On the density of the set of primes which are related to decimal expansion of rational numbers (Analytic Number Theory and Surrounding Areas)

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On the density of the set of primes which are related to decimal expansion of rational numbers

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We give several conjectures on the set of prime numbers which are closely related to 10-adic decimal expansion of rational numbers. The starting point is the following theorem.

**Theorem 1** Let \( p \neq 2,5 \) be a prime number. \( 1/p \) has a purely periodic decimal expansion

\[
1/p = 0.c_1\cdots c_e = 0.c_1\cdots c_e c_1\cdots c_e \cdots, \quad (0 \leq c_i \leq 9)
\]

where we assume that \( e \) is the minimal length of periods, i.e. \( e = \text{the order of } 10 \mod p \). Suppose \( e = nk \) for natural numbers \( n > 1, k \). We divide the period to \( n \) parts of equal length and add them. Then we have

\[
c_1\cdots c_k + c_{k+1}\cdots c_{2k} + \cdots + c_{(n-1)k+1}\cdots c_{nk} = 9\cdots 9 \times \begin{cases} n/2 & \text{if } n \text{ is even,} \\ s(p) & \text{if } n \text{ is odd,} \end{cases}
\]

where \( 9\cdots 9 = 10^k - 1 \) and \( s(p) \) is an integer such that \( 1 \leq s(p) \leq n - 2 \).

We are concerned with the density of the set of primes for given \( n \) and \( s = s(p) \). Hereafter we assume that \( n \geq 3 \) is an odd natural number and \( 1 \leq s \leq n - 2 \). Put

\[
P(n, s, x) = \frac{\#\{p | p \leq x, n|e, s(p) = s\}}{\#\{p | p \leq x, n|e\}},
\]

where \( p \neq 2,5 \) stands for a prime number and \( e = \text{the order of } 10 \mod p \).

The following table of \( P(n, s, 10^9) \) is made by computer.
As a matter of fact, the graph of \( P(n, s, x) \) in \( x \) is almost straight line. The ratios are symmetric at \((n - 1)/2\). In the table, 0.0000 means that primes which take the values \( s = 1, 9 \) are very rare in the case of \( n = 11 \), and 0 for \( n = 9 \) means that the set is empty, which can be proven. The first conjecture is

**Conjecture 1** \( \lim_{x \to \infty} P(n, s, x) \) exists, and by denoting it by \( P(n, s) \)

\[
P(n, s) = P(n, n - 1 - s) \text{ for } 1 \leq s \leq n - 2.
\]

Moreover \( P(n, s) > 0 \) holds if \( n \) is an odd prime number.

Moreover the table above looks like normal distribution. Let us recall notations of statistics. For the table of frequency distribution

<table>
<thead>
<tr>
<th>value</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( \ldots )</th>
<th>( x_m )</th>
<th>sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>relative frequency</td>
<td>( r_1 )</td>
<td>( r_2 )</td>
<td>( \ldots )</td>
<td>( r_m )</td>
<td>1</td>
</tr>
</tbody>
</table>

define the average \( \mu \) and the standard deviation \( \sigma \) by

\[
\mu = \sum_{i=1}^{m} x_i r_i, \quad \sigma = \sqrt{\sum_{i=1}^{m} x_i^2 r_i - \mu^2}.
\]

Then we get

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \mu )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.0001</td>
<td>0.5774</td>
</tr>
<tr>
<td>9</td>
<td>4.0002</td>
<td>0.7070</td>
</tr>
<tr>
<td>11</td>
<td>5.0002</td>
<td>0.9132</td>
</tr>
<tr>
<td>37</td>
<td>18.0010</td>
<td>1.7325</td>
</tr>
</tbody>
</table>

This table suggests
Conjecture 2
\[ \lim_{x \to \infty} \mu = (n - 1)/2. \]

To formulate being normal distribution, we denote the density function of normal distribution of average \( \mu \) and standard deviation \( \sigma \) by
\[ f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right) \]
and compare the ratio with it. The table is

| n   | \max_{1 \leq s \leq n-2} |P(n, s, x) - f_{\mu,\sigma}(s)| |
|-----|--------------------------|-----------------------------|
| 5   | 0.0243                   |                             |
| 9   | 0.0641                   |                             |
| 11  | 0.0067                   |                             |
| 37  | 0.0006                   |                             |

This table and more general table for odd \( n \leq 101 \) suggest

Conjecture 3
\[ \lim_{n \to \infty} \lim_{x \to \infty} \max_{1 \leq s \leq n-2} |P(n, s, x) - f_{\mu,\sigma}(s)| = 0. \]

We considered 10-adic expansion. But in the proof of Theorem 1, the number 10 is not important. It is generalized as follows:

**Theorem 2** Let \( a \neq 0, \pm 1 \) be an integer and \( p \) a prime number. Put \( e = \text{the order of } a \text{ mod } p \) and suppose \( e = nk \), where \( n \geq 3 \) and \((a^k - 1, p) = 1\). Define an integer \( r_i \) by
\[ r_i \equiv a^{ki} \mod p, \quad 0 \leq r_i < p. \]

Then \( s(p) = (\sum_{i=0}^{n-1} r_i)/p \) is an integer such that \( 1 \leq s(p) \leq n - 2 \).

The former part is the case of \( a = 10 \). Similarly as above, we put
\[ P_a(n, s, x) = \frac{\# \{p \mid p \leq x, n|e, s(p) = s \}}{\# \{p \mid p \leq x, n|e \}}. \]

The numerical data suggest the final

Conjecture 4
\[ \lim_{x \to \infty} P_a(n, s, x) = \lim_{x \to \infty} P_{10}(n, s, x)(= P(n, s)). \]

The proof of theorems are easy and other probably new observations will be included in 本格的に代数を学ぶ前に.