeq 0$

For calcium, ionization formulae are described, for the condensed regions, by

$$N^{(c)}_{\mathrm{CaIII}}/N^{(c)}_{\mathrm{CaII}} = \frac{1}{n^{(c)}_e}f_{\mathrm{CaII}}+\phi_{\mathrm{CaII}}, \tag{3.47}$$

$$N^{(c)}_{\mathrm{CaII}}/N^{(c)}_{\mathrm{CaI}} = \frac{1}{n^{(c)}_e}f_{\mathrm{CaI}}+\phi_{\mathrm{CaI}}, \tag{3.48}$$

$$N^{(c)}_{\mathrm{H}} = (N^{(c)}_{\mathrm{CaI}}+N^{(c)}_{\mathrm{CaII}}+N^{(c)}_{\mathrm{CaIII}})A_{\mathrm{Ca}}, \tag{3.49}$$

$$n^{(c)}_e = n^{(c)}_{\mathrm{metal}}+n^{(c)}_{\mathrm{H+}} = \frac{1}{s_c}(D_{\mathrm{Ca}}N^{(c)}_{\mathrm{Ca}}+N^{(c)}_{\mathrm{H+}}), \tag{3.50}$$

$$N^{(c)}_{\mathrm{H+}}/N^{(c)}_{\mathrm{H_0}} = \frac{1}{n^{(c)}_e}f_{\mathrm{H}}+\phi_{\mathrm{H}}, \tag{3.51}$$

while, for the ground region, we have

$$N^{(g)}_{\mathrm{CaIII}}/N^{(g)}_{\mathrm{CaII}} = \frac{1}{n^{(g)}_e}f_{\mathrm{CaII}}+\phi_{\mathrm{CaII}}, \tag{3.52}$$

$$N^{(g)}_{\mathrm{CaII}}/N^{(g)}_{\mathrm{CaI}} = \frac{1}{n^{(g)}_e}f_{\mathrm{CaI}}+\phi_{\mathrm{CaI}}, \tag{3.53}$$

$$N^{(g)}_{\mathrm{H}} = (N^{(g)}_{\mathrm{CaI}}+N^{(g)}_{\mathrm{CaII}}+N^{(g)}_{\mathrm{CaIII}}), \tag{3.54}$$

$$n^{(g)}_e = n^{(g)}_{\mathrm{metal}}+n^{(g)}_{\mathrm{H+}} = \frac{1}{s_g}(D_{\mathrm{Ca}}N^{(g)}_{\mathrm{Ca}}+N^{(g)}_{\mathrm{H+}})A_{\mathrm{Ca}}, \tag{3.55}$$

$$N^{(g)}_{\mathrm{H+}}/N^{(g)}_{\mathrm{H_0}} = \frac{1}{n^{(g)}_e}f_{\mathrm{H}}+\phi_{\mathrm{H}}. \tag{3.56}$$

A_{Ca} is the normal abundance ratio of hydrogen to calcium, namely $\log A_{\mathrm{Ca}}=5.6$. $N_{\mathrm{H_0}}$ means the total number of neutral hydrogen atom. As done before, N_{CaI} can be neglected compared with $N_{\mathrm{CaII}}+N_{\mathrm{CaIII}}$. Since the critical density of hydrogen derived from Strömgren's criterion is $\log n_{c,\,\mathrm{crit}}=8$ and is smaller than the density derived from the observation of calcium, the first terms on the right-hand sides of (3.51) and (3.56) are both nearly equal to zero.

Taking account of these approximations, the family of equations for collisional ionization can be solved. Thus, we have, for the condensed region,

$$I_{\mathrm{CaI}}^{(c)} \equiv \frac{N_{\mathrm{CaII}}^{(c)}}{N_{\mathrm{CaI}}^{(g)}} = f_{\mathrm{CaI}}\left[\frac{J}{s_c} N_{\mathrm{CaII}}^{(c)} + \frac{1}{2}\left\{\frac{J}{s_c} N_{\mathrm{CaII}}^{(c)} (\phi_{\mathrm{CaII}}-1) + \left[\left\{\frac{J}{s_c} N_{\mathrm{CaII}}^{(c)} (\phi_{\mathrm{CaII}}-1)\right\}^2 + 4\frac{J}{s_c} N_{\mathrm{CaII}}^{(c)}\left\{f_{\mathrm{CaII}} + \frac{J}{s_c} N_{\mathrm{CaII}}^{(c)2} \phi_{\mathrm{CaII}}\right\}\right]^{1/2}\right\}\right]^{-1} + \phi_{\mathrm{CaI}}, \tag{3.57}$$

$$n_{\mathrm{e}}^{(c)} = \frac{J}{s_c} N_{\mathrm{CaII}}^{(c)} + \frac{1}{2}\left\{\frac{J}{s_c} N_{\mathrm{CaII}}^{(c)} (\phi_{\mathrm{CaII}}-1) + \left[\left\{\frac{J}{s_c} N_{\mathrm{CaII}}^{(c)} (\phi_{\mathrm{CaII}}-1)\right\}^2 + 4\frac{J}{s_c} N_{\mathrm{CaII}}^{(c)}\left\{f_{\mathrm{CaII}} + \frac{J}{s_c} N_{\mathrm{CaII}}^{(c)} \phi_{\mathrm{CaII}}\right\}\right]^{1/2}\right\}, \tag{3.58}$$

where

$$J = D_{\mathrm{Ca}} + \phi_{\mathrm{H}}' A_{\mathrm{Ca}}, \tag{3.59}$$

with

$$\phi_{\mathrm{H}}' = 1 - \frac{1}{\phi_{\mathrm{H}}}, \tag{3.60}$$

and for the ground region we have

$$I_{\mathrm{CaI}}^{(g)} \equiv \frac{N_{\mathrm{CaII}}^{(g)}}{N_{\mathrm{CaI}}^{(g)}} = f_{\mathrm{CaI}}\left[\frac{J}{s_g} N_{\mathrm{CaII}}^{(g)} + \frac{1}{2}\left\{\frac{J}{s_g} N_{\mathrm{CaII}}^{(g)} (\phi_{\mathrm{CaII}}-1) + \left[\left\{\frac{J}{s_g} N_{\mathrm{CaII}}^{(g)} (\phi_{\mathrm{CaII}}-1)\right\}^2 + 4\frac{J}{s_g} N_{\mathrm{CaII}}^{(g)}\left\{f_{\mathrm{CaII}} + \frac{J}{s_g} N_{\mathrm{CaII}}^{(g)} \phi_{\mathrm{CaII}}\right\}\right]^{1/2}\right\}\right]^{-1} + \phi_{\mathrm{CaI}}, \tag{3.61}$$

$$n_{\mathrm{e}}^{(g)} = \frac{J}{s_g} N_{\mathrm{CaII}}^{(g)} + \frac{1}{2}\left\{\frac{J}{s_g} N_{\mathrm{CaII}}^{(g)} (\phi_{\mathrm{CaII}}-1) + \left[\left\{\frac{J}{s_g} N_{\mathrm{CaII}}^{(g)} (\phi_{\mathrm{CaI}}-1)\right\}^2 + 4\frac{J}{s_g} N_{\mathrm{CaII}}^{(g)}\left\{f_{\mathrm{CaII}} + \frac{J}{s_g} N_{\mathrm{CaII}}^{(g)}\right\}\right]^{1/2}\right\}. \tag{3.62}$$

(*b*) *Case II.* $\tau_{\mathrm{CaII}} \gg 1$

B star's radiation is not effective to ionize CaII to CaIII, and the corresponding terms can be neglected. The family of equations is simplified and is then solved using approximations as applied to *Case I.* Thus, we have, for the condensed region,

$$N_{\mathrm{CaIII}}^{(c)}/N_{\mathrm{CaII}}^{(c)} = \phi_{\mathrm{CaII}}, \tag{3.63}$$

$$N_{\mathrm{CaII}}^{(c)}/N_{\mathrm{CaI}}^{(c)} = \frac{f_{\mathrm{CaI}}}{n_{\mathrm{e}}^{(c)}} + \phi_{\mathrm{CaI}}, \tag{3.64}$$

$$N_{\mathrm{H}}^{(c)} = (N_{\mathrm{CaI}}^{(c)} + N_{\mathrm{CaII}}^{(c)} + N_{\mathrm{CaIII}}^{(c)}) A_{\mathrm{Ca}}, \tag{3.65}$$

$$n_{\mathrm{e}}^{(c)} = \frac{1}{s_c}(D_{\mathrm{Ca}} N_{\mathrm{Ca}}^{(c)} + \phi_{\mathrm{H}}' N_{\mathrm{H}^+}), \tag{3.66}$$

while, for the ground region, we get

$$N_{\mathrm{CaIII}}^{(g)}/N_{\mathrm{CaII}}^{(g)} = \phi_{\mathrm{CaII}}, \tag{3.67}$$

$$N_{\mathrm{CaII}}^{(g)}/N_{\mathrm{CaI}}^{(g)} = \frac{f_{\mathrm{CaI}}}{n_{\mathrm{e}}^{(g)}} + \phi_{\mathrm{CaI}}, \tag{3.68}$$

$$N_{\mathrm{H}}^{(g)} = (N_{\mathrm{CaI}}^{(g)} + N_{\mathrm{CaII}}^{(g)} + N_{\mathrm{CaIII}}^{(g)}) A_{\mathrm{Ca}}\,, \tag{3.69}$$

$$n_{\mathrm{e}}^{(g)} = \frac{1}{s_g}(D_{\mathrm{Ca}} N_{\mathrm{Ca}}^{(g)} + \phi_{\mathrm{H}}' N_{\mathrm{H}})\,. \tag{3.70}$$

Consequently, the final solutions become as follows. Namely, for the condensed region, we have

$$I_{\mathrm{CaI}}^{(c)} = \frac{N_{\mathrm{CaII}}^{(g)}}{N_{\mathrm{CaI}}^{(g)}} = f_{\mathrm{CaI}} \Big/ \left[\frac{N_{\mathrm{CaII}}^{(c)}}{s_c}\{D_{\mathrm{Ca}} + A_{\mathrm{Ca}}(1+\phi_{\mathrm{CaII}})\phi_{\mathrm{H}}'\}\right] + \phi_{\mathrm{CaI}}\,, \tag{3.71}$$

$$n_{\mathrm{e}}^{(c)} = \frac{N_{\mathrm{CaII}}^{(c)}}{s_c}\{D_{\mathrm{Ca}} + A_{\mathrm{Ca}}(1+\phi_{\mathrm{CaII}})\phi_{\mathrm{H}}'\}\,, \tag{3.72}$$

while, for the ground region, we have

$$I_{\mathrm{CaI}}^{(g)} \equiv \frac{N_{\mathrm{CaII}}^{(g)}}{N_{\mathrm{CaI}}^{(g)}} = f_{\mathrm{CaI}} \Big/ \left[\frac{N_{\mathrm{CaII}}^{(g)}}{s_g}\{D_{\mathrm{Ca}} + A_{\mathrm{Ca}}(1+\phi_{\mathrm{CaII}})\phi_{\mathrm{H}}'\}\right] + \phi_{\mathrm{CaI}}\,, \tag{3.73}$$

$$n_{\mathrm{e}}^{(g)} = \frac{N_{\mathrm{CaII}}^{(g)}}{s_g}\{D_{\mathrm{Ca}} + A_{\mathrm{Ca}}(1+\phi_{\mathrm{CaII}})\phi_{\mathrm{H}}'\}\,. \tag{3.74}$$

(*c*) *Ionization of iron*

For iron we consider *Case II* only. We can use $n_{\mathrm{e}}^{(c)}$ and $n_{\mathrm{e}}^{(g)}$ as derived from calcium, and simplified formulae are obtained. Thus,

$$N_{\mathrm{FeII}}^{(c)}/N_{\mathrm{FeI}}^{(c)} = \frac{f_{\mathrm{FeI}}}{n_{\mathrm{e}}^{(c)}} + \phi_{\mathrm{FeI}}\,, \tag{3.75}$$

$$N_{\mathrm{FeII}}^{(g)}/N_{\mathrm{FeI}}^{(g)} = \frac{f_{\mathrm{FeI}}}{n_{\mathrm{e}}^{(g)}} + \phi_{\mathrm{FeI}}\,. \tag{3.76}$$

We take (3.72) and (3.74) for $n_{\mathrm{e}}^{(c)}$ in (3.75) and $n_{\mathrm{e}}^{(g)}$ in (3.76).

3.5 Throughout subsections (*a*), (*b*) and (*c*) in §3.4, the connecting relations between both regions and the mean degree of ionization are the same as those already expressed by (3.21)~(3.27), (3.28), (3.33) and (3.39). Degree of ionization for each case (*a*), (*b*) and (*c*) are calculated and shown in Fig. 6 as functions of N_{Ca}/s_s or N_{Fe}/s_s with T_ε as a parameter.

Following the discussion in §3.4, let us discuss the atmosphere with constant packing fraction. With the formulae derived, the mean degree of ionization is expressed for both calcium and iron as follows:

$$I_{\mathrm{total}} = \Phi\left(T_\varepsilon,\ \frac{N_{\mathrm{CaII}}}{s_s},\ \alpha,\ \beta\right). \tag{3.77}$$

Since heights for given N_{Ca}/s_s or N_{Fe}/s_s are derived from observational data shown in Fig. 3, (3.77) has a form:

$$I_{\mathrm{total}} = \Psi(T_\varepsilon,\ h,\ \alpha,\ \beta)\,. \tag{3.78}$$

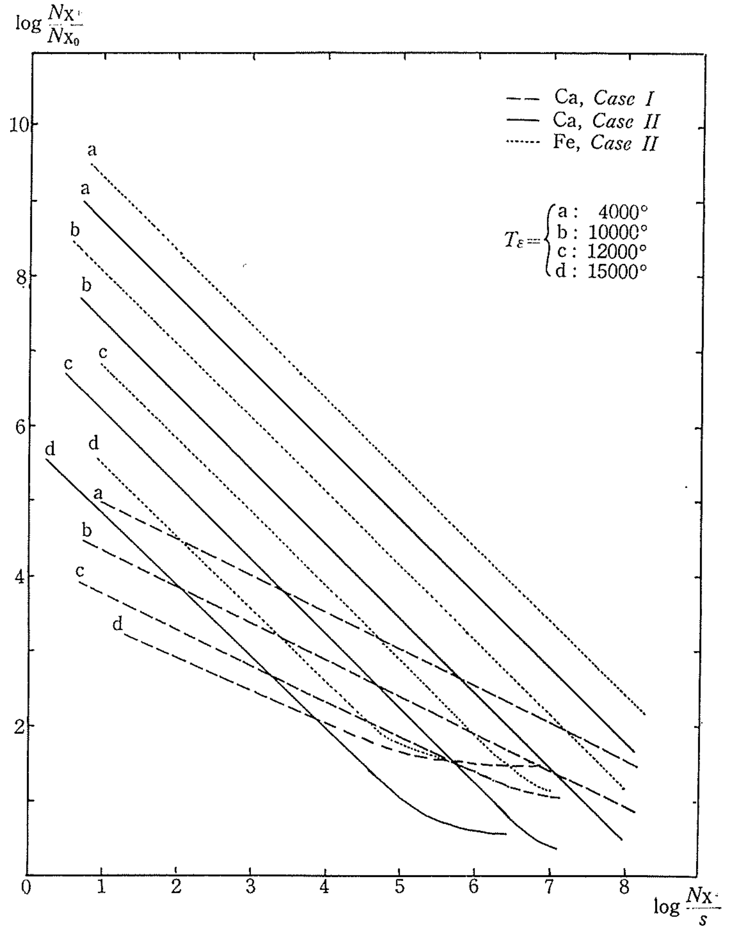


Fig. 6. Zeta Aurigae: Degree of ionization as a function of central density.

By the discussion in §3.3, taking fixed values of I_{total}, namely, 5000 for calcium and 500 for iron, the sufficient conditions for electron temperature are required as

$$T_\varepsilon = \Psi'(h, \alpha, \beta)\,. \qquad (3.79)$$

Ψ' is given in Figs. 7a and 7b as a function of height with α and β as parameters. Electron densities in both regions are determined by (3.58), (3.62), (3.72) and (3.74). They are directly affected by α and β. In Fig. 8 we plotted electron density *versus* atmospheric height for the case of uniformly smoothed distribution.

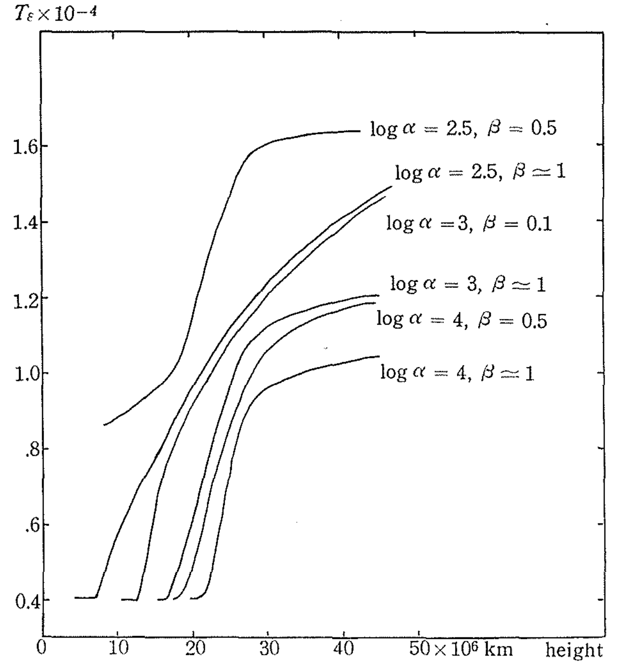


Fig. 7a. Zeta Aurigae: T_ε *versus* height for *Case I*.

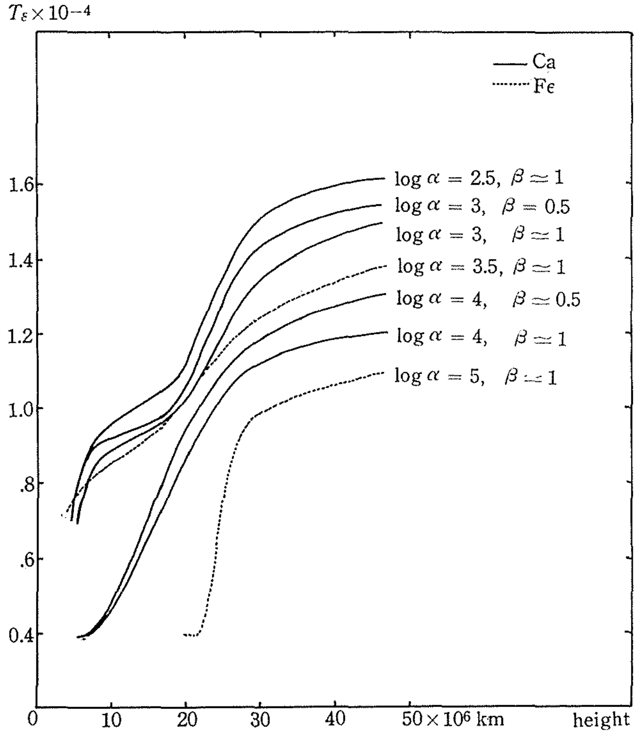


Fig. 7b. Zeta Aurigae: T_ε *versus* height for *Case II*.

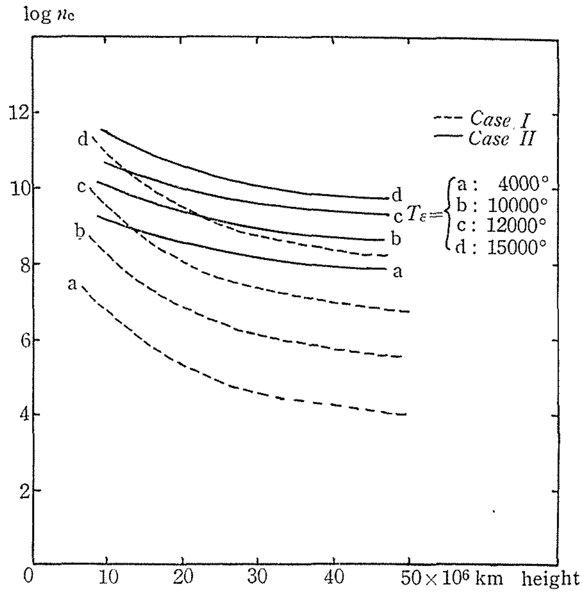


Fig. 8. Zeta Aurigae: Smoothed electron density *versus* height.

4. The atmosphere of the K-type component of 31 Cygni

4.1 Because of quite recent discovery of the nature of atmospheric eclipse of 31 Cygni by McLaughlin in 1950 (**17**), discussions on this star are not so much as that for Zeta Aurigae. Recently, detailed analysis was done by Wright (**2**) for 1951 eclipse, and he suggested the condensed structure of the atmosphere.

Let us estimate the reciprocal of packing fraction which may agree with the observed degree of metallic ionization with normal temperature of the K-type atmosphere. Simplifying assumptions used for Zeta Aurigae, i.e. (i)~(iii) in §3.2, are also adopted for 31 Cygni. We use the observed data for 1951 eclipse by Wright. His data are rather more detailed than those for Zeta Aurigae; the total number along the ray and the degree of ionization are given for neutral and ionized iron, calcium, titanium and scandium and so on. Although there are some scatters from each other in the estimations, it will be convenient to know the whole feature of the star. The total number for each element along the line of sight and their degree of ionization observed by Wright are shown in Figs. 9 and 10. As in the case of Zeta Aurigae, it is necessary to examine whether the ionization of CaII to CaIII proceeds or not. Applying (2.31) with $T_B=18600°K$, $T_\varepsilon=3900°K$, $W=7.4\times10^{-7}$, $a_{CaII}=2.4\times10^{-19}$ and $\gamma=1\times10^{-12}\,cm^{-1}$, we see that $\log n_{c,crit}=8.96$, which is much larger than the observed central density of CaII

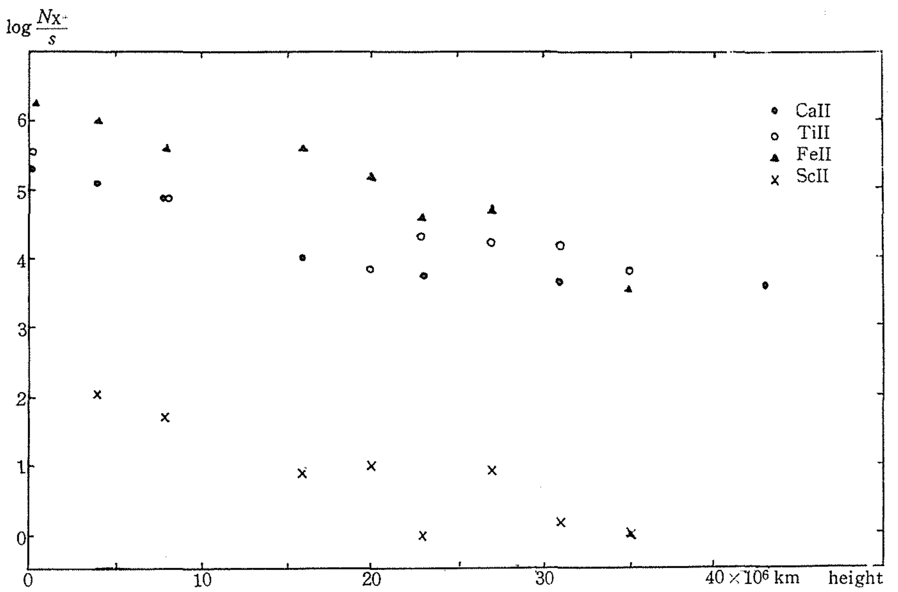


Fig. 9. 31 Cygni: Central density *versus* height for singly ionized elements. (Calculated from Wright's data)

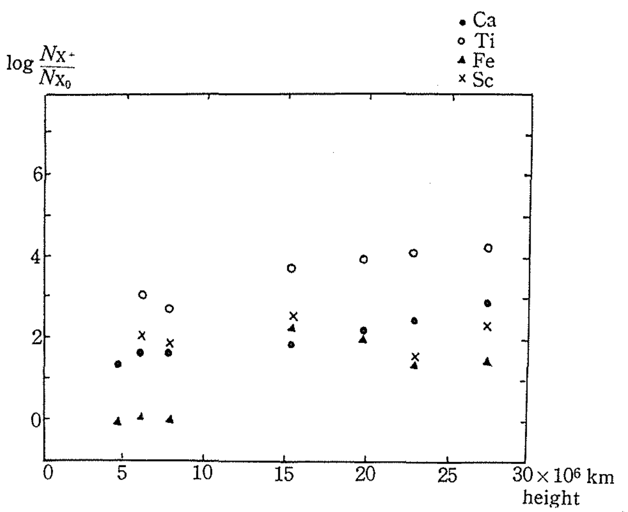


Fig. 10. 31 Cygni: Metallic ionization as a function of height observed by Wright.

at any height. Hence we consider that K-atmosphere is transparent for radiation to ionize CaII. Here we examine the suppression effect due to Lyman Beta absorption. The optical depth for Lyman Beta at the wavelength corresponding to the absorption edge of CaII is already given in (3.2). For the total number of neutral hydrogen we take tentatively the mean of values estimated from the abundance of each element obtained by Wright. Then we find that τ_{CaII} is of the order of unity at the height of 27×10^6 km and several ten times larger at the lower atmosphere. Then if we also take account of the scatter of observed value of $N(\mathrm{H})$, the suppression of radiation by Lyman Beta absorption is large and the central density lies in critical value. Consequently, in this section too, we consider two extreme cases, $\tau_{\mathrm{CaII}}\simeq 0$ and $\tau_{\mathrm{CaII}}\gg 1$.

By a similar method to that in §3, let us derive the reciprocal of packing fraction with normal K star's temperature. Solutions for the two cases of calcium are obtained with the use of Eqs. (3.4)∼(3.15) and (3.29)∼(3.32) directly. For other element x, denoting the quantities with suffix x_0 for neutral atom and x^+ for singly ionized atom, we have the family of equations as follows. Thus, for the condensed region:

$$N_{\mathrm{x}^+}^{(c)}/N_{\mathrm{x}_0}^{(c)} = f_{\mathrm{x}_0}/n_{\mathrm{e}}^{(c)}, \tag{4.1}$$

$$n_{\mathrm{e}}^{(c)} = n_{\mathrm{metal}}^{(c)} = D_{\mathrm{x}} n_{\mathrm{x}}^{(c)}, \tag{4.2}$$

$$n_{\mathrm{x}}^{(c)} = N_{\mathrm{x}}/s_c, \tag{4.3}$$

$$N_{\mathrm{x}}^{(c)} = N_{\mathrm{x}_0}^{(c)} + N_{\mathrm{x}^+}^{(c)}, \tag{4.4}$$

and for the ground region:

$$N_{\mathrm{x}^+}^{(g)}/N_{\mathrm{x}_0}^{(g)} = f_{\mathrm{x}_0}/n_{\mathrm{e}}^{(g)}, \tag{4.5}$$

$$n_{\mathrm{e}}^{(g)} = n_{\mathrm{metal}}^{(g)} = D_{\mathrm{x}} n_{\mathrm{x}}^{(g)}, \tag{4.6}$$

$$n_{\mathrm{x}}^{(g)} = N_{\mathrm{x}}^{(g)}/s_g, \tag{4.7}$$

$$N_{\mathrm{x}}^{(g)} = N_{\mathrm{x}_0}^{(g)} + N_{\mathrm{x}^+}^{(g)}, \tag{4.8}$$

where

$$f_{\mathrm{x}_0} = 10^{15.38-\chi_{\mathrm{x}_0}\theta} T_B T_\varepsilon^{1/2} W. \tag{4.9}$$

Here, the notations are the same as those in §3. We take $\log D_{\mathrm{x}}=3.3$ for Ti, 4.9 for Sc, and 1.1 for Fe. Relations between the condensed region and ground region are given by (3.21)∼(3.27).

Wright confirmed that the degrees of ionization for the metal are nearly constant throughout the atmosphere as in Zeta Aurigae. We estimate the constant values of the degree of ionization from Fig. 10 as follows:

$$\left.\begin{aligned} \log N_{\mathrm{TiII}}/N_{\mathrm{TiI}} &\simeq 3.5\,, \\ \log N_{\mathrm{FeII}}/N_{\mathrm{FeI}} &\simeq 1.5\,, \\ \log N_{\mathrm{ScII}}/N_{\mathrm{ScI}} &\simeq 2.2\,, \\ \log N_{\mathrm{CaII}}/N_{\mathrm{CaI}} &\simeq 2.2\,. \end{aligned}\right\} \tag{4.10}$$

Using (3.4)~(3.15), (3.29)~(3.32) and (4.1)~(4.10), we get the relation between α and h by the same method as used in §3.3. Results are presented in Fig. 11. They are plotted for $\beta \simeq 1$. In the case of 31 Cygni, the increase of α with height is not so abrupt as in the case of Zeta Aurigae.

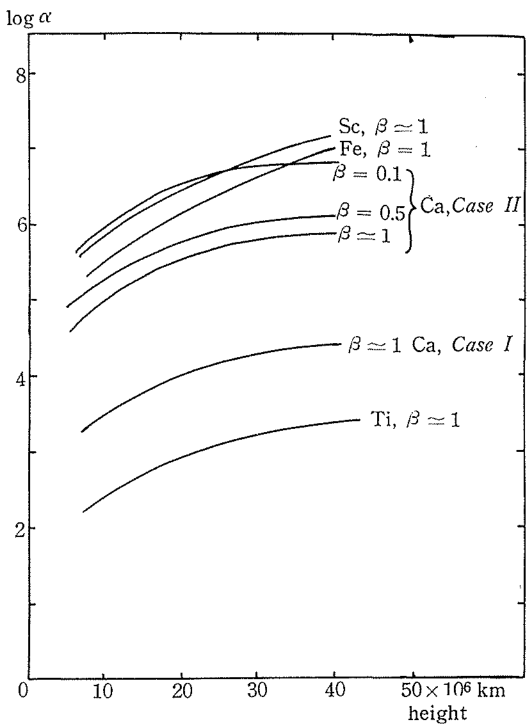


Fig. 11. Cygni: log α *versus* height.

4.2 As discussed in §3.4, we again consider the atmospheric model to get observed values of the degree of ionization, taking α as constant throughout the atmosphere. Formulations are performed as done in §3.4. For both *Case I* and *Case II* of calcium, (3.47)~(3.56) and (3.63)~(3.70) can be applied. For another element x, we get, for the condensed region,

$$N_{\mathrm{x}^+}^{(c)}/N_{\mathrm{x}_0}^{(c)} = \frac{1}{n_e^{(c)}} f_{\mathrm{x}_0} + \phi_{\mathrm{x}_0}\,, \tag{4.11}$$

$$N_{\mathrm{H}}^{(c)} = (N_{\mathrm{x}_0}^{(c)}+N_{\mathrm{x}+}^{(c)})A_{\mathrm{x}}\,, \tag{4.12}$$

$$n_{\mathrm{e}}^{(c)} = \frac{1}{s_c}(D_{\mathrm{x}}N_{\mathrm{x}}^{(c)}+\phi_{\mathrm{H}}'N_{\mathrm{H}}^{(c)})\,, \tag{4.13}$$

and, for the ground region,

$$N_{\mathrm{x}+}^{(g)}/N_{\mathrm{x}_0}^{(g)} = \frac{1}{n_{\mathrm{e}}^{(g)}}f_{\mathrm{x}_0}+\phi_{\mathrm{x}_0}\,, \tag{4.14}$$

$$N_{\mathrm{H}}^{(g)} = (N_{\mathrm{x}_0}^{(g)}+N_{\mathrm{x}+}^{(g)})A_{\mathrm{x}}\,, \tag{4.15}$$

$$n_{\mathrm{e}}^{(g)} = \frac{1}{s_g}(D_{\mathrm{x}}N_{\mathrm{x}}^{(g)}+\phi_{\mathrm{H}}'N_{\mathrm{H}}^{(g)})\,, \tag{4.16}$$

where A_{x} is the abundance ratio of hydrogen to element x, and we take $\log A_{\mathrm{x}} = 7.1$ for Ti, 8.6 for Sc and 4.9 for Fe.

Then, for the condensed region, we have

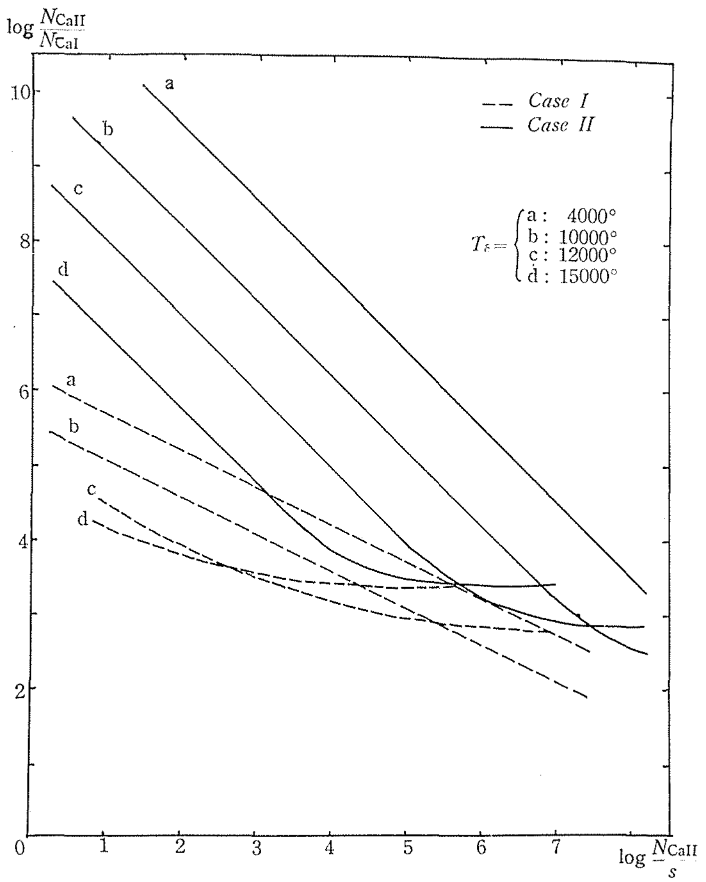


Fig. 12a. 31 Cygni: Degree of ionization for Ca as a function of central density.

$$N_{x^+}^{(c)}/N_{x_0}^{(c)} = \frac{f_{x_0}}{(N_x^{(c)}/s_c)\{D_x + A_x\phi_H'\}} + \phi_{x_0}, \tag{4.17}$$

$$n_e^{(c)} = \frac{N_x^{(c)}}{s_c}\{D_x + A_x\phi_H'\}, \tag{4.18}$$

and for the ground region, we get

$$N_{x^+}^{(g)}/N_{x_0}^{(g)} = \frac{f_{x_0}}{(N_x^{(g)}/s_g)\{D_x + A_x\phi_H'\}} + \phi_{x_0}, \tag{4.19}$$

$$n_e^{(g)} = \frac{N_x^{(g)}}{s_g}\{D_x + A_x\phi_H'\}, \tag{4.20}$$

where we have also assumed that $N_{x_0} \ll N_{x^+}$. In Figs. 12a∼12c the computed degree of ionization is shown as a function of particle density N/s for each element with T_ε as a parameter.

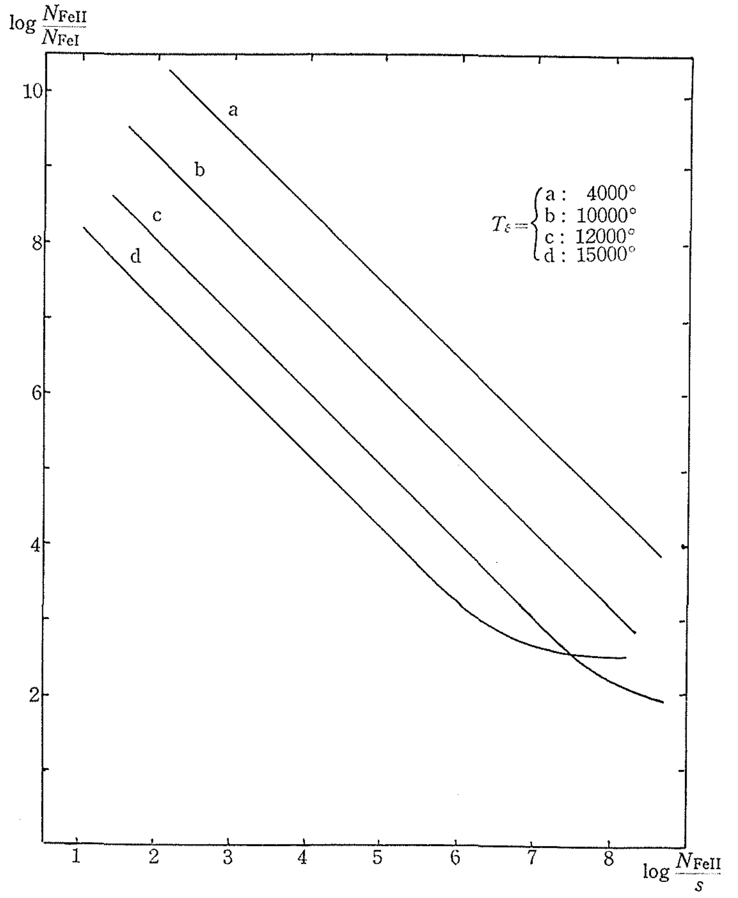


Fig. 12b. 31 Cygni: Degree of ionization for iron as a function of central density.

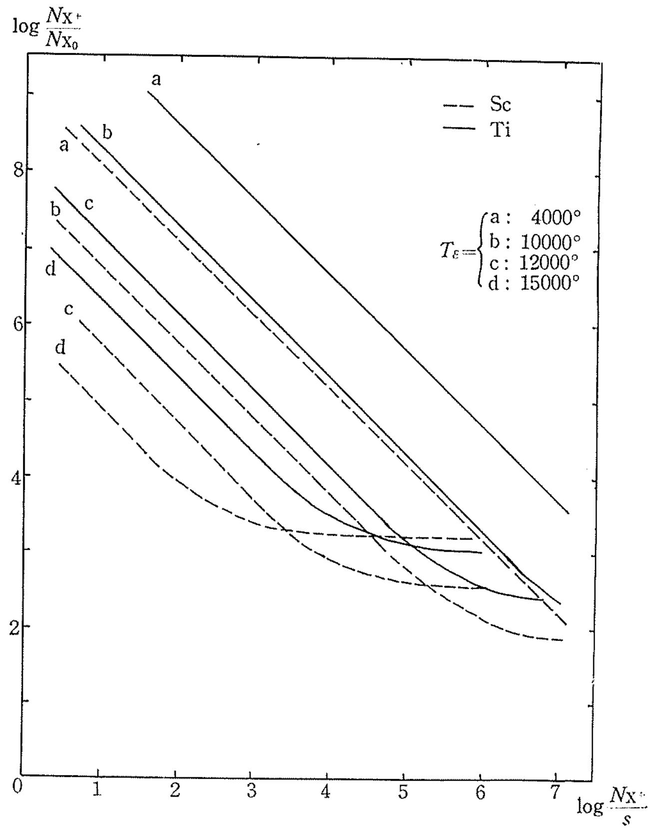
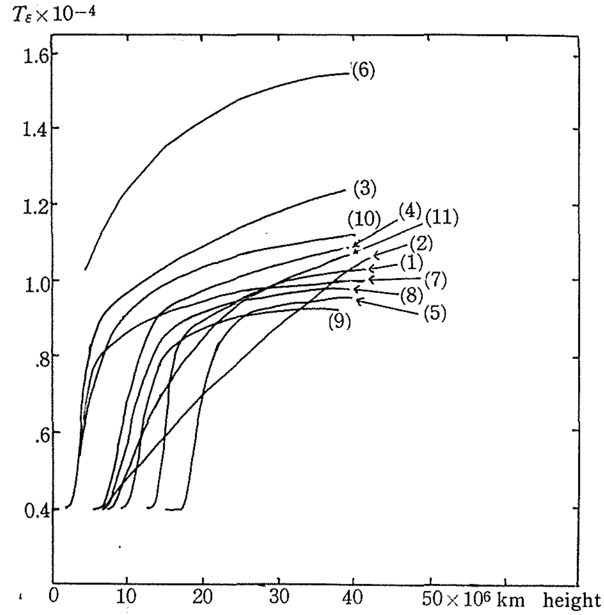


Fig. 12c. 31 Cygni: Degrees of ionization for Ti and Sc as functions of central density.

Now, we construct an atmosphere with constant α throughout the layers which may give the observed values of the degree of ionization as given in (4.10). The curves of electron temperature plotted against height, namely the curves of the function Ψ' defined by (3.79), are presented in Fig. 13 with α and β as parameters. The uniformly smoothed values of electron density calculated from each element are given in Figs. 14a and 14b. In Fig. 14c geometrical mean of those values are given. It will be seen that local density varies with α. Looking at Figs. 13 and 14a~14c, it is found that numerical values for each element are somewhat scattered, especially in the case of titanium.



No.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Element	Ca (*Case I*)	Ca (*Case II*)	Fe	Fe	Fe	Ti	Ti	Ti	Ti	Sc	Sc
log α	3	4.5	4.5	5	6	0	2	2.5	2.5	5	6
β	1	1	1	1	1	0	1	0.5	1	1	0.5

Fig. 13. 31 Cygni: T_ε *versus* height.

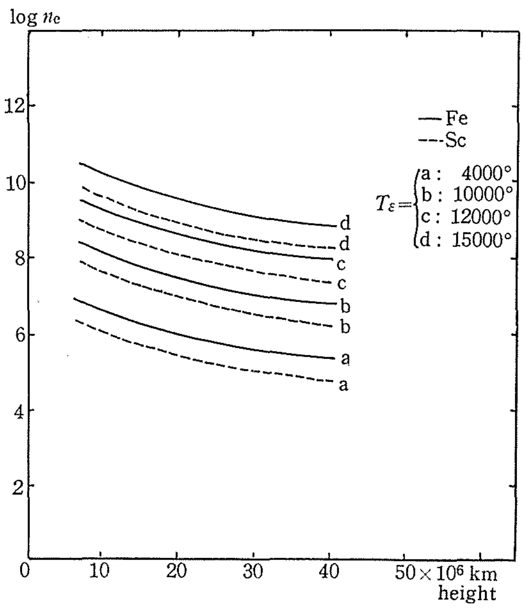


Fig. 14a. 31 Cygni: Smoothed electron density *versus* height, deduced from Ca and Ti.

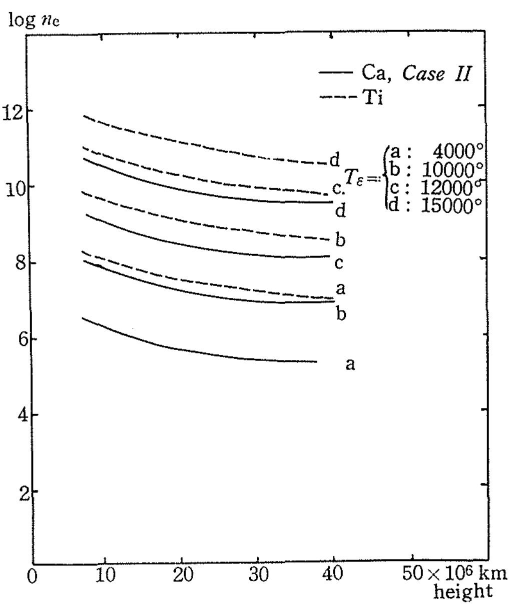


Fig. 14b. 31 Cygni: Smoothed electron density *versus* height, deduced from Fe and Sc.

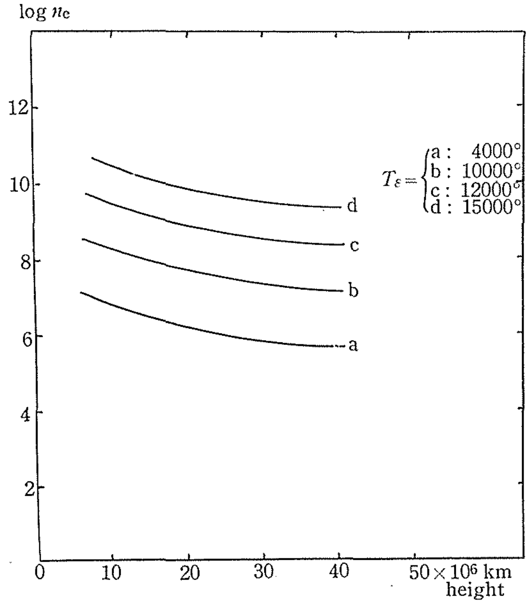


Fig. 14c. 31 Cygni: Mean value of smoothed electron density, deduced from Fe, Ti, Sc and *Case II* of Ca.

5. The atmosphere of the M-type component of VV Cephei

Owing to its large dimension of VV Cephei, the scale of atmospheric eclipse is much larger than the other three stars, and the last observed data are obtained at the eclipse in 1936. Although a number of observations are reported, discussions on the atmospheric structure of M-type component are still difficult, especially because of its intrinsic variable character. In this section we use the data obtained by Goedicke (**18**) at 1936 eclipse, and try to discuss the state of ionization in the atmosphere in connection with the results in the preceding sections.

From the analysis of absorption lines observed in partial phase, Goedicke derived the ratio of total number of atoms of neutral and ionized elements in the column of unit cross section from the B star to the observer, to that in the photosphere of the M star. Denoting the former by N and the latter by N_M, his results are reproduced in Fig. 15a for calcium and for titanium. Although the observed value of degree of ionization cannot be determined directly without knowing the total abundance of ionized and neutral atoms in the column from the B star, we can know the order of degree of ionization from Goedicke's data.

The ratio of N/N_M of x^+ to that of x_0 is expressed by

$$\frac{N_{x^+}/N_{M,x^+}}{N_{x_0}/N_{M,x_0}}=\frac{N_{x^+}}{N_{x_0}}\frac{1}{N_{M,x^+}/N_{M,x_0}}. \tag{5.1}$$

The denominator in the second factor on the right-hand side in (5.1) is the degree of ionization of element x in the atmosphere of M star along a ray from B star. Ionization due to M star's radiation can be obtained from

$$\frac{N_{M,x^+}}{N_{M,x_0}}n_e=10^{15.38-\chi_{x_0}\theta}T_M^{3/2}, \tag{5.2}$$

where T_M is the effective temperature of M star. We adopt T_M=2700°K. The left-hand side of (5.1) is estimated with the use of Fig. 15a and is given in Fig. 15b. Hence N_{x^+}/N_{x_0} can be deduced with n_e as a parameter. Next, we

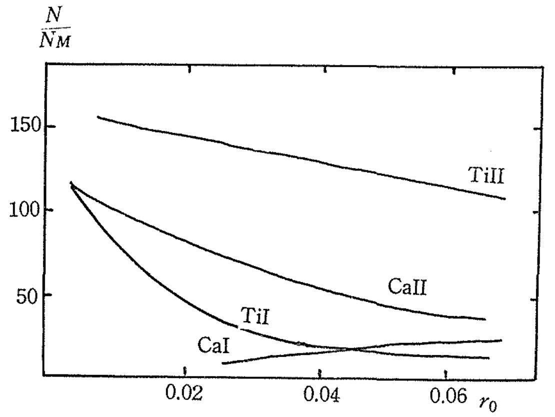


Fig. 15a. VV Cephei: N/N_M *versus* height. (By Goedicke)

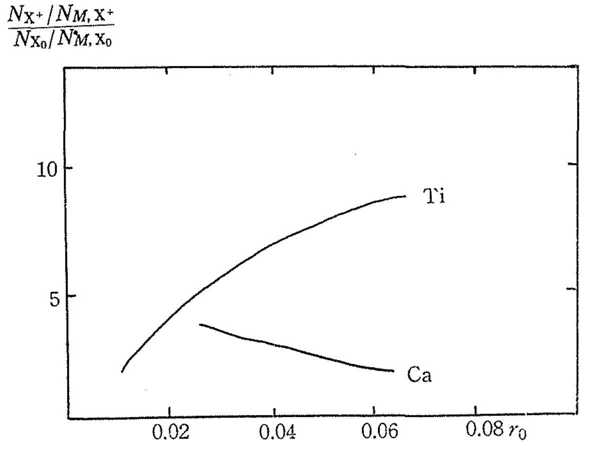


Fig. 15b.

consider the ionization in the M star's atmosphere due to radiation from B star. The ionization formula can be expressed by

$$\frac{N_{x^+}}{N_{x_0}} n_e = 10^{15.38-\chi_{x_0}\theta} T_B T_\varepsilon^{1/2} W, \tag{5.3}$$

and we can estimate N_{x^+}/N_{x_0} with n_e as a parameter, taking values of $T_B = 12000°K$ and $W = 2\times10^{-6}$. If we assume that the electron density along the ray from the B star is equal to that above the photosphere of M star, (5.2) is much larger than (5.3), and the radiation from B star has dominant role in the ionization of metal. The degree of ionization in the column along the ray derived from (5.1) is obviously much lower than that derived from B star's radiation. From these differences we can deduce the packing fraction or its reciprocal α in the atmosphere as derived in §3. The results are shown in Fig. 16. But we do not want to proceed to further discussions, because of the lack of quantitatively accurate data.

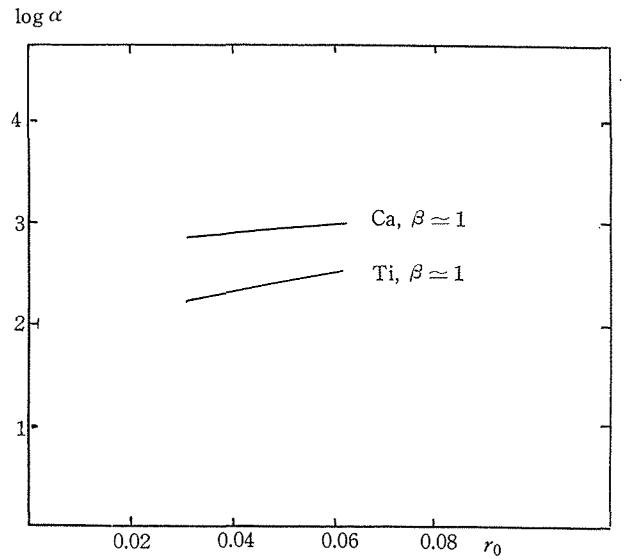


Fig. 16. VV Cephei: log α *versus* height.

6. Treatment of r^{-m}-distribution of the matter in the atmosphere

In §§3, 4 and 5, our considerations are based on the assumption of the hydrostatic distribution of matter. As described in §2, recent observations show evidences of prominence-like motion of cloud which expands to the outer atmosphere or slow ejection of matter of late supergiant stars.

These circumstances suggest that we should treat the density distribution as of expanding form. As the simplest treatment for this problem we consider

r^{-m} density distribution and develop similar discussions as done in §§3 and 4 with respect to *Case II* of calcium at the upper atmosphere of Zeta Aurigae and 31 Cygni. For the case of this distribution we have already prepared the expressions for the number of atoms in the column from the secondary star during the atmospheric eclipse, (2.19), and critical central density, (2.33) in §2. To discuss the observational data we must start again from the discussion of the variation of total number in the column with height, i.e., Fig. 3 for Zeta Aurigae and Fig. 9 for 31 Cygni. Instead of γ for the case of exponential form, we must find the exponent m from those curves. For the case $m=2$, the continuity of matter holds. When m is greater than 2, the physical interpretation of flow will become complicated. However, in this section we shall determine m formally, and proceed our discussions.

We consider a column whose minimum distance from the center of the K star is r_0. (see Fig. 1.) The total number of atoms along the ray from the secondary is given by

$$N(r_0) = n_0 r_0 \sqrt{\pi}\, \Gamma\left(\frac{m-1}{2}\right) \Big/ \Gamma\left(\frac{m}{2}\right), \tag{6.1}$$

and the total number at the atmospheric height r is given by

$$N(r) = n_0 \left(\frac{r}{r_0}\right)^{-m} r \sqrt{\pi}\, \Gamma\left(\frac{m-1}{2}\right) \Big/ \Gamma\left(\frac{m}{2}\right). \tag{6.2}$$

Consequently we obtain

$$\frac{d \log N(r)}{dr} = \frac{-(m-1)}{r}, \tag{6.3}$$

and

$$\frac{d^2 \log N(r)}{dr^2} = \frac{m-1}{r^2}. \tag{6.4}$$

As long as $m>1$, (6.4) is always positive. Hence, curves of $\log N$ *versus* h are concave upward, while curves for the case of exponential distribution are straight line with single gradient. As the observed curve has upward concave nature, the assumption of r^{-m}-distribution law seems favorite to the case. But, in general observed curve cannot be expressed with single value of m-exponent. Now, if we assume that single value of m-exponent can be applied to fit the curve between r_0 and r, then from (6.1) and (6.2) we obtain

$$\frac{N(r_0)}{N(r)} = \left(\frac{r}{r_0}\right)^{m-1},$$

and therefore

$$m = \frac{\log N(r_0) - \log N(r)}{\log (r/r_0)} + 1. \tag{6.5}$$

With the observed values of $N(r_0)$ and $N(r)$, we can determine m which describes the decrease of density. We apply this interpretation to upper atmosphere of $1.2\sim1.35\,r_0$. They correspond to the atmospheric heights of 28×10^6 km$\sim48\times10^6$ km for Zeta Aurigae, and 24×10^6 km$\sim41\times10^6$ km for 31 Cygni.

Using the data in Figs. 3 and 9, and the expression (6.5), we find $m=13$ for Zeta Aurigae and $m=3$ for 31 Cygni. With these values of m, we repeat discussions as done in §§3 and 4. In this case, s_s and n_c defined as uniformly smoothed distance and density in §3 should be replaced by

$$s_s = r_c\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)\Big/\Gamma\left(\frac{m}{2}\right), \tag{6.6}$$

and

$$n_c = N_{\text{total}}\Big/\left\{r_c\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)\Big/\Gamma\left(\frac{m}{2}\right)\right\}, \tag{6.7}$$

respectively.

We divide, as done in §3, the atmosphere into two parts, namely condensed and ground regions and derive the reciprocal of packing fraction α with normal K star's temperature for *Case II* with (3.29)$\sim$(3.33). Results are presented in Fig. 17. In this case also α increases towards the upper atmosphere. As done in §3.5, in this section again we calculate the electron temperature which can explain the observed degree of ionization with constant α, by (3.63)$\sim$(3.74). For the expression of distance s_s and density n_c, (6.6) and (6.7) are applied

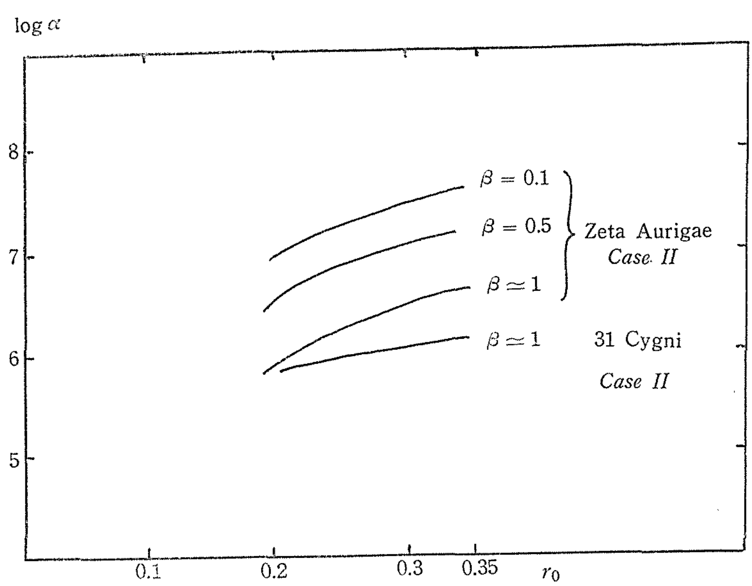


Fig. 17. $\log\alpha$ *versus* height for the case of r^{-m}-distribution.

instead of (3. 3) and (2. 14). Finally, we get the expression of T_ε *versus* h, namely Ψ', which should be compared with (3. 79). Results are shown in Fig. 18 with r instead of h.

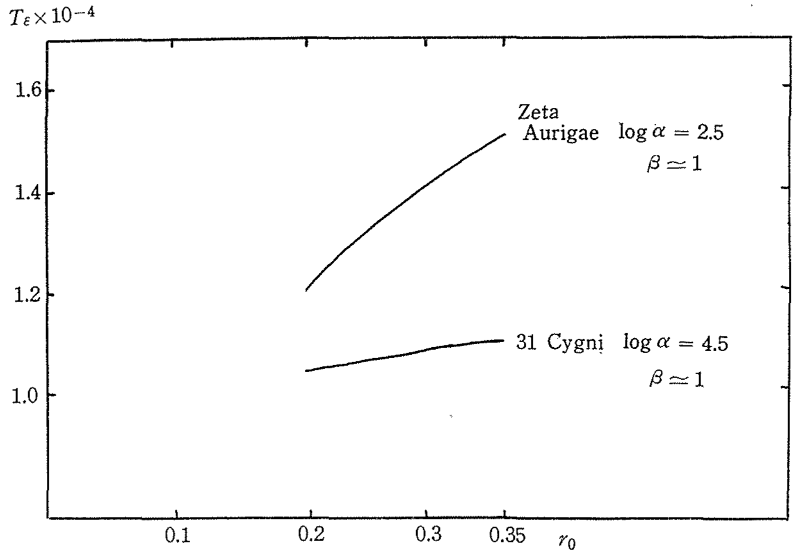


Fig. 18. T_ε *versus* height for the case of r^{-m}-distribution.

7. Summary and discussions

In the previous sections, we obtained the reciprocal of packing fraction, α, with the assumption of Wilson and Abt's condensation structure that divides the atmosphere into condensed and ground regions. From our results, for example in the *Case II* of calcium for $\beta \simeq 1$, we find that for Zeta Aurigae $\log \alpha$ ranges from 3 to 5 in the lower atmosphere and from 5 to 6.5 in the upper, and for 31 Cygni, $\log \alpha$ ranges from 4.5 to 5.5 throughout the atmosphere. For VV Cephei $\log \alpha$ for calcium and titanium is of the order of 3~4 and shows slight increase in the upper atmosphere. In the case of Zeta Aurigae we used the degree of ionization at the upper atmosphere as constant value throughout the atmosphere, and when we take account of the anomalous depression of calcium ionization in the lower atmosphere, the increment of $\log \alpha$ from the lower to upper atmosphere is reduced by the order of 2. But taking account of Groth's data the drop of degree of ionization in the lower part is not so conspicuous, and hence our increment of $\log \alpha$ seems to have sound value. Next, we derived electron temperature distribution with height that agreed with observed degree of ionization by the geneneral ionization formula with local condensations and high electron temperature, under the several assumptions described in §3. The results depend on the assumed values of α and β and the trend of temperature increase of curves differs appreciably from one another. Hence, we cannot say definite solution, but a crude estimation of our curves shows that electron temperature may change from 4000°K~8000°K in the lower level to 12000°K~16000°K in the upper in the case of Zeta Aurigae, and from 4000°K~8000°K in the lower level to 10000°K~12000°K in the upper in the case of 31 Cygni. The results of 31 Cygni show larger scatter than those of Zeta Aurigae owing to the variety of considering element.

It goes without saying that the curves corresponding to the smaller values of α lie above those corresponding to greater values of α. Consequently, we can imagine a family of solutions as a family of trajectories on which the value of α decreases as the value of h increases.

We prefer these trajectories as the solution to those which are given by constant values of α, since the former seems more persuasive from the dynamical point of view. But, choice of practical solution for the given conditions implies further results of observations.

The value of α cannot be smaller than that in Figs. 7a, 7b or 13, unless the value of fixed degree of ionization applied to Eq. (3.78) is altered. In other words, calculation gives necessarily greater values to the degree of ionization than observation does, so far as the value of α is smaller than that given in the figure. Owing to the scatter of observed data the values of observed degree of ionization have some range of error. However, for the smaller α higher value of electron temperature is required. For higher electron temperature than that of maximum value we obtained, spectral lines of higher order ions could be observed. The upper limit of electron temperature should be determined. Therefore, suitable value of $\log\alpha$ for Zeta Aurigae is 2.5~4 for calcium and 3.5~5 for iron. For 31 Cygni, $\log\alpha$ is 2~5 if we take account of the scatter of observational data. Comparing those values of α with those for $T_e=T_\varepsilon$, higher electron temperature hypothesis can reduce α by a order of 10^2~10^4. With respect to the ionization of CaII to CaIII, we have treated two cases. For Zeta Aurigae, Wilson and Abt showed that if the atmosphere was transparent for CaII ionizing radiation, namely in *Case I*, the abundance of each element increased remarkably, so that they preferred *Case II* in which the ionization was suppressed by Lyman Beta absorption. For 31 Cygni Wright considered only *Case II*. If the Lyman Beta absorption suppresses the CaII ionization, then $N(\mathrm{H})$ satisfying $\tau_{\mathrm{CaII}}\simeq 1$ is about 10^{23}. $N(\mathrm{H})$ should not be affected largely by our high electron temperature hypothesis. Owing to the scatter of $N(\mathrm{H})$ we cannot derive any decisive conclusion about Lyman Beta absorption. Let us examine the abundance of element for both cases. Comparison between *Case I* and *Case II* is presented in Table 2 for both stars when $T_\varepsilon=4000°K$. Even the assumption of high electron temperature cannot remove off such quantitative differences between *Case I* and *Case II*. Particularly for 31 Cygni, n_e and n_H as derived from *Case I* are increased by a factor of 10^3 and present conspicuous difference from the results for other metals. Furthermore, for Zeta Aurigae, as discussed in §3.3, degree of ionization for iron for *Case I* much exceeds that of calcium, though ionization potential of iron is higher than calcium. Particle density in the K-type supergiant atmosphere is still unknown exactly, but the abundance derived from *Case I* seems too large and hence it will be reasonable that we prefer *Case II* to *Case I*.

Table 2. Comparison between *Case I* and *Case II*.

	Zeta Aurigae				31 Cygni			
	$0.057\ r_0$		$0.34\ r_0$		$0.057\ r_0$		$0.034\ r_0$	
	$\log n_e$	$\log n_H$	$\log n_e$	$\log n_H$	$\log n_e$	$\log n_H$	$\log n_e$	$\log n_H$
Case I	9.4	13.9	7.8	11.6	9.5	13.3	8.9	12.7
Case II	7.2	10.9	4.1	7.8	6.4	10.3	5.3	9.1

We obtained smaller value of α with the assumption of high electron temperature, but even our values of α seem still too large.

During the eclipse of Zeta Aurigae-type stars prominence-like motions of CaII clouds are observed, and it is said that those clouds may correspond to the condensations originally proposed by Wilson and Abt. Up to the present, the most detailed analysis of calcium cloud was done for 31 Cygni by McKellar *et al.* (**6**). Let us discuss briefly this star, though the atmospheric region in which satellite lines are observed is higher than that discussed in §4. Following their observations, examples of CaII satellite line produced in the outer atmosphere accompanied by principal line are cited in Table 3. Total number of CaII along the line of

Table 3. Equivalent width and number of CaII atoms in the line of sight for double *K*-lines. (Reproduced from McKellar *et al.*'s Table)

Phase	Equivalent width		Log No. Absorbing atoms	
	Violet component	Red component	Violet component	Red component
40.6 days	0.19 A	0.15 A	12.42	12.25
41.6	0.12	0.09	12.14	12.02
71.6	0.09	0.07	12.02	11.95
87.6	0.12	0.04	12.14	11.79

sight estimated with main line is shown Fig. 19. According to their estimations, discrete cloud of CaII of dimension of 2×10^7 km can exist. We can find from Fig. 19 the density gradient in the outer atmosphere, which is $7.2\times10^{-14}\ \mathrm{cm}^{-1}$

between $h=14\times10^7$ km and 33×10^7 km. For example, we take tentatively calcium cloud at phase of 40.6 days. If we assume that red component corresponds to moving cloud as in the inner atmosphere, then we can find from Table 3 the value of β defined in §3, i.e. $\beta\simeq0.4$.

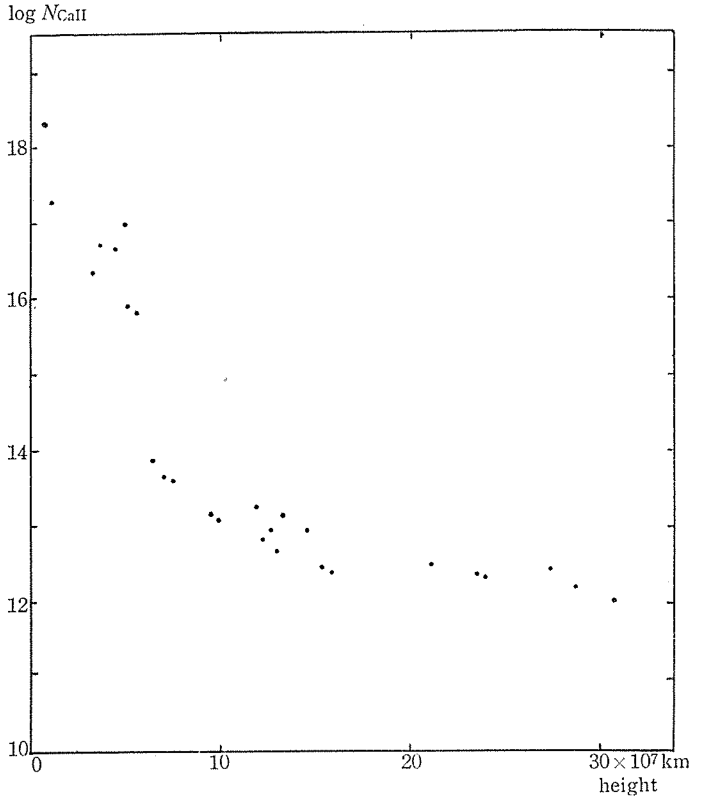


Fig. 19. 31 Cygni: CaII distribution in the outer atmosphere. (Plotted by McKellar *et al.*'s Table)

Applying the density gradient γ obtained from Fig. 19, we find that s_s at this atmospheric height is 4.7×10^{13} cm and hence we have $n_{c,\,\rm Ca}=0.09$. If we assume that the constant degree of ionization in the inner atmosphere obtained by Wright is continued to this layer, and that T_ε is the normal effective temperature, we can find that α becomes $10^{5.4}$ for *Case I* and $10^{9.5}$ for *Case II*. On the other hand, the linear dimension has been estimated to be of the order of 20×10^6 km. Even if we assume that only one cloud of this scale exists, its density is $n_{\rm cloud}=0.9$. Consequently α is only of the order of ten. Neither increase of degree of ionization in the outer atmosphere nor that of electron temperature can remove off this discrepancy of α. Hence it seems difficult to identify the condensation which has been assumed to do some role in the interpretation of the metallic ionization with calcium cloud observed in the outer atmosphere of K star. These difficulties will remain also in the case of r^{-m}-distribution of matter. In his discussion of matter ejection from red supergiant star as observed in Alpha Herculis, Wilson (**19**) considered the radiation pressure of Lyman Alpha and suggested that the non-uniformity of the effect of radiation pressure would produce the fluctuation of density in the upper atmosphere, i.e., condensation. As well as the presence of condensation, mechanism of its formation is still remained unknown.

We derived high electron temperature in the atmosphere with several assumptions. Atmospheric height we are considering ranges from the base to 0.35 r_0. If we compare our results with the temperature increase from solar chromosphere to corona, temperature increase of our case seems not so absurd. On the analogy with the solar corona, if we assume the presence of granulation and the production of a stream of acoustic noise from granulation, the heating of the upper atmosphere would be explained. However, because of the lack of detailed data near the photospheric base, we do not proceed to further discussions.

Let us consider the temperature derived by the method of scale height. Of course, the atmosphere of red supergiant star cannot be considered to be in hydrostatic equilibrium, but as a crude approximation we may apply formally to our case in the lower atmosphere. The density gradient of ionized calcium is $1.01\times10^{-11}\,\mathrm{cm}^{-1}$ for Zeta Aurigae (**1**) and $1.3\times10^{-12}\,\mathrm{cm}^{-1}$ for 31 Cygni (**2**). As turbulent velocities, V_t, we take the upper limits of observed values, namely, 10 km/sec for Zeta Aurigae and 20 km/sec for 31 Cygni. Applying the usual formula:

$$\gamma = \frac{mg}{kT+mV_t^2/3}, \tag{7.1}$$

we find $T=5\times10^5\,{}^\circ\mathrm{K}$ for Zeta Aurigae and $T=5\times10^6\,{}^\circ\mathrm{K}$ for 31 Cygni. They are very high values and are of the order of the temperature of solar corona. γ decreases at the upper atmosphere for both stars, and these temperatures increase much. Of course this is formal discussion, giving only the upper limits of temperature.

In §6 we have considered r^{-m}-distribution of matter and discussed the state of upper atmosphere with respect to calcium. Results show that for Wilson and Abt's model of *Case II*, α ranges from 10^6 to $10^{7.5}$ for Zeta Aurigae and from $10^{5.8}$ to $10^{6.1}$ for 31 Cygni. These values are slightly higher than that for hydrostatic case. For the assumptions to reduce α, the results with the assumption of high electron temperature are shown in Fig. 18 for both stars. Comparing these results with those in §§3 and 4, we find that there is no remarkable difference between the two forms of density distribution.

Acknowledgments

The author wishes to express his hearty thanks to Prof. S. Miyamoto, Kyoto University, for suggesting the present problem as well as for his constant guidance and encouragement during the course of this work. He also wishes to thank particularly Prof. S. Ueno, Dr. M. Kanno and Dr. T. Kogure, Kyoto University, and Dr. Y. Inoue, National Defence Agengy, for their valuable discussions and encouragements. Grateful thanks are also due to Dr. S. Saito, Kyoto University, for his reading, and commenting on, the manuscript.

REFERENCES

1. O. C. WILSON and H. ABT: Ap. J., Suppl. No. 1 (1954).
2. K. O. WRIGHT: Puls. Dominion Astrophys. Obs. Victoria, **11** (1959), 77.
3. A. MCKELLAR, L. H. ALLER, G. J. ODGERS and D. B. MCLAUGHLIN: Nature, **169** (1952), 990.
4. H. G. GROTH: Zs. f. Ap., **43** (1957), 185.
5. A. J. DEUTSCH: Ap. J., **123** (1956), 210.
6. A. MCKELLAR, L. H. ALLER, G. J. ODGERS and E. H. RICHARDSON: Publs. Dominion Astrophys. Obs. Victoria, **11** (1959), 35.
7. A. MCKELLAR and R. M. PETRI: M. N. **112** (1952), 641.
8. D. B. MCLAUGHLIN, Ap. J., **103** (1946), 35.
9. S. KAWABATA: Publ. Astron. Soc. Japan, **9** (1957), 72.
10. C. P. GAPOSCHKIN and S. GAPOSCHKIN: Ap. J., **101** (1945), 56.
11. A. P. LINNEL: Ap. J., **127** (1958), 211.
12. B. STRÖMGREN, G. P. KUIPER and O. STRUVE: Ap. J., **86** (1937), 588.
13. W. H. CHRISTIE and O. C. WILSON: *Ap. J.*, **81** (1935), 426.
14. F. E. ROACH and F. B. WOOD: Ann. d'Ap., **15** (1952), 21.
15. A. UNSÖLD: *Physik der Sternatmosphären* (Berlin, 1955), XXI Kapitel, § 143, F.
16. G. ELWERT: Z. Naturforsch., **7a** (1952), 202.
17. D. B. MCLAUGHLIN: Publs. Astron. Soc. Pacific, **62** (1950), 13.
18. V. GOEDICKE: Publs. Obs. Univ. Michigan, 8 (1943), 1.
19. O. C. WILSON: Ap. J., **131** (1960), 75.