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ON THE TURBULENT DIFFUSION IN THE ATMOSPHERE

BY

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ABSTRACT

In Part I, a generalized theory of turbulent diffusion is developed by a modification of Inoue's turbulon model theory, and it is shown, with some experimental verifications, that the results in a reasonably assumed case may be applied to certain problems of turbulent diffusion.

In Part II, a new governing equation for turbulent diffusion in the atmospheric surface layer with shear velocity is derived from the viewpoint of mixing-length concept, and the solutions to an important two-dimensional problem are given for large and small dispersion times respectively, with some experimental verifications.

1. Introduction

The theoretical investigation of turbulent diffusion based on pure dynamical system has been on the rocks owing to its complexity, and especially in the atmosphere the situation becomes almost intractable because of the enormous range in the scale of turbulence and the frequent co-existence of natural convection. So far, two lines of attack have been developed for the understanding of the tubulent diffusion; one is the "statistical theory of turbulence" pioneered by G. I. Taylor (**1**), and the other is the so-called K-theory.

Taylor's statistical theory can predict universal formulae for the mean dispersion and the rate of its change with respect to time without making any assumption concerning the mechanism of turbulence, but in order to determine the values of these quantities it is necessary to know the form of Lagrangian correlation function $R(\xi)$, the correlation between the velocity of a particle at any instant and that of the same particle after a time-interval of ξ. It is an important subject, therefore, to find the form of the correlation function appropriate for the atmospheric turbulent diffusion, and Part I of the present paper deals with this subject.

In K-theory the flux of diffusing matter (or property) across a fixed surface is assumed to be proportional to the mean concentration gradient of diffusing

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matter along the normal in the direction of the flux. This hypothesis obviously comes from the analogy with the kinetic theory of gases. The treatment of K-theory with a simple model in the field of atmospheric turbulent motion utilizing the mixing-length idea was first undertaken by G. I. Taylor (**2**). In Part II, the fundamental equation of two-dimensional diffusion in the lower atmosphere with shear velocity is derived from the viewpoint of mixing-length model in K-theory. Referring to the result obtained in Part I, the fundamental equation is extended to the case of small dispersion time, and some problems concerning the distribution of conservative concentration in the lower atmosphere are treated with more or less success.

PART I. MODIFIED THEORY OF TURBULON MODEL OF ATMOSPHERIC TURBULENCE

2. Historical note

For the case of homogeneous turbulence with zero mean velocity, Taylor (**1**) derived the very important formulae:

$$\frac{1}{2}\frac{d\overline{y^2}}{d\xi} = \overline{v^2}\int_0^\xi R(\xi')\,d\xi' , \tag{1}$$

and

$$\overline{y^2} = 2\overline{v^2}\int_0^\xi\int_0^{\xi'} R(\xi'')\,d\xi''d\xi' , \tag{2}$$

where bars imply mean values, v is the velocity component in the y-direction and $R(\xi)$ is the correlation coefficient between the value of v of a particle at any instant (t) and that of the same particle after the time-interval of ξ, i. e.,

$$R(\xi) = \frac{\overline{v(t)\,v(t+\xi)}}{\sqrt{\overline{v^2(t)}}\sqrt{\overline{v^2(t+\xi)}}} = \frac{\overline{v(t)\,v(t+\xi)}}{\overline{v^2}} . \tag{3}$$

Here we have assumed the relations:

$$\overline{v^2(t)} = \overline{v^2(t+\xi)} = \overline{v^2} ,$$

and

$$R(-\xi) = R(\xi) .$$

The present author considers t to be an instant at which any particle starts from the origin, and that for practical purpose bar notation implies the time mean with respect to starting time t, which, strictly speaking, requires some deep reconsideration of the ergodic hypothesis. In the field where $\overline{v^2}$ is defined, the problem of turbulent diffusion may be settled by the very determination of the form of $R(\xi)$. Various investigators have thus tried to determine the form of $R(\xi)$. Most of them, however, have determined it without inquiring into the detailed mechanism of turbulence, but have taken it rather hypothetically or as an assumption. For reference, the forms of $R(\xi)$ used by various authors are tabulated in Table 1 below.

Table 1.

Form of $R(\xi)$	Authors
$\exp(-\xi/A)$	G. I. Taylor (**1**), H. Gebelein (**3**), F. N. Frenkiel (**4**), J. Sakagami (**5**)
$\left(\frac{a}{\overline{v^2}\xi}\right)^n$, $\left(\frac{\nu}{\nu+\overline{v^2}\xi}\right)^n$, $\left(\frac{\nu+N}{\nu+N+\overline{v^2}\xi}\right)^n$	O. G. Sutton (**6, 7, 8, 9, 10, 11**)
$\exp(-\xi^2/A)$	G. Dedeband and P. Wehlré (**12**), F. N. Frenkiel (**4**)

The experimental approach to the determination of $R(\xi)$ by utilizing the definition of $R(\xi)$ or the relation:

$$R(\xi) = \frac{1}{2\overline{v^2}} \frac{d^2\overline{y^2}}{d\xi^2}, \tag{4}$$

which follows from (1), has recently been started by such investigators as Edinger (**13**) and Barad (**14**). But, it is very difficult to put this method in practice apart from laboratory works. On the basis that the turbulent energy spectrum function in terms of frequency may be closely related to $R(\xi)$ in the Lagrangian time-space, just as the Eulerian correlation function $R(t)$ is related to the turbulent energy spectrum function in terms of frequency in the Eulerian time-space (Taylor (**15**)), Gifford (**16**) attempted another experimental approach to determining the kinetic enegy spectrum in the Lagrangian time-space. But, all of the experimental results above mentioned do not appear to lead to the conclusive form of $R(\xi)$.

Inoue's investigations (**17, 18, 19**) attract the author in that he takes into account the mechanism of turbulence theoretically and proves the possibility of further development. He leans largely upon the modern statistical theory of locally homogeneous and isotropic turbulence initiated by Kolmogorov (**20, 21**) and developed, in more or less different ways, by other authors such as Obkhov (**22**), Onsager (**23**), Weizsäcker (**24**), Heisenberg (**25, 26**) and von Kármán (**27**). Inoue's basic assumption is that the turbulence consists of disintegrating turbulons of various ranks and turbulons of any rank conserve their energy during their proper life-time. In his theory the expression for $R(\xi)$ is derived as follows.

In statistically (locally) homogeneous, isotropic and stationary turbulent field, velocity component $v(t)$ for any one materialized point at any time t in the y-direction, which is perpendicular to the x-axis taken along the steady uniform mean flow $(\bar{u})$, can be represented by

$$v(t) = v_{\infty}(t) + \cdots + v_n(t) + \cdots + v_0(t) , \tag{5}$$

where $v_n(t)$ is the y-component of the proper velocity of turbulon of rank n which surrounds the materialized point under consideration. Then, the velocity component at an interval of time ξ later is given by

$$v(t+\xi) = v_{\infty}(t+\xi) + \cdots + v_n(t+\xi) + \cdots + v_0(t+\xi) . \tag{6}$$

By the presumption of the individuality of every rank of turbulon, the Lagrangian correlation function $R(\xi)$ defined by (3) can be represented as follows:

$$R(\xi) = R_{\infty}(\xi) \frac{\overline{v_{\infty}^2}}{\overline{v^2}} + \cdots + R_n(\xi) \frac{\overline{v_n^2}}{\overline{v^2}} + \cdots + R_0(\xi) \frac{\overline{v_0^2}}{\overline{v^2}} . \tag{7}$$

The assumption of the conservation of the original proper velocity of the disintegrating turbulons during the proper life-time τ_n led Inoue to conclude that

$$R_n(\xi) = \begin{cases} 1 & (\xi \leqq \tau_n) , \\ 0 & (\xi \geqq \tau_n) , \end{cases} \tag{8}$$

and under the assumption of uniform distribution of kinetic energy of turbulons such that

$$F(\tau) = \begin{cases} 0 & (\tau < \tau_{\infty}) , \\ 1/(\tau_0 - \tau_{\infty}) & (\tau_{\infty} \leqq \tau \leqq \tau_0) , \\ 0 & (\tau > \tau_0) , \end{cases} \tag{9}$$

he derived

$$R(\xi) = \begin{cases} 1 & (\xi \leqq \tau_{\infty}) , \\ 1 - (\xi - \tau_{\infty})/(\tau_0 - \tau_{\infty}) & (\tau_{\infty} \leqq \xi \leqq \tau_0) , \\ 0 & (\xi > \tau_0) , \end{cases} \tag{10}$$

or, in the case of $\tau_{\infty} \ll \tau_0$, approximately,

$$R(\xi) = \begin{cases} 1 - (\xi/\tau_0) & (0 \leqq \xi \leqq \tau_0) , \\ 0 & (\xi \geqq \tau_0) . \end{cases} \tag{10'}$$

3. Generalization of Inoue's assumption of deriving $R(\xi)$

Inoue supposed that the material point considered at time t was surrounded only by the fresh turbulons having full life-time, but it seems to the present author that there is no reason to consider all of the turbulons surrounding the material point to be fresh; it is more reasonable to assume that some of them are old and accordingly have no full original life-time. Then, the turbulons of rank n surrounding the point considered at t may be classified into two groups by the remaining life-time "e" or by the age of the turbulons $\tau_n - e$. For the turbulons of $e > \xi$, we have $R_n(\xi) = 1$, while for the turbulons of $e \leqq \xi$ we have $R_n(\xi) = 0$. If we put down the ratio of the number of turbulons belonging to the first group to that of the second group as $\nu_n(\xi) : \mu_n(\xi)$ (where $\nu_n(\xi) + \mu_n(\xi) = 1$), the correlation function $R_n(\xi)$ becomes

$$R_n(\xi) = \begin{cases} \nu_n(\xi) & (0 \leqq \xi \leqq \tau_n), \\ 0 & (\xi \geqq \tau_n). \end{cases} \tag{11}$$

Since the generalized representation of (7) is expressed by

$$R(\xi) = \int_0^{\tau_0} R_\tau(\xi)\, dG(\tau), \tag{12}$$

where $R_\tau(\xi)$ is the Lagrangian correlation function of the turbulons of life-time τ and $G(\tau)$ is the distribution of the energy of turbulons with respect to the same life-time such that $dG(\tau)/d\tau = F(\tau)$ when $F(\tau)$ is a continuous function and the integration on the right-hand side is taken in the sense of Stieltjes, we get, from (11) and (12),

$$R(\xi) = \int_0^{\tau_0} \nu_\tau(\xi)\, dG(\tau). \tag{13}$$

To settle the form of $\nu_n(\xi)$ is a problem of experimental physics. Under the present state where no experimental knowledge is obtained, equal appearance of the chance may be reasonably assumed for the turbulons of different ages. Then we have $\nu_n(\xi) = 1 - (\xi/\tau_n)$, $\mu_n(\xi) = \xi/\tau_n$; thus we have

$$R_n(\xi) = \begin{cases} 1 - (\xi/\tau_n) & (0 \leqq \xi \leqq \tau_n), \\ 0 & (\xi \geqq \tau_n). \end{cases} \tag{14}$$

Moreover, if we assume with some reason (Inoue (**17**)) such a form of $F(\tau)$ as the form (9) or approximately, in the case of $\tau_\infty \ll \tau_0$,

$$F(\tau) = \begin{cases} 1/\tau_0 & (\tau \leqq \tau_0), \\ 0 & (\tau > \tau_0), \end{cases} \tag{15}$$

we get respectively

$$R(\xi) = \begin{cases} 1 + \dfrac{\xi}{\tau_0 - \tau_\infty} \log \dfrac{\tau_\infty}{\tau_0} & (0 \leqq \xi \leqq \tau_\infty), \\ \dfrac{\tau_0}{\tau_0 - \tau_\infty} - \dfrac{\xi}{\tau_0 - \tau_\infty} + \dfrac{\xi}{\tau_0 - \tau_\infty} \log \dfrac{\xi}{\tau_0} & (\tau_\infty \leqq \xi \leqq \tau_0), \\ 0 & (\xi > t_0), \end{cases} \tag{16}$$

or approximately, in the case of $\tau_\infty \ll \tau_0$,

$$R(\xi) = \begin{cases} 1-\dfrac{\xi}{\tau_0}+\dfrac{\xi}{\tau_0}\log\dfrac{\xi}{\tau_0} & (0 \leqq \xi \leqq \tau_0)\,, \\ 0 & (\xi \geqq \tau_0)\,. \end{cases} \tag{17}$$

This form of $R(\xi)$ is in perfect agreement with that of Grant (28) derived in an apparently different, but essentially the same, way of thinking.

It is of interest that if we take $A=\tau_0/4$ to give the same Lagrangian time scale $\int_0^\infty R(\xi)\,d\xi$, our expression of $R(\xi)$ yields a very good numerical agreement with the familiar expression $R(\xi)=\exp(-\xi/A)$. These are illustrated in Fig. 1.

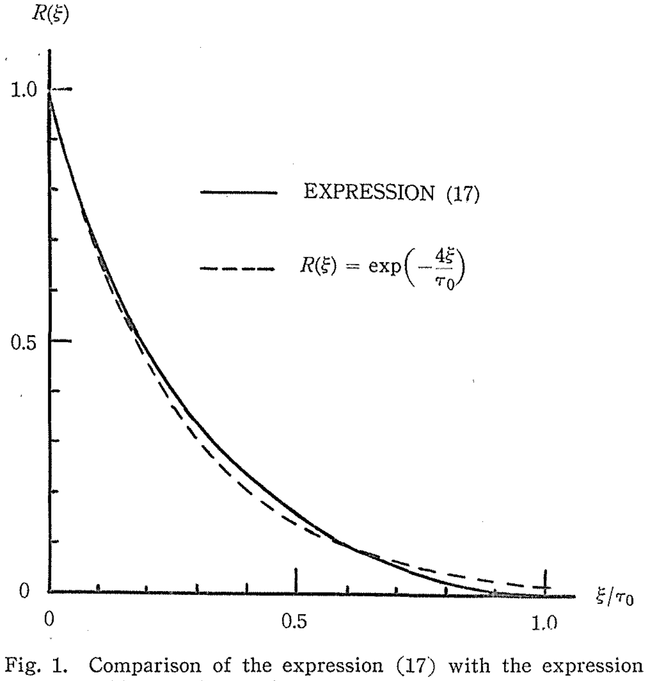

Fig. 1. Comparison of the expression (17) with the expression $R(\xi)=\exp(-4\xi/\tau_0)$.

Inoue's form of $R(\xi)$, i.e., (10′) is a limiting case of $\nu_n \rightarrow 1$, and Grant-Matsuoka's is also a special case corresponding to $\nu_n(\xi)=1-(\xi/\tau_n)$. Further experimental evidences might lead to other form or forms of $\nu_n(\xi)$. The form of $F(\tau)$ may also have room for modification, because the form (9) or the approximate form (15) is derived by a consideration of the range of life-times corresponding to the so-called "inertial subrange" in the modern statistical theory of turbulence, and the τ_0 must be taken as a virtual life-time of the equivalently effective largest turbulon in the case when the real spectrum is replaced by the form (15).

It may be added that the investigation of the general mathematical character of $R(\xi)$ is also important. For instance, the fact that, if $R''(\xi)$ exists for $\xi>0$, the relation (14) and any non-negative continuous spectral function $F(\tau)$ lead, as is easily seen, to $R''(\xi)\geqq 0(\xi>0)$—a fact which was proved from another approach by Kac (**29**, p. 149).

4. Horizontal diffusion from a continuous point-source of fixed type in the atmosphere with no shear velocity

By substituting the form $R(\xi)$ expressed by (17) into Taylor's formulae (1) and (2), we get

$$K(\xi)\equiv\frac{1}{2}\frac{d\overline{y^2}}{d\xi}=\begin{cases}\overline{v^2}\xi\left(1-\frac{3}{4}\frac{\xi}{\tau_0}+\frac{\xi}{2\tau_0}\log\frac{\xi}{\tau_0}\right) & (\xi\leqq\tau_0)\,,\\ \frac{\overline{v^2}\tau_0}{4} & (\xi\geqq\tau_0)\,,\end{cases}\tag{18}$$

and

$$\overline{y^2}(\xi)=\begin{cases}\overline{v^2}\xi^2\left(1-\frac{11}{18}\frac{\xi}{\tau_0}+\frac{\xi}{3\tau_0}\log\frac{\xi}{\tau_0}\right) & (\xi\leqq\tau_0)\,,\\ \overline{v^2}\tau_0\left(\frac{\xi}{2}-\frac{\tau_0}{9}\right) & (\xi\geqq\tau_0)\,,\end{cases}\tag{19}$$

for the diffusion along the y-direction.

By using our formula (19), we can evaluate the lateral width of the cloud of diffusing matter downwind from an elevated continuous point-source. Here, the width is defined by the root mean square distance travelled by diffusing matter in the lateral (y-) direction at the distance (x) measured in the direction of steady and horizontally uniform motion of mean velocity from the source, and τ_0 and $\bar{u}$ are respectively the life-time of effective largest turbulon within the layer in which the diffusion occurs and the mean velocity at the height of the source. If we put

$$\bar{u}\tau_0=x_0,\quad \sqrt{\overline{v^2}}/\bar{u}=\sigma_A\,,\quad r=x/x_0\quad\text{and}\quad \bar{u}\xi=x\,,\tag{20}$$

we get

$$\overline{y^2}=\begin{cases}\sigma_A^2x_0^2r^2\left(1-\frac{11}{18}r+\frac{1}{3}r\log r\right) & (r\leqq 1)\,,\\ \sigma_A^2x_0^2\left(\frac{1}{2}r-\frac{1}{9}\right) & (r\geqq 1)\,,\end{cases}\tag{21}$$

or

$$\sqrt{\overline{y^2}}/(\sigma_Ax_0)=\begin{cases}r\left(1-\frac{11}{18}r+\frac{1}{3}r\log r\right)^{1/2} & (r\leqq 1)\,,\\ \left(\frac{1}{2}r-\frac{1}{9}\right)^{1/2} & (r\geqq 1)\,.\end{cases}\tag{22}$$

Allowing the approximation $\bar{u}\xi = x$, namely, the released particles of dispersion time ξ being always displaced by the distance x in the x-direction, and assuming that the vertical diffusion does not affect the probability distribution of the travelled particles in the horizontal layer as Sutton (**11**), Cramer (**30**), Inoue (**31**) and others put, we can verify our result by comparing it with some available experimental data (Cramer (**30**)). Cramer's data in near-neutral cases shown in his Figs. 8 (O'Neill daytime experiments) and 9 (O'Neill nighttime experiments) determined by the data of concentration measurements at 50 m, 100 m, 200 m, 400 m and 800 m downwind from the source point at 1.5 m height,

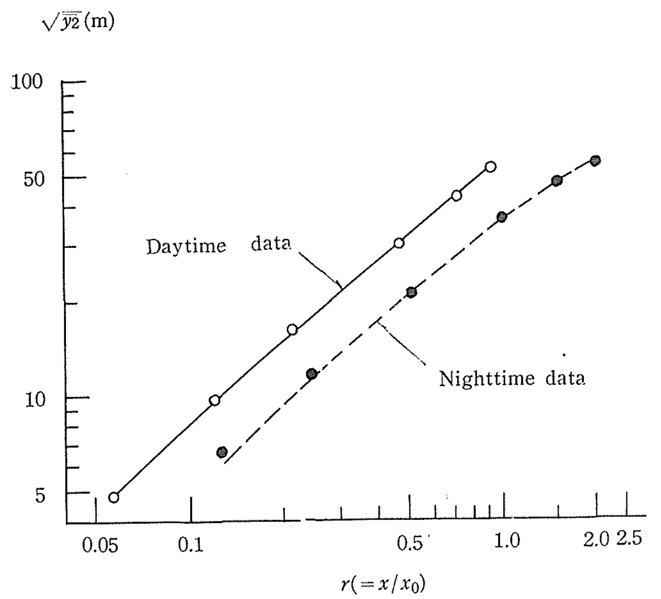

Fig. 2. Cramer's experimental curves of $\sqrt{\overline{y^2}}$ *versus* r in near neutral conditions at daytime and nighttime and the plotted theoretical values of $\sqrt{\overline{y^2}}$ (○ for daytime, ● for nighttime).

are reproduced in Fig. 2 to compare with our result (22). The values of τ_0 are determined by the method of trial and error, and the values of $\sqrt{\overline{y^2}}$ in our expression (22), corresponding respectively to $x=50$ m, $x=100$ m, $x=200$ m, $x=400$ m, $x=600$ m and $x=800$ m, are plotted in the same figure. The excellent fittness of our plotted points to Cramer's curves is indicated in Fig. 2. Here, from Cramer's data, we have taken $\bar{u}=5$ m/sec and $\sigma_A=6°$ for daytime and $\bar{u}=5$ m/sec and $\sigma_A=8°$ for nighttime, and, further, assumed $x_0=850$ m (i.e., $\tau_0=170$ sec) for daytime and $x_0=400$ m (i.e., $\tau_0=80$ sec) for the nighttime.

5. Two-dimensional diffusion of a cluster released instantaneously at a point-source in the atmosphere with no shear velocity

As we have treated in the previous section, Taylor's analysis is concerned only with the statistical behavior of a single particle and is generally applicable to the diffusion problem of continuous fixed source type. In this case, the concentration distribution does not give the instantaneous concentration in a diffusing cloud, implying that the turbulons of all ranks or the eddies of all sizes are effective at all dispersion times. On the other hand, in the case of the problem involving the relative spread of particles initially given in a fluid volume (e. g. a puff of smoke), we must treat the problem with careful distinction from the former problem. In the case of smoke puff, the relative diffusion is clearly controlled by the small scale eddies as long as the parcels are fairly close together. The basic idea of relative diffusion was first proposed by L. F. Richardson (**32**) and was later developed by Batchelor (**33, 34**). Richardson's idea that only the comparatively small scale turbulence components contribute mainly to the turbulent diffusion process while comparatively large scale turbulence components contribute mainly to the displacement of the cluster as a whole or of the center of the cluster mass when the growing cluster is of intermediate size, is introduced in a simplified form by Inoue (**19**), and by treating the horizontal diffusion of fixed source type he obtained a result similar to Batchelor's.

A presumption similar to Inoue's is adopted in the present study. Referring to the possibility of labeling the size of turbulons by their life-time (τ's) and that of the size of growing cluster by dispersion time (ξ), the author assumes that the turbulons with $\tau < k\xi$ effectively contribute only to the diffusion of cluster particles and the turbulons with $\tau \geqq k\xi$ effectively contribute only to the translation of the cluster as a whole, where k is a certain constant such that $0 < k < 1$.

It must be noticed that the above-mentioned assumption has no physical reality, but it may be valid in the convenient sense, and no serious error may be expected as far as we discuss the resultant statistical nature.

Let $y_i(\xi)$ be the y-coordinate of each particle of cluster at any dispersion time relative to the fixed coordinate system whose origin is the original source point, designating any one particle by suffix i, $y_{r_i}(\xi)$ the y-coordinate of any one particle designated by suffix i at that instant relative to the center of cluster, and $y_g(\xi)$ the y-coordinate of the center of the cluster at that time relative to the fixed coordinate system whose origin is the original source point, and then we have

$$y_i(\xi) = y_g(\xi) + y_{r_i}(\xi) ,$$

and, employing the commonly used bracket notation to represent the average values of the whole particles composing the cluster, i. e. $[Q]=\frac{1}{n}\sum_{i=1}^{n}Q_i$, where n is the total number of particles composing the cluster, we have

$$[y^2(\xi)] = y_g^2(\xi)+[y_r^2(\xi)], \tag{23}$$

by inserting the definition of the center of cluster or $[y_r]=0$.

The above equation (23) is valid for a cluster of single puff. Repeating the release of the such puff in sufficiently large number so as to be capable of estimating the mean value of each term of Eq. (23), making a single puff a sample, we have

$$[\overline{y^2}] = \overline{y_g^2}+[\overline{y_r^2}], \tag{24}$$

where bars mean the mean values of all samples. It is clear that $[\overline{y^2}(\xi)]$ is identical with $\overline{y^2}$ which is the mean square distance travelled by the trial particle at ξ from the fixed origin and identical with $\overline{y^2}$ represented by Eq. (19) in the previous section. By the assumed condition of the uniformity of the turbulence, we know that each $\overline{y_r^2}$ taken at any moving origin gives the same value independent of the location of the moving origin itself. Accordingly, we have

$$[\overline{y_r^2}] = \overline{y_r^2}. \tag{25}$$

Consequently, we have from (24)

$$\overline{y^2} = \overline{y_g^2}+\overline{y_r^2}, \tag{26}$$

and

$$\frac{1}{2}\frac{d\overline{y^2}}{d\xi} = \frac{1}{2}\frac{d\overline{y_g^2}}{d\xi}+\frac{1}{2}\frac{d\overline{y_r^2}}{d\xi}, \tag{27}$$

or

$$K(\xi) = K_g(\xi)+K_r(\xi).$$

The evaluation of each term of Eq. (27) can be carried out as follows.

(i) $K(\xi)$ is already given by (18), i. e.,

$$K(\xi) = \begin{cases} \overline{v^2}\xi\left(1-\frac{3}{4}\frac{\xi}{\tau_0}+\frac{\xi}{2\tau_0}\log\frac{\xi}{\tau_0}\right) & (\xi\leqq\tau_0), \\ \frac{\overline{v^2}\tau_0}{4} & (\xi\geqq\tau_0). \end{cases} \tag{28}$$

(ii) According to our assumption the effective turbulons for K_r at ξ are limited by $\tau\leqq k\xi$. Accordingly, by (12) and (15) we have when $k\xi\leqq\tau_0$

$$K_r(\xi) = \overline{v'^2}\int_0^{\xi}\int_0^{k\xi}\frac{1}{k\xi}R_\tau(\xi')\,d\tau\,d\xi', \tag{29}$$

where $\overline{v'^2}$ is the mean total kinetic enery of effective turbulons for K_r and by the same assumption as (15) we have $\overline{v'^2}=(k\xi/\tau_0)\overline{v^2}$. The evaluation of (29) can be made by putting $\overline{v'^2}$ and $k\xi$ in place of $\overline{v^2}$ and τ_0 in (28) respectively and taking the case of $\xi \geqq \tau_0$ in (28) since $\xi > k\xi$, we have

$$K_r(\xi) = \frac{\overline{v'^2}k\xi}{4} = \frac{\overline{v^2}k^2\xi^2}{4\tau_0}. \tag{30}$$

On the other hand, when $k\xi \geqq \tau_0$, it is easily seen that

$$K_r(\xi) = K(\xi) = \frac{\overline{v^2}\tau_0}{4}. \tag{31}$$

(iii) By the aid of the relation (27), i. e., $K_g = K - K_r$, the expression for $K_g(\xi)$ is obtained as follows:

$$K_g(\xi) = \begin{cases} \overline{v^2}\xi\left(1 - \frac{3+k^2}{4}\frac{\xi}{\tau_0} + \frac{\xi}{2\tau_0}\log\frac{\xi}{\tau_0}\right) & (\xi \leqq \tau_0)\,, \\ \frac{\overline{v^2}}{4\tau_0}(\tau - k^2\xi^2) & (\tau_0 \leqq \xi \leqq \tau_0/k)\,, \\ 0 & (\xi \geqq \tau_0/k)\,. \end{cases} \tag{32}$$

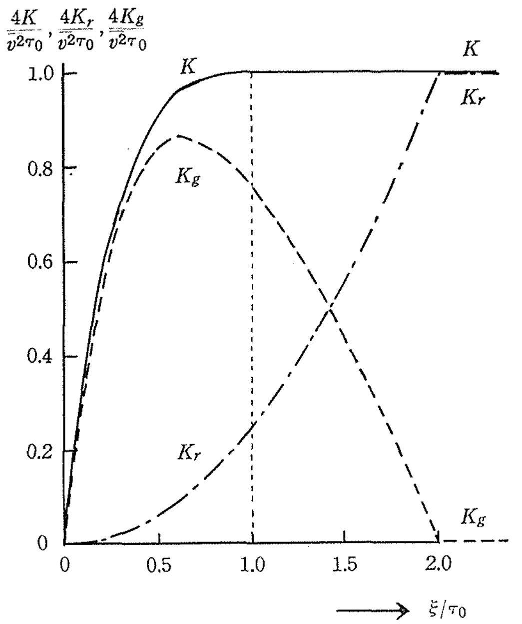

Fig. 3. Graphical illustration of $K(\xi)$, $K_r(\xi)$ and $K_g(\xi)$.

Next, the evaluation of each term of Eq. (26) can be carried out as follows: $\overline{y^2}(\xi)$ is given by (19), and integrating (30), (31) and (32), we have

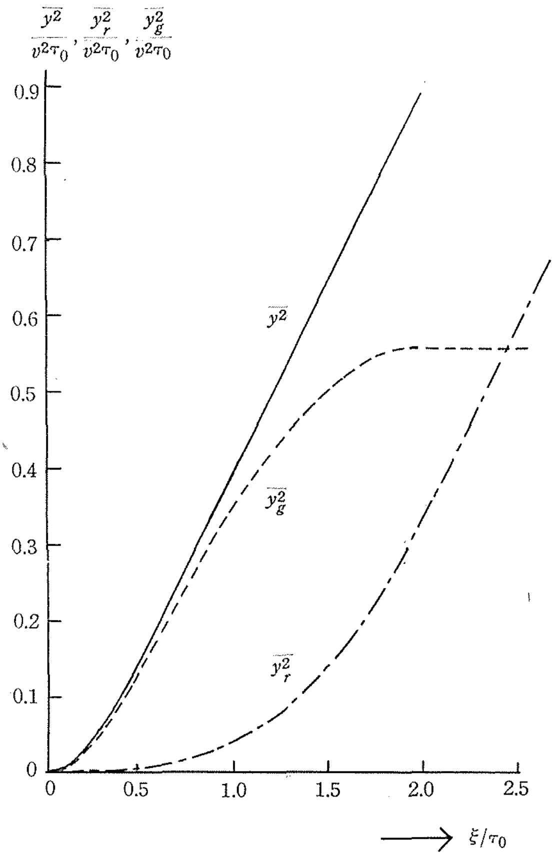

Fig. 4. Graphical illustration of $\overline{y^2}(\xi)$, $\overline{y_r^2}(\xi)$ and $\overline{y_g^2}(\xi)$.

$$\overline{y_r^2}(\xi) = \begin{cases} \dfrac{\overline{v^2}k^2\xi^3}{6\tau_0} & (\xi < \tau_0/k)\,, \\ \dfrac{\overline{v^2}\tau_0^2}{6}\left(3\dfrac{\xi}{\tau_0} - \dfrac{2}{k}\right) & (\xi \geqq \tau_0/k)\,, \end{cases} \tag{33}$$

and

$$\overline{y_g^2}(\xi) = \begin{cases} \dfrac{\overline{v^2}\tau_0^2}{18}\left(\dfrac{\xi}{\tau_0}\right)^2\left\{18 - (3k^2+11)\dfrac{\xi}{\tau_0} + 6\dfrac{\xi}{\tau_0}\log\dfrac{\xi}{\tau_0}\right\} & (\xi \leqq \tau_0)\,, \\ \dfrac{\overline{v^2}\tau_0^2}{18}\left\{9\dfrac{\xi}{\tau_0} - 2 - 3k^2\left(\dfrac{\xi}{\tau_0}\right)^3\right\} & (\tau_0 \leqq \xi \leqq \tau_0/k)\,, \\ \dfrac{\overline{v^2}\tau_0^2}{18}\left(\dfrac{6}{k} - 2\right) & (\xi \geqq \tau_0/k)\,. \end{cases} \tag{34}$$

Graphical illustrations of the above results are shown in Figs. 3 and 4, where k is taken to be 0.5 as an example.

Now, it is easily shown that the relations (30) and (33) lead to the famous

Richardson's law $K \propto L^{4/3}$ if it is allowed to regard $\sqrt{\overline{y_r^2}}$ as a representative length scale of the diffusion phenomenon; that is, eliminating ξ from (30) and (33), we have

$$K_r(\xi) = \frac{1}{4}\left(36\frac{k^2\overline{v^2}}{\tau_0}\right)^{1/3}\left(\sqrt{\overline{y_r^2}}\right)^{3/4}. \tag{35}$$

Next, the above relations may be compared with Batchelor's (**33**, **34**) given by

$$\frac{d\overline{y_r^2}}{d\xi} = c\varepsilon\xi^2 \quad \text{and} \quad \overline{y_r^2} = \frac{c}{3}\varepsilon\xi^3 .$$

These expressions are respectively identical with the expressions (30) and (33) if we put

$$c\varepsilon = \frac{\overline{v^2}k^2}{2\tau_0}. \tag{36}$$

The rough order of magnitude of $\overline{v^2}k^2/2\tau_0$ can be known by the experimental data. The resumé of the estimates of $c\varepsilon$ is given by Lettau (**35**).

PART II. TWO-DIMENSIONAL TURBULENT DIFFUSION IN THE LOWER ATMOSPHERE WITH SHEAR VELOCITY

6. Preliminary

The general three-dimensional equation of turbulent diffusion is usually written in the form:

$$\frac{d\overline{\chi}}{dt} = \frac{\partial}{\partial x}\left(K_x\frac{\partial\overline{\chi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial\overline{\chi}}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial\overline{\chi}}{\partial z}\right), \tag{37}$$

(e. g., Sutton (**11**), p. 273). If this equation is correct, it must be applicable also to the field of stationary homogeneous turbulence with steady zero mean flow as a special case. In such a field, Taylor's analysis can be extended to determine the general dispersion tensor, i. e.,

$$\overline{x_i x_j}(\xi)\,,^{*} \tag{38}$$

in place of the left-hand side of (2), and the left-hand side of (1) may be extended to

$$\frac{1}{2}\frac{d\overline{x_i x_j}}{d\xi}, \tag{39}$$

and, in this case, $\overline{v^2}R(\xi)$ in (1) and (2) must be replaced by

* Hereafter we will, for the sake of brevity, make use of x_1, x_2, x_3– coordinate system in place of x, y, z– coordinate system.

$$S_{ij}(\xi)\,, \tag{40}$$

where

$$\overline{x_i x_j}(\xi) = 2\int_0^{\xi}\int_0^{\xi'} S_{ij}(\xi'')\,d\xi''\,d\xi'\,,$$

$$\frac{1}{2}\frac{d\overline{x_i x_j}}{d\xi} = \int_0^{\xi} S_{ij}(\xi')\,d\xi'$$

$$S_{ij}(\xi) = \left\{\overline{v_i(t)\,v_j(t+\xi)} + \overline{v_i(t+\xi)\,v_j(t)}\right\}/2\,.$$

Such an analysis was carried out by Gebelein (**3**), Batchelor (**36**) and Corrsin (**37**), and such consideration was applied to the flow with shear by Corrsin (**37**, **38**) and Hinze (**39**).

The tensor (40) is not always of diagonal type and therefore the non-diagonal components of tensor (39) do not always vanish. This suggests that the diffusion equation (37) may be extended in some way. In fact, Ertel (**40**) introduced the generalized "Austauschtensor" with nine components. However, his derivation seems to be too formal. In the atmospheric shearing layer there seems to be some characteristic features in turbulent motion like horizontal shearing eddy stress. In the following sections, the derivation of such a diffusion equation and the application of the equation to certain two-dimensional diffnsion problems are given.

7. Derivation of the fundamental equation of turbulent diffusion in the atmosphere with shear velocity

The equation of conservative concentration is generally given by

$$\frac{\partial \chi}{\partial t} = -\nabla\cdot(\mathbf{V}\chi)\,,$$

where χ and $\mathbf{V}$ denote respectively the instantaneous values of concentration and velocity at a point in a turbulent flow field, and it is assumed that no sinks and sources exist.

Denoting the velocity component, in the x_1–, x_2– and x_3– directions by v_1, v_2 and v_3, we put

$$v_i = \overline{v_i} + v_i'\,,\quad \chi = \overline{\chi} + \chi'\,,\quad \overline{v_i'} = 0\,,\quad \overline{\chi'} = 0\,, \tag{42}$$

where bars denote the mean values and prime dedotes the deviation from the mean value. We then obtain

$$\frac{\partial \overline{\chi}}{\partial t} + v_i\,\frac{\partial \overline{\chi}}{\partial x_i} = -\frac{\partial}{\partial x_i}\overline{v_i'\chi'}\,, \tag{43}$$

where the equation of no divergence $\nabla\cdot\mathbf{V} = 0$ has been used.

We introduce the quantities x_1', x_2', x_3' such that

$$\chi' \sim x_i' \frac{\partial \chi}{\partial x_i}\,. \tag{44}$$

Then x_1', x_2', x_3' may be regarded as components of a radius vector of turbulent trajectory.

Substituting (44) into (43), we have

$$\frac{\partial \chi}{\partial t} + v_i \frac{\partial \chi}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\delta \overline{v_i' x_j'} \frac{\partial \chi}{\partial x_j}\right). \tag{45}$$

This is the general diffusion equation, and in this equation we can put $K_{ij} = \delta \overline{v_i' x_j'}$. Here δ is a proportional constant nearly equal to unity.

Now, if, in the atmospheric surface layer having vertical shear of steady horizontal motion, we take the x_1-axis along the direction of mean wind ($\overline{v_1}$) and the x_3-axis vertically upward, $d\overline{v_1}/dx_3 \neq 0$ and $\overline{v_2} = \overline{v_3} = 0$, and Eq. (45) becomes

$$\frac{\partial \chi}{\partial t} + \overline{v_1} \frac{\partial \chi}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\delta \overline{v_1' x_1'} \frac{\partial \chi}{\partial x_1} + \delta \overline{v_1' x_3'} \frac{\partial \chi}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\delta \overline{v_2' x_2'} \frac{\partial \chi}{\partial x_2}\right)$$
$$+ \frac{\partial}{\partial x_3}\left(\delta \overline{v_3' x_1'} \frac{\partial \chi}{\partial x_1} + \delta \overline{v_3' x_3'} \frac{\partial \chi}{\partial x_3}\right). \tag{46}$$

Here we have assumed that

$$\overline{v_1' x_2'} = \overline{v_2' x_1'} = \overline{v_2' x_3'} = \overline{v_3' x_2'} = 0\,, \tag{47}$$

which can be derived when we assume the statistical symmetry of turbulent motion with respect to any vertical plane involving the straight line parallel to the x_1-axis.

Table 2. Correlation coefficient (r) and regression coefficient ($-\lambda$).

	r	λ	$S.R. \times 10^2$
A	−0.66	1.53	0 — −1
	−0.62	1.14	−1 — −2
	−0.49	0.81	−3 — −4
B	−0.35	0.80	0 — −1
	−0.39	0.72	−1 — −2
	−0.41	0.68	−2 — −3
	−0.53	0.56	−4 — −5
	−0.29	0.54	−6 — −7

	r	λ	$R_i \times 10^3$
C	−0.36	0.65	125 — 100
	−0.37	0.86	100 — 75
	−0.36	0.72	75 — 50
	−0.37	0.82	50 — 25
	−0.50	1.41	0 — −25
	−0.41	0.93	−25 — −50
	−0.31	0.71	−75 — −100
	−0.28	0.74	−100 — −125
	−0.33	0.73	−125 — −150
	−0.21	0.49	−175 — −200
	−0.39	0.53	−200 — −225
	−0.06	1.23	−325 — −350
	−0.15	0.40	−450 — −475

Now, in view of the fact that there exists some definite correlation between v_1' and v_3' as shown in Table 2, we will consider the regression coefficient of v_1' on v_3'. Then the coefficient $(-\lambda)$ becomes

$$-\lambda = \frac{\overline{v_1'v_3'}}{\overline{v_3'^2}}. \tag{48}$$

In this table, A is McCormick's (**41**) data of air of land trajectory at 91 m level, B those of air of sea trajectory at 91 m level, and C Swinbank's (**42**) data at 1.5 m level, and

$$r = \frac{\frac{1}{N}\sum v_1'v_3'}{\sqrt{\frac{1}{N}\sum v_1'^2}\sqrt{\frac{1}{N}\sum v_3'^2}}, \quad \lambda = \frac{-\frac{1}{N}\sum v_1'v_3'}{\frac{1}{N}\sum v_3'^2},$$

where, for A and B, $N=3420$ (every 10 sec reading over 57 minuites), and for C, $N=300$ (every 1 sec reading over 5 minuites). Also, *S. R.* is a stability index defined by $(T_{125}-T_{11})/U_{91}$ (deg sec^2 m^{-2}) where T_{125} and T_{11} are the mean temperatures (°C) at the heights of 125 m and 11 m respectively and U_{91} the mean wind speed (m sec^{-1}) at 91 m level, and Ri is the Richardson number.

Thus we can put

$$v_1' = v_{11}' + v_{12}', \quad v_{11}' = -\lambda v_3', \tag{49}$$

and hereafter we will put v_{12}' out of consideration and simply take $v_1'=v_{11}'$. Then we have

$$v_1' = -\lambda v_3'. \tag{50}$$

Furthermore, if we adopt a plausible assumption of the constancy of vertical velocity (v_3') of an individual eddy during the time needed to bring the eddy to the reference point from its starting point and denote the time by t', we have

$$x_1' = (\overline{v_1}+v_1')\,t', \quad x_3' = v_3't'. \tag{51}$$

Thus, the quantities $\overline{v_1'x_3'}$, $\overline{v_1'x_1'}$, $\overline{v_3'x_1'}$ and $\overline{v_3'x_3'}$ in Eq. (46) becomes, from (50) and (51),

$$\begin{aligned} \overline{v_1'x_3'} &= \overline{v_3'x_1'} = -\lambda\overline{v_3'^2t'} \quad (<0), \\ \overline{v_1'x_1'} &= \lambda^2\overline{v_3'^2t'} \quad (>0), \\ \overline{v_3'x_3'} &= \overline{v_3'^2t'} \quad (>0), \end{aligned} \tag{52}$$

where we have taken $\overline{x_3'}=0$. Accordingly, putting

$$K_3 = \delta\overline{v_3'x_3'}, \tag{53}$$

the general equation (46) is reduced to

$$\frac{\partial \bar{\chi}}{\partial t}+\bar{v}_1\frac{\partial \bar{\chi}}{\partial x_1}=\frac{\partial}{\partial x_1}\left(\lambda^2K_3\frac{\partial \bar{\chi}}{\partial x_1}-\lambda K_3\frac{\partial \bar{\chi}}{\partial x_3}\right)+\frac{\partial}{\partial x_2}\left(K_{22}\frac{\partial \bar{\chi}}{\partial x_2}\right)$$
$$+\frac{\partial}{\partial x_3}\left(-\lambda K_3\frac{\partial \bar{\chi}}{\partial x_1}+K_3\frac{\partial \bar{\chi}}{\partial x_3}\right), \tag{54}$$

or, in two dimensions,

$$\frac{\partial \bar{\chi}}{\partial t}+\bar{v}_1\frac{\partial \bar{\chi}}{\partial x_1}=\frac{\partial}{\partial x_1}\left(\lambda^2K_3\frac{\partial \bar{\chi}}{\partial x_1}-\lambda K_3\frac{\partial \bar{\chi}}{\partial x_3}\right)+\frac{\partial}{\partial x_3}\left(-\lambda K_3\frac{\partial \bar{\chi}}{\partial x_1}+K_3\frac{\partial \bar{\chi}}{\partial x_3}\right). \tag{55}$$

This is the required fundamental equation of two-dimensional diffusion in the lower atmosphere with shear velocity.

It is important to give the values of λ which, from Table 2, appears to be in the range from 1.5 to 0.5 and depends upon the stability of the fluid layer, the surface roughness and probably on height. For the sake of simplicity, we assume in the following section that λ is a certain constant throughout the shearing layer considered.

8. Application of the diffusion equation to large dispersion times

If, in Eq. (55), we put

$$x_* = x_1+\lambda x_3, \tag{56}$$

we have

$$\frac{\partial \bar{\chi}(x_*,\ x_3)}{\partial t}+v_1\frac{\partial \bar{\chi}(x_*,\ x_3)}{\partial x_*}=\frac{\partial}{\partial x_3}\left(K_3\frac{\partial \bar{\chi}(x_*,\ x_3)}{\partial x_3}\right), \tag{57}$$

or, in usual conventional notations,

$$\frac{\partial \bar{\chi}(x_*,\ z)}{\partial t}+\bar{u}\frac{\partial \bar{\chi}(x_*,\ z)}{\partial x_*}=\frac{\partial}{\partial z}\left(K_z\frac{\partial \bar{\chi}(x_*,\ z)}{\partial z}\right). \tag{57'}$$

Hereafter we will use such notations.

In this section we will treat the important problem of diffusion from a line source of infinite length along $x=z=0$, which emits matter at a constant rate Q for a long time, taking the origin of coordinate system at the source.

Then the diffusion equation (57′) becomes

$$\bar{u}(z)\ \frac{\partial \bar{\chi}(x_*,\ z)}{\partial x_*}=\frac{\partial}{\partial z}\left(K_z\frac{\partial \bar{\chi}(x_*,\ z)}{\partial z}\right). \tag{58}$$

The accessory conditions are:

(i) $\bar{\chi}(x,z)\to 0$ as $x, z\to\infty$;

(ii) $\left[-\lambda K_z\frac{\partial \bar{\chi}(x,z)}{\partial x}+K_z\frac{\partial \bar{\chi}(x,z)}{\partial z}\right]_{z=-h}=0$;

(iii) $\bar{\chi}(x,z)\to 0$ as $x+\lambda z\to 0$ $(z\neq 0)$, and
$\bar{\chi}(x,z)\to\infty$ as $x+\lambda z\to 0$ $(z=0)$;

(iv) The condition of constancy for the total flux of diffusing matter from the bottom to the top of the layer through any line with $x+\lambda z=C$ $(=\text{const}>0)$, viz.,

$$\int_{-h/\cos\theta}^{\infty}\bar{u}(z)\cos\theta\,\bar{\chi}(x,z)\,ds$$
$$+\int_{-h/\cos\theta}^{\infty}\left\{-\left(\lambda^2K_z\frac{\partial\bar{\chi}(x,z)}{\partial x}-\lambda K_z\frac{\partial\bar{\chi}(x,z)}{\partial z}\right)\cos\theta\right.$$
$$\left.-\left(-\lambda K_z\frac{\partial\bar{\chi}}{\partial x}+K_z\frac{\partial\bar{\chi}}{\partial z}\right)\sin\theta\right\}ds$$
$$=Q\ (=\text{const})\ \text{for all}\ C>0,$$

where θ is the intersecting angle between the line $x+\lambda z=C$ and the z-axis, $s=z/\cos\theta$, and the integrations have been taken along the line $x+\lambda z=C$. Using the new parameter x_* as defined by (56), these conditions are reduced respectively to

(i) $\bar{\chi}(x_*,z)\to 0$ as $x_*, z\to\infty$;

(ii) $\left[K_z\frac{\partial\bar{\chi}(x_*,z)}{\partial z}\right]_{z=-h}=0$;

(iii) $\bar{\chi}(x_*,z)\to 0$ as $x_*\to 0$ $(z\neq 0)$ and
$\bar{\chi}(x_*,z)\to\infty$ as $x_*\to 0$ $(z=0)$;

(iv) $\int_{-h}^{\infty}\bar{u}(z)\,\bar{\chi}(x_*,z)\,dz=Q$ for all $x_*>0$.

Thus, we arrive at the same mathematical problem as those treated by Frost (**43**), Calder (**44**), Sutton (**8, 9**), Rounds (**45**) and Smith (**46**). Therefore, for instance, in the case when

$$\bar{u}=u_0(z+h)^{\alpha},\quad K_z=K_0(z+h)^{1-\alpha},\tag{59}$$

we get, after Smith,

$$\bar{\chi}(x_*,z)=\frac{Q}{2\alpha+1}\frac{(zh+h^2)^{\alpha/2}}{K_0x_*}\exp\left[-\frac{u_0(z+h)^{1+2\alpha}+u_0h^{1+2\alpha}}{K_0(2\alpha+1)^2x_*}\right]$$
$$\times I_{-\alpha/(1+2\alpha)}\left[\frac{2u_0(zh+h^2)^{(1+2\alpha)/2}}{K_0(2\alpha+1)^2x_*}\right],$$

and accordingly we have

$$\bar{\chi}(x,z)=\frac{Q}{2\alpha+1}\frac{(zh+h^2)^{\alpha/2}}{K_0(x+\lambda z)}\exp\left[-\frac{u_0(z+h)^{1+2\alpha}+u_0h^{1+2\alpha}}{K_0(2\alpha+1)^2(x+\lambda z)}\right]$$
$$\times I_{-\alpha/(1+2\alpha)}\left[\frac{2u_0(zh+h^2)^{(1+2\alpha)/2}}{K_0(2\alpha+1)^2(x+\lambda z)}\right].$$

Thus, Smith's figure giving the concentration at the ground level should be modified and can be utilized only by altering the abscissa scale of $K_0x/(u_0h^{1+2\alpha})$ into $K_0(x-\lambda h)/(u_0h^{1+2\alpha})$ (Fig. 5).

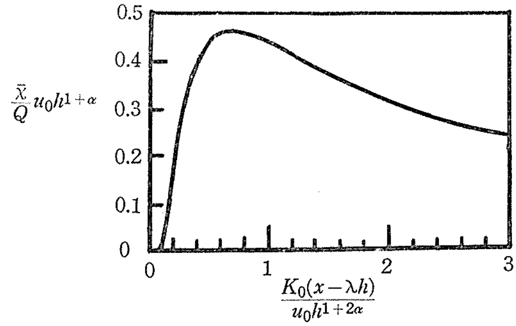

Fig. 5. The concentration at ground level, as function of the distance downstream due to an elevated source when $\alpha=1/7$.

9. Distribution of conservative concentration for small dispersion times

As was treated in §§ 3 and 4 of Part I, special consideration must be given to small dispersion times in the problems of diffusion of matter emitted from a continuous source of fixed type. Therefore such consideration is to be given to the problem treated in the preceding section. In this case, since the form of K_z is unknown, we suppose tentatively that, in the small domain within the thin layer at the source level except when the source is on the ground (the reasoning of this exception will be taken up later.), the form of Eq. (18) is still applicable, though the effective life-time τ_0 in this case may differ from the effective life-time of horizontal motion, such as treated in § 4 in Part I, which does not correlate to vertical eddy motion. If this supposition is allowed, the eddy diffusivity along the vertical ($K_z(\xi)$) is given by

$$K_z(\xi) = \overline{w'^2}\xi\left(1-\frac{3}{4}\frac{\xi}{\tau_0}+\frac{\xi}{2\tau_0}\log\frac{\xi}{\tau_0}\right), \tag{60}$$

where $\overline{w'^2}$ is the mean square vertical component of eddy velocity, and the dispersion time, ξ, can be taken as $\xi\approx x_*/\bar{u}(0)$ (where $\bar{u}(\neq 0)$ is the mean velocity at the source level within the shearing layer). Here we also suppose that the variation of $\bar{u}$ and τ_0 with height does not affect seriously the resultant distribution of $\bar{\chi}$ so far as the dispersion time ξ is much smaller than τ_0. The method of estimating τ_0 will be given later.

Then we have the following diffusion equation for small dispersion times,

$$\bar{u}(0)\frac{\partial\bar{\chi}(x_*, z)}{\partial x_*} = K_z(x_*/\bar{u}(0))\frac{\partial^2\bar{\chi}(x_*, z)}{\partial z^2}. \tag{61}$$

Since the ground surface lies far below the layer under consideration, the effect resulting from its boundary condition may be removed, and referring to a theorem due to Chandrasekhar (**47**, p. 34), we have the solution satisfying the condition:

$$\int_{-\infty}^{\infty} \bar{u}(0)\,\bar{\chi}(x_*, z)\,dz = Q = \text{const.}$$

Thus,

$$\left.\begin{aligned} \bar{\chi}(x_*, z) &= \frac{Q/\bar{u}(0)}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{z^2}{2\sigma^2}\right), \\ \text{where} \quad \sigma^2 &= \overline{w'^2}\left(\frac{x_*}{\bar{u}(0)}\right)^2\left\{1-\frac{11}{18}\frac{x_*}{\tau_0\bar{u}(0)}+\frac{x_*}{3\tau_0\bar{u}(0)}\log\frac{x_*}{\tau_0\bar{u}(0)}\right\}, \\ \text{and} \quad x_* &= x+\lambda z. \end{aligned}\right\} \quad (62)$$

	λ	$\sqrt{\overline{w'^2}}/\bar{u}$	τ_0 (sec)
a ···············	0	0.07	508
b — — — —	0	0.07	415
c - - - - - - -	1	0.06	485
d ━━━━━━	1	0.07	415
e — - - - —	1	0.08	364
f — - — - —	1.5	0.07	508
g ————	1	0.1	291

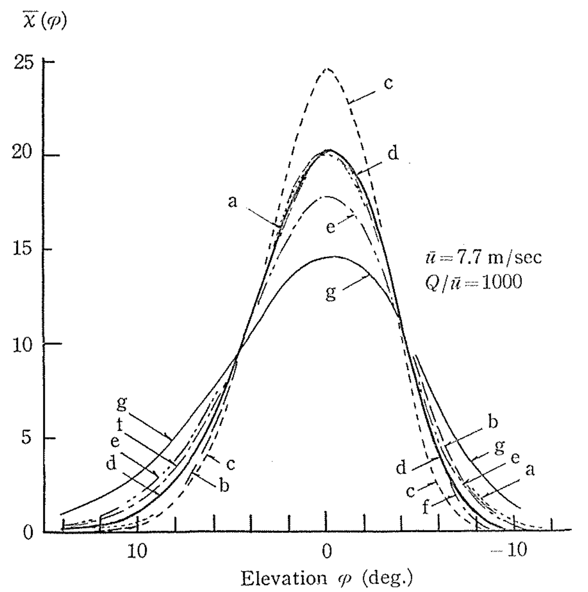

Fig. 6. Graphical illustration of theoretical values of $\bar{\chi}$ in terms of elevation angle (φ) for fixed value of x (=300 m), $\bar{u}$ (=7.7 m/sec), h (=140 m) and $Q/\bar{u}$ (=1000).

This solution is graphically illustrated in Fig. 6 in terms of angular elevation with respect to the source (i. e., the angle φ given by $\varphi=\tan^{-1}(z/x)$ instead of z) for fixed x. Here we have taken $h=140$ m, $x=300$ m, $\bar{u}(0)=7.7$ m/sec and $Q/\bar{u}(0)=1000$. The values of τ_0's are estimated from λ and $\sqrt{\overline{w'^2}}$ in the following manner.

If we assume δ in Eq. (53) to be unity, then the eddy diffusivity K_z under the condition of neutral stability becomes identical with the eddy viscosity, and the eddy viscosity at the height of h can, as it is well established, be expressed as $K_z=v_*\kappa h$, where v_* is the friction velocity and κ is the Kármán constant nearly equal to 0.4. On the other hand, the eddy diffusivity (K) for large dispersion times in the locally homogeneous, isotropic turbulence field can be expressed as $K=\overline{w'^2}\tau_0/4$, where τ_0 is the life-time of effective largest eddy as was shown in Part I in this paper (see the expression (18)). Accordingly, it seems plausible under near-neutral conditions that $v_*\kappa h\approx\overline{w'^2}\tau_0/4$ and then $\tau_0\approx 4v_*\kappa h/\sqrt{\overline{w'^2}}$ $=4\sqrt{\lambda}\ \kappa h/\sqrt{\overline{w'^2}}$. It must be noticed that this expression for τ_0 leads to the fact that $K_z(\xi)$ expressed by Eq. (60) must be a function of height. However, we suppose that the variation of τ_0 with height can be neglected, as stated before.

The theoretical result of Eq. (62) can be verified, at least partly, by Hay and Pasquill's experimental data (48) in the following way.

Hay and Pasquill's experiment on the frequency distribution of diffusing spores is concerned with the flux of spores, as they state that "the collection efficiency in the number of particles impacting the cylinder, expressed as a percentage of the number which would have *passed through the same cross-sectional area** if the cylinder had not been there". Hence, their results may be compared with the theoretical values of the mean longitudinal flux of diffusing materials.

Now, the flux $\overline{u\chi}$ can be evaluated as follows:

$$F(x,z)\equiv\bar{u}\bar{\chi}+\overline{u'\chi'}=\bar{u}\bar{\chi}-\lambda K_z\bar{\chi}/\sigma^2$$
$$=\bar{u}\bar{\chi}\left(1-\frac{\lambda K_z z}{\bar{u}(0)\sigma^2}\right),$$

and such a function $G(x,\varphi)$ as $F(x,z)\,dz=G(x,\varphi)\,d\varphi$ becomes

$$\begin{aligned}G(x,\varphi)&=F(x,x\tan\varphi)\,x\sec^2\varphi\\&=\frac{Q}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2\tan^2\varphi}{2\sigma^2}\right)\left\{1-\frac{\lambda K_z x\tan\varphi}{\bar{u}(0)\,\sigma^2}\right\}x\sec^2\varphi\\&=\bar{u}\bar{\chi}\left\{1-\frac{\lambda K_z x\tan\varphi}{\bar{u}(0)\,\sigma^2}\right\}x\sec^2\varphi,\end{aligned}$$

where

* The italics here are the present author's.

$$K_z = \frac{\overline{w'^2}x(1+\lambda\tan\varphi)}{\bar{u}(0)}\left\{1-\frac{3}{4}\frac{x(1+\lambda\tan\varphi)}{\tau_0\bar{u}(0)} + \frac{x(1+\lambda\tan\varphi)}{2\tau_0\bar{u}(0)}\log\frac{x(1+\lambda\tan\varphi)}{\tau_0\bar{u}(0)}\right\},$$

and

$$\sigma^2 = \frac{\overline{w'^2}x^2(1+\lambda\tan\varphi)^2}{\bar{u}^2(0)}\left\{1-\frac{11}{18}\frac{x(1+\lambda\tan\varphi)}{\tau_0\bar{u}(0)} + \frac{x(1+\lambda\tan\varphi)}{3\tau_0\bar{u}(0)}\log\frac{x(1+\lambda\tan\varphi)}{\tau_0\bar{u}(0)}\right\}.$$

Then we can compare $G(x,\varphi)\Delta\varphi$ (in case of $\Delta\varphi=1°=1.745\times10^{-2}$ rad, $Q=100$) with Hay and Pasquill's experimental results. We will choose the experiments in case of $x=300$ m because their experiments involve the largest number of experiments in this case and are rather successful as they state. Among these experiments, the experiment 12 seems to be in the near-average state of the source height and meteolorogical conditions. Therefore, we choose the experiment 12 as a representative case of all the experiments of $x=300$ m. Then, from their Table 1, we have $x=300$ m, $h=140$ m, $\bar{u}(0)=7.7$ m/sec. Assuming different values for $\sqrt{\overline{w'^2}}/\bar{u}(0)$ and λ, and using the estimated rough values of τ_0, the corresponding theoretical values of $G(300,\varphi)\Delta\varphi$ are graphically illustrated in Fig. 7 as a function of φ and are compared with the results of all experiments in case of $x=300$ m, including the result of the experiment 12, the experimental values having been read from Fig. 3 in Hay and Pasquill's paper. Fig. 7 shows that the theoretical and the observed values are statistically in good agreement with each other, if we take suitable vaules for λ, $\sqrt{\overline{w'^2}}/\bar{u}(0)$ and τ_0.

In addition to the above verification, the theoretical vertical distribution of concentration of diffusing matter (or gas) given by Eq. (62) could be compared with Stewart *et al.*'s experimental results (**49**). According to them, the characteristic feature of the vertical concentration distribution is that "the adiabatic and unstable curves are of approximately Gaussian form, somewhat extended in the direction of increasing height." This feature is manifested by our theoretical curves in Fig. 6.

However, they state that their experiments were conducted with warm air containing a small amount of radioactive argon (^{41}A) emitted from 61-m chimney of BEPO reactor and so the air cloud has considerably great buoyancy, and therefore the agreement of the feature of the experimental results with our theoretical result cannot be so conclusive.

The characteristic feature of such skewness of the vertical distribution of concentration was also discussed and manifested by Hinze (**39**) in shear flow of laboratory scale.

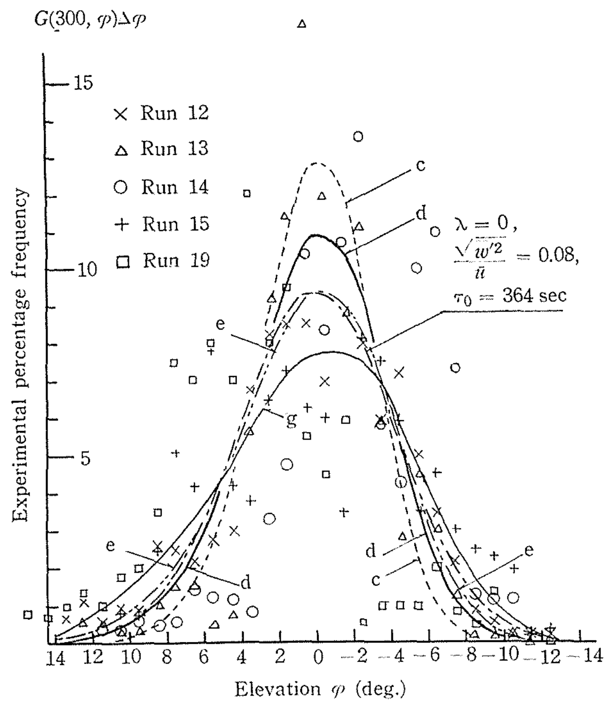

Fig. 7. Graphical illustration of theoretical values of flux $G(300, \varphi)\Delta\varphi$ in case of $\Delta\varphi = 1.745\times10^{-2}$ rad and $Q=100$ and the experimental percentage frequency of particles captured. Theoretiocal curves indicated as *c*, *d*, *e* and *g* correspond to those in *Fig.* 6.

10. Discussions

(i) Our basic concept of modification of the conventional general diffusion equation comes essentially from the fact that in the atmospheric surface layer u' and w' are negatively correlated not only in the Eulerian way but also in the Lagrangian way at least during certain periods, and seems to be essentially in agreement with Hinze's concept (**39**). The two new terms introduced in the ordinary diffusion equation by the author signify such interrelation, while Lettau (**50**) introduced only one of these terms. That is, he concluded, without giving any explanation, that $\overline{u'z'} \neq 0$ but $\overline{w'x'}=0$. He (**50, 51**) called this term the "shearing advection term". However, the other term which has been newly derived by the author does not mean the "shearing advection". The author believes that the two new terms should be interpreted as an effect of the change of structure of turbulence associated with shear velocity.

(ii) From §§ 8 and 9, it is clear that in the general application of the present theory K_z must be a function of height above the ground as well as the dispersion time even in the case of large dispersion times. Therefore, some correction may be necessary to give the corrected value of concentration at great distances downstream. One method of correction is to use a virtual origin at some distance downwind from the real source. Quantitative or qualitative treatments were made by Inoue (e. g. **19**) and Grant (**28**). The correction does not seem necessary in the case of grounded source, since τ_0 for the ground level may be zero judging from our method of estimating τ_0, and in that case the problem seems to be always in the case of large dispersion times in spite of Grant's assertion.

(iii) The estimation of the value of λ together with $\sqrt{\overline{w'^2}}/\bar{u}(0)$ may be made by comparing the probability distribution of wind inclination derived by the present model of turbulence structure with the observed frequency distribution of wind inclination, in the following way.

If we assume that the probability distribution of w' at a point is normal, and we put the probability that the value of w' lies between w' and $w'+dw'$ to be $p(w')\,dw'$, then, referring to the fact that $\tan\theta = w'/(\bar{u}(0)+u')$ where θ is the angle of wind inclination, we have

$$p(w')\,dw' = \frac{\bar{u}}{\sqrt{2\pi}\sqrt{\overline{w'^2}}} \exp\left(-\frac{\bar{u}^2/(\cot\theta+\lambda)^2}{2\overline{w'^2}}\right) \frac{\operatorname{cosec}^2\theta}{(\cot\theta+\lambda)^2}\,d\theta\,.$$

Therefore, if we put the probability that the angle of wind inclination lies between θ and $\theta+d\theta$ to be $f(\theta)d\theta$, we have

$$f(\theta)\,d\theta = \frac{\bar{u}}{\sqrt{2\pi}\sqrt{\overline{w'^2}}} \exp\left(-\frac{\bar{u}^2/(\cot\theta+\lambda)^2}{2\overline{w'^2}}\right) \frac{\operatorname{cosec}^2\theta}{(\cot\theta+\lambda)^2}\,d\theta\,.$$

In Fig. 8, the theoretical distributions of $f(\theta)\,\Delta\theta$ for $\Delta\theta=1°$ are shown in a few cases of combination of λ and $\sqrt{\overline{w'^2}}/\bar{u}$ which correspond to some cases in Fig. 6 or Fig. 7, with the observed values corresponding to the data illustrated in Fig. 7. Fig. 8 shows that the optimum values of λ and $\sqrt{\overline{w'^2}}/\bar{u}$ to fit the observed data seem to be very close to those in Fig. 7. Accordingly, if we prepare a catalogue of correspondence of the frequency distribution of wind inclination to λ and $\sqrt{\overline{w'^2}}/\bar{u}$, we can readily determine appropriate values of λ and $\sqrt{\overline{w'^2}}/\bar{u}$ from the observed frequency distribution of wind inclination.

(iv) The theoretical results in §9 may be compared with Sutton's famous semi-empirical formula (**11, 52**) in two dimensions for the elevated source in the case of sufficiently small x,

$$\bar{\chi}(x, z) = \frac{Q}{\sqrt{\pi}\ C_z \bar{u} x^{1-n/2}} \exp\left(-\frac{z^2}{C_z^2 x^{2-n}}\right),$$

where the coordinate system is taken as in §9. His argument in the course of derivation of this formula seems to partly confuse this with the problem of relative diffusion, but lastly he applies this formula to the problem of continuous cross-wind infinite line source of fixed type.

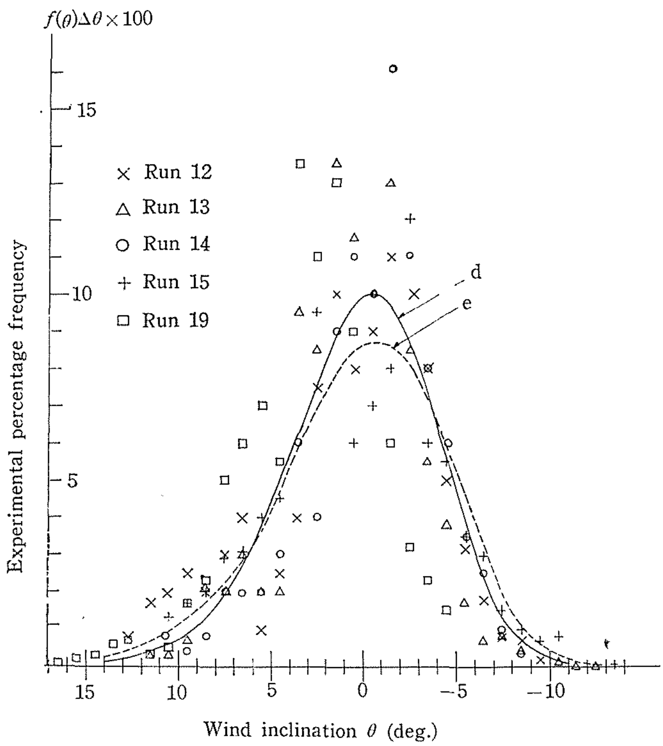

Fig. 8. Graphical illustration of theoretical values of wind inclination frequency $f(\theta)\Delta\theta$ in the case of $\Delta\theta=1.745\times10^{-2}$ rad and the values of λ and $\sqrt{\overline{w'^2}}/\bar{u}$ identical with those in the case of d and e in Figs. 6 and 7, and the experimental percentage frequency of wind inclination corresponding to the experimental results of particle elevation in Fig. 7.

Comparing his formula with the present one (62), we have the following correspondence:

$$C_z \leftrightarrow \sqrt{\overline{w'^2}}/\bar{u}\,,$$

and

$$x^{1-n/2} \leftrightarrow (x+\lambda z)\left\{1-\frac{11}{18}\frac{x+\lambda z}{\tau_0\bar{u}}+\frac{x+\lambda z}{\tau_0\bar{u}}\log\frac{x+\lambda z}{\tau_0\bar{u}}\right\}^{1/2},$$

where, Sutton's n having rather obscure physical meaning corresponds to the effect of τ_0 and the effect of λ is not taken into account by Sutton. His recommended value of $n=0.25$ seems to correspond to the value of

$$\frac{d}{d\log(\xi/\tau_0)}\log\left\{\frac{\xi}{\tau_0}\left(1-\frac{11}{18}\frac{\xi}{\tau_0}+\frac{\xi}{3\tau_0}\log\frac{\xi}{\tau_0}\right)^{1/2}\right\},$$

at the value of $\xi\approx 0.3\ \tau_0$ (i.e., $x\approx 0.3\ \tau_0\bar{u}$) in the formula of σ, putting formally $\lambda=0$.

(v) Since the solution of the new equation gives, on the level of source, the same values of $\bar{\chi}$ with the values of the solution of the classical diffusion equation:

$$\bar{u}(z)\frac{\partial\bar{\chi}}{\partial x}=\frac{\partial}{\partial z}\left(K_z\frac{\partial\bar{\chi}}{\partial z}\right),$$

the verification can be made by comparing the theoretical values of the concentration at that level with the experimental values. Such verification was successfully done in the case of grounded source by Sutton (8) and Calder (44). Therefore, their is no contradiction between our theoretical result and their experimental result in that case. However, in the case of elevated source, as was stated before, the solution of our new equation must not be in agreement with that of the classical equation. It is highly desirable that experimental data for the verification in such cases are published. The author is now making preparations for certain experiments which are capable of verifying our proposal.

It may be added here that, Davies (**53**) solved Lettau's (**50**) equation which contains only one new term additively and obtained the solution for the case of grounded source, and that he thus found that there is a little, though practically negligible, difference between the values of concentration at the ground level and those given by the classical equation, in contrast to the exact agreement between the values of concentration at the ground level given by the solution of the classical equation and those given by the solution of the present new equation.

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