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CONVECTIVE GROWTH RATES IN THE CASE OF VARYING SUPERADIABATIC GRADIENT

BY

Schichiro YAMAGUCHI

Institute of Applied Mathematics, Gifu University and Institute of Astrophysics, University of Kyoto

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ABSTRACT

The problem to seek for the convective growth rates of initially given disturbances in the thin fluid layer of varying superadiabatic temperature gradient is investigated. For some model variations of the gradient the growth rates and the eigenfunctions are obtained by solving a second order differential equation where the viscosity is neglected and the disturbances are assumed to be optically thin or adiabatic. The validity of the variational method is examined. When a suitable trial function (including only one variational parameter) is adopted, the approximate eigenvalue aggrees with the exact value for the fundamental mode within the limit of error of a few percent (except for very small wavenumbers of disturbances) for a variety of the superadiabatic gradient.

1. Introduction

In stellar convection zone the superadiabatic gradient β varies with depth z and even changes its sign due to the ionization of the elements composing of the stellar gas, while the temperature gradient is constant in the initial state of the parallel-plate convection. To obtain an image of the stellar convection, we should discuss first how much grade of instability the gradient $\beta(z)$ implies. Spiegel (1958) discussed the marginal stability of the convection zone of early-type stars, approximating the actual profile of the gradient by box-type profile of equal area and neglecting the penetration of the convective flow into upper and lower stable regions. This approximation should be examined. In a paper to be published (Yamaguchi 1967) the effect of varying superadiabatic gradient on the marginal stability problem is studied. In this paper the problem to seek for the convective growth rates of initially given disturbances in the thin fluid layer is solved for some model variations of the gradient, when the viscosity is neglected and the disturbances are assumed to be optically thin or adiabatic.

2. Equation of the problem

We consider a thin layer in the envelope of a star. Let the superadiabatic temperature gradient (along the acceleration of gravity) $\beta_s(\zeta)(\zeta=z/d)$ be positive in the layer (thickness d) and be negative in the layers extending above and below. The system is supposed to be disturbed by the disturbance of small amplitude whose horizontal wavenumber is $k=a/d$. This disturbance will grow or damp as $\exp \sigma t$. We seek for the growth rate σ for given $\beta_s(\zeta)$ and $a=kd$. This problem is described by the following equation,

$$(D^2-a^2)W(\zeta)=-R(\zeta)a^2W(\zeta), \qquad \cdots\cdots\cdots(1)$$

where W expresses the dependence of the vertical velocity component w on depth $z=\zeta d$, i. e.

$$w=W(z)F(x,\ y)\exp \sigma t,\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F=-k^2F. \qquad \cdots\cdots\cdots(2)$$

The other notation is listed below;

$$R(\zeta)=\Delta(\zeta)\frac{g}{d}\frac{1}{(\sigma+q)\sigma}, \qquad \cdots\cdots\cdots(3)$$

$$\Delta(\zeta)=\left(1-\frac{\partial \ln \mu}{\partial \ln T}\right)(\nabla-\nabla_{AD})\frac{d}{H_p}=\left(1-\frac{\partial \ln \mu}{\partial \ln T}\right)\beta_s(\zeta)\frac{d}{T_o}, \qquad \cdots\cdots\cdots(4)$$

μ=the mean molecular weight of the stellar gas,
H_p=the pressure scale height,

$$\nabla=\frac{d \ln T}{d \ln P},$$

q=the characteristic inverse time for radiative cooling (see Spiegel 1964, Unno and Spiegel 1966),

$$D=\frac{d}{d\zeta}.$$

In deriving equation (1) we have assumed that the disturbance is optically thin ($a \gg \tau$, τ is the optical thickness of the unstable layer) or adiabatic, that the Boussinesq approximation is valid (i. e. $H_p \gg d$ and the velocity field is solenoidal; see Spiegel and Veronis 1961, Mihaljan 1962), and that the viscosity can be neglected. When we put $q=0$, equation (1) describes the case where heat exchange is absent and the disturbance is adiabatic. The superadiabatic gradient or $\Delta(\zeta)$ is given by the model of stellar envelope for given T_e (effective temperature) and g (surface gravity). In this paper we study the model case where $R(\zeta)$ is expressed by the following equation,

$$R(\zeta)=Af(\zeta\,;\ \alpha,\ p), \qquad \cdots\cdots\cdots(5)$$

$$f(\zeta)=Q[\exp \alpha(1-|\zeta|^p)-1],\ Q\equiv\frac{1}{e^\alpha-1}. \qquad \cdots\cdots\cdots(6)$$

In equation (5) $f(\zeta)$ represents the variation of superadiabatic gradient with depth and the parameter A is the maximum value of $R(\zeta)$. We seek for the eigenvalue of A for given α and p. Since A^{-1} is proportional to $\sigma(\sigma+q)$, the behavior of the growth rate $\sigma(>0)$ as a function of the wavenumber a can be seen from the dependence of A on a.

We consider first the case where the medium extends from $\zeta=0$ to $\zeta=\infty$ and a fixed boundary is set at $\zeta=0$ (semi-infinite case). Next we treat the case where the medium extends from $\zeta=-\infty$ to $\zeta=\infty$ (infinite case). In the infinite case the thickness of the unstable layer is $2d$, and the nondimensional wavenumber should be replaced by $2a$ in comparing the results obtained in both cases, since the nondimensional wavenumber a is proportional to the thickness of the unstable layer by the definition ($a=kd$). The eigenvalue A needs not be calibrated, since it does not include the quantity d (see equations (3)−(5)).

The boundary conditions are taken to be

$$W=0, \text{ at } \zeta=0 \text{ and } \infty \text{ (semi-infinite case)}, \qquad \cdots\cdots(7)$$

$$W=0, \text{ at } \zeta=\pm\infty \text{ (infinite case)}. \qquad \cdots\cdots(8)$$

3. Methods giving the exact and approximate solutions

We can apply various methods to solve equation (1) with boundary conditions (7) or (8) (see Spiegel 1965). We list them below.

(1) Exact solution.

For special sets of values of p and α, equation (1) is reduced to the well-known equation whose solution was studied.

(2) The variational method.

Multiplying equation (1) by $W(\zeta)$ and integrating over the entire region, we obtain an equation,

$$\langle (DW)^2\rangle + a^2\langle W^2\rangle = Aa^2\langle fW^2\rangle, \qquad \cdots\cdots(9)$$

where the symbol $\langle\ \ \rangle$ denotes the integration over entire region. We introduce the following trial functions,

$$W=\zeta \exp\left(-\frac{s^2\zeta^2}{2}\right), \qquad \cdots\cdots(10)$$

or

$$W=\begin{cases}\sin b\zeta & \left(0<\zeta<\frac{\pi}{b}\right),\\ 0 & \left(\zeta>\frac{\pi}{b}\right),\end{cases} \qquad \cdots\cdots(11)$$

for semi-infinite case, and

$$W=\exp\left(-\frac{s^2\zeta^2}{2}\right), \qquad \cdots\cdots(12)$$

or

$$W=\begin{cases}\cos b\zeta & \left(|\zeta|<\frac{\pi}{2b}\right),\\ 0 & \left(|\zeta|>\frac{\pi}{2b}\right),\end{cases} \qquad \cdots\cdots(13)$$

for infinite case, where s and b are variational parmeters. Introducing these trial functions into equation (9) and minimizing the quantity A^{-1} with respect to the parameter s or b, we obtain an approximate eigenvalue $A(a\,;\,\alpha,\,p)$.

(3) The WKB method.

In equation (1) the function $R(\zeta)-1$ has a zero $\zeta_0 \leqq 1$ (turning point), at which $R(\zeta_0)=1$, in the semi-infinite case. We seek for asymptotic solution $W_I(\zeta)$ in the region $0\leqq\zeta\ll\zeta_0$ which satisfies the condition $W_I(0)=0$, and look for the asymptotic solution $W_{II}(\zeta)$ in the region $\zeta\gg\zeta_0$ which satisfies $W_{II}(\infty)=0$. We can connect both solutions by the connecting formula in the WKB method. Then the eigenvalue and the eigenfunction for the fundamental mode are obtained as below,

$$a\int_0^{\zeta_0}\sqrt{P(\zeta)}d\zeta=\frac{3}{4}\pi\,, \qquad \cdots\cdots(14)$$

$$W=2P^{-\frac{1}{4}}\sin\left[a\int_0^{\zeta}P^{\frac{1}{2}}d\zeta\right] \qquad (0\leqq\zeta\ll\zeta_0)\,, \qquad \cdots\cdots(15)$$

$$W=(-P)^{-\frac{1}{4}}\exp\left[-a\int_{\zeta_0}^{\zeta}(-P)^{\frac{1}{2}}d\zeta\right] \qquad (\zeta\gg\zeta_0)\,, \qquad \cdots\cdots(16)$$

$$P(\zeta)=\mathrm{R}(\zeta)-1=Af(\zeta)-1\,. \qquad \cdots\cdots(17)$$

These formulae are valid if the behavior of $P(\zeta)$ near the turning point is linear (i. e. $P(\zeta)\propto\zeta-\zeta_0$ near $\zeta=\zeta_0$). Therefore they can not be applied to the case $p=\infty$ having a jump at the turning point $\zeta=\zeta_0=1$.

The validity and usefulness of the WKB method were already pointed out by Spiegel (1965) in the study of convective instability in a compressible atmosphere. It is, however, rather tedious in calculating the eigenvalue especially in the case of higher than second order differential equation. On the other hand the variational method is convenient to obtain the approximate eigenvalue (not the eigenfunction) and was used in studying the marginal stability of a layer of varying superadiabatic gradient with the same trial functions as given by equations (10)–(13) (Yamaguchi 1967). In this paper we will examine the validity of the variational method with these trial functions in evaluating the eigenvalue of A.

4. Results

First we list the exact solution for some special sets of values of the parameters p and α.

(1) The case $p=\infty$ (semi-infinite).

$$A=\frac{a^2+\theta^2}{a^2},\quad W=\begin{cases}\sin\theta\zeta & (0\leqq\zeta\leqq1)\,,\\ \sin\theta\cdot\exp\beta(1-\zeta) & (\zeta\geqq1)\end{cases} \qquad \cdots\cdots(18)$$

where θ's are the the positive roots of the following equation,

$$\tan\theta=-\frac{\theta}{\beta}\,,\quad \beta=\sqrt{Q(a^2+\theta^2)+a^2}\,. \qquad \cdots\cdots(19)$$

The fundamental mode is obtained by substituting the smallest positive root θ_1, into equation (18). The smallest positive root θ_1 is the increasing function of the wave number a, and $\pi/2\leqq\theta_1(a,\ Q)\leqq\pi$.

(2) The case $p=2$ and $\alpha=0$ (semi-infinite).

$$A^{\frac{1}{2}}=\frac{\sqrt{\left(n+\frac{1}{2}\right)^2+a^2}+n+\frac{1}{2}}{a},\quad W=\exp\left(-\frac{s^2\zeta^2}{2}\right)H_n(s\zeta) \quad (n=1,\ 3,\ 5,\cdots\cdots),\ \cdots(20)$$

where $s=(a^2A)^{\frac{1}{4}}$ and H_n are Hermite polynomials of odd order.

(3) The case $p=1$ and $\alpha=0$ (semi-infinite).

The eigenvalues of A are given by the equation,

$$\frac{(1-\delta^2)^3}{\delta^2}=\left(\frac{3\mu}{2a}\right)^2,\quad \delta^2=\frac{1}{A}\,, \qquad \cdots\cdots(21)$$

and the eigenfunctions are

$$W=\begin{cases} x^{\frac{1}{2}}Z(cx^{\frac{3}{2}}), & x=1-\delta^2-\zeta \\ x^{\frac{1}{2}}X(cx^{\frac{3}{2}}), & x=\zeta+\delta^2-1 \end{cases} \qquad \cdots\cdots(22)$$

where Z and X are expressed by the Bessel functions as below,

$$Z(t)=J_{\frac{1}{3}}(t)+J_{-\frac{1}{3}}(t)\,, \qquad \cdots\cdots(23)$$

$$X(t)=i^{\frac{1}{3}}J_{-\frac{1}{3}}(it)-i^{-\frac{1}{3}}J_{\frac{1}{3}}(it)=\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}(t). \qquad \cdots\cdots(24)$$

In equation (21) μ's are the positive zeros of Z. The smallest value of μ is $\mu_1=2.376$. In equation (22), $c=2a/3\delta$. (When the WKB method is applied, the equation for eigenvalue is obtained in the same form as eqaution (21) if $3\mu/2=3.564$ in the right-hand side is replaced by $9\pi/8=3.534$. Thus the WKB mehtod gives a good approximation.)

(4) The case $p=1$ and $\alpha\neq 0$ (semi-infinite).

The eigenvalues $A(a)$ are obtained by the equation,

$$A=(1-e^{-\alpha})\left(\frac{\alpha\mu}{2a}\right)^2, \qquad \cdots\cdots(25)$$

where μ's are the positive zeros of the Bessel function J_ν. The order of the Bessel function, ν, is determined by the following equation for given a and α,

$$a=\frac{\alpha}{2}\sqrt{\nu^2-\mu^2e^{-\alpha}}\,. \qquad \cdots\cdots(26)$$

The eigenfunctions are given by the equation,

$$W=J_\nu(\mu e^{-\frac{\alpha\zeta}{2}})\,. \qquad \cdots\cdots(27)$$

When $\nu\to\infty$, the smallest positive zero of J_ν is $\mu_1=\nu+O(\nu^{\frac{1}{3}})$. Therefore, when $a\to\infty$, $A\to 1$ (for the fundamental mode).

(5) The case $p=\infty$ (infinite).

The eigenvalues A and the eigenfunctions W are given by the equations,

$$A=\frac{a^2+\theta^2}{a^2}\,,\quad W=\begin{cases}\cos\theta\zeta & (|\zeta|\leqq 1) \\ \cos\theta\cdot\exp\beta(1-|\zeta|) & (|\zeta|\geqq 1)\,,\end{cases} \qquad \cdots\cdots(28)$$

where θ are given by the following equation,

$$\tan\theta=\frac{\beta}{\theta}\,,\quad \beta=\sqrt{Q(a^2+\theta^2)+a^2}. \qquad \cdots\cdots(29)$$

(6) The case $p=2$ and $\alpha=0$ (infinite).

The expression for the eigenfunctions and the eigenvalues are the same as the semi-infinite case (2), but in the present case n includes also even integers ($n=0$, 1, 2, 3, 4, ……).

The relation between $A^{-1}\propto\sigma(\sigma+q)$ and the wavenumber a is plotted in Figure 1 in the semi-infinite case, while the relation between A^{-1} and $a_*=2a$ is plotted in the infinite case to compare both cases.

Secondly, we apply the variational method and obtain the lowest eigenvalue $A(a)$. In this method, the exponential trial function (10) or (12) gives the results which agree with the exact eigenvalues within the error of drawing (a few percent when $a\gtrsim 1$) except the neighborhood of the case $p=\infty$ and $Q=\infty$ where the trigo-

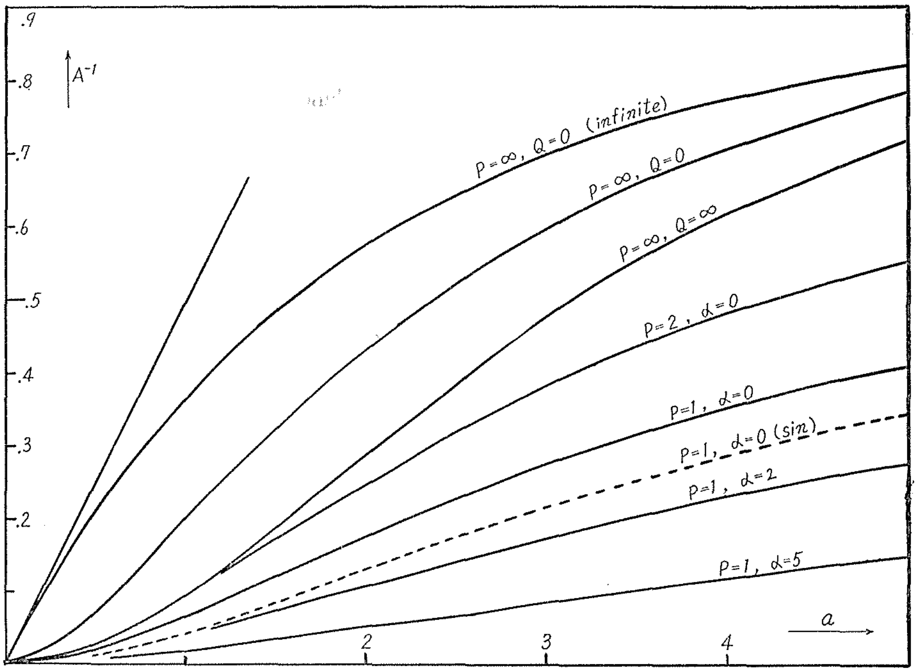

Fig. 1 The eivgenvalues A^{-1} as functions of wavenumber of disturbance, a, for some sets of values of the parameters p and α contained in the superadiabatic gradient $f(\zeta)$. All curves represent the results in the semi-infinite case except the curve labeled by "infinite". The dashed curve labeled by "sin" is the result by using the variational method with the trigonometric function (11).

nometric function (11) or (13) is the exact solution.

5. Discussion

First, we discuss the semi-infinite case. As A^{-1} is proportional to $\sigma(\sigma+q)$, the behavior of $A^{-1}(a)$ corresponds to that of the growth rate $\sigma(a)$.

(1) The case $p=\infty$.

The parameter Q means the depth of the superadiabatic gradient $f(\zeta)$ in the stable region ($\zeta>1$). When Q decreases from ∞ (the classical case where definite boundaries exist at $\zeta=0$ and $\zeta=1$) to 0, the superadiabatic gradient $f(\zeta)$ becomes 0 in the stable region and remains constant ($f=1$) in the unstable region ($0<\zeta<1$). Therefore the system will become more unstable and the flow penetrates more deeply into the stable region, as Q decreases. Correspondingly, the eigenvalue $A^{-1}(a)$ or the growth rate σ increases when Q decreases (Figure 1).

(2) The case $p=1$.

When α increases ($\alpha\gg1$), the width of the region where the superadiabatic gradient $f(\zeta)$ is large ($f\approx1$) decreases as inversely proportional to α and the other region becomes convectively neutral ($f\approx0$). Both effects act in opposite direction on the instability of the system. As a whole, the contraction of the unstable region has the dominant effect on the instability of the system and it becomes less unstable when α increases.

The case $p=2$ is guessed to have a similar tendency to the case $p=1$ when α varies.

Secondly, in the infinite case the system is more unstable than in the semi-infinite case. Especially, the behavior of $A^{-1}(a)$ near $a=0$ is linear in the case $p=\infty$ and $Q=0$, while it is parabolic in all other cases. This may be due to the fact that the region where $DW\neq 0$ goes away to infinity in that case (see equation (9)).

Now we examine an approximation in which the actual profile of $f(\zeta)$ is replaced by a box-type profile (the case $p=\infty$ and $Q=\infty$) of equal area $\int_0^1 f(\zeta)d\zeta$. In Table 1 the values of Aa^2 in the limit $a\to 0$ estimated by this approximation (third column) are compared with the exact values (last column).

Table 1. The approximate and exact values of Aa^2 in the limit $a\to 0$.

cases	area $=\int_0^1 f\,d\zeta$	Aa^2, approximate	Aa^2, exact
$p=1$, $\alpha=0$	1/2	$2\cdot\pi^2=20$	13
$p=2$, $\alpha=0$	2/3	$3/2\cdot\pi^2=15$	9
$p=2$, $\alpha=0$, infinite	2/3($\times$2)	$3/2\cdot\pi^2=15$	$Aa_*^2=4$
$p=\infty$, $Q=\infty$	1	π^2	π^2

Both values of Aa^2 differ from each other by a factor 2~4. Thus this approximation is not so good. Another approximation taking into account the profile of $f(\zeta)$ in the stable region should be adopted. For example, the following approximation is suggested.

$$Af(\zeta) \to \begin{cases} A\dfrac{\langle fW^2\rangle_{f>0}}{\langle W^2\rangle_{f>0}}=\bar{A} & (0\leq|\zeta|<1) \\ A\dfrac{\langle fW^2\rangle_{f<0}}{\langle W^2\rangle_{f<0}}=-\bar{A}\bar{Q} & (|\zeta|>1) \end{cases} \qquad \cdots\cdots(30)$$

The validity of this approximation is examined by the author in the study of the marginal stability problem (Yamaguchi 1967). The function W should be guessed appropriately, e. g. the function given by equation (10) or (12) may be useful.

When $a\to\infty$, the eigenvalues of A approach to a common limit $A=1$ in all cases. This is due to the fact that when the scale of disturbance becomes small $(a\to\infty)$ the equality $\langle fW^2\rangle=\langle W^2\rangle$ is realized in equation (9). The equality $\langle fW^2\rangle=\langle W^2\rangle$ means that the flow is confined to a small region near $\zeta=0$ where the gradient is large $(f\approx 1)$.

Finally, we mention the behavior of the eigenfunction. It is shown from equations in section 4 that the penetration of the flow into stable regions increases when $f(\zeta)$ becomes 0 in the stable region (α increases). When the wavenumber of disturbance a increases, the flow is confined to a smaller region near $\zeta=0$. This agrees with the consequence that $A\to 1$ when $a\to\infty$.

6. Concluding Remarks

In this paper we got the exact solutions (the convective growth rates σ and the corresponding eigenfunctions) for some profiles of the superadiabatic gradient. The behavior of the eigenvalue $A^{-1}(a)$ at small wavenumbers ($0<a\lesssim 5$) depends fairly sensitively on the shape of the gradient (Figure 1). Therefore, in discussing the stellar convection the variation and the changes of sign of the superadiabatic gradient should be taken into account.

The variational method gives a good approximtion to the eigenvalue within the limit of the error of a few percent (except very small wavenumbers of disturbance), if a suitable trial function (exponential function as given by equation (10) or (12)) is used, though it includes only one parameter.

Equation (1) is not of ordinary Sturm-Liouville type, since the parameter A is multiplied by the function $f(\zeta)$ which is not positive definite. It is noted that, when we expand the field variables into Fourier-like series in the non-linear theory (see Ledoux, Schwarzschild, and Spiegel 1961), the eigenfunctions obtained in this paper should be used carefully. If a function can be expanded by the system of the eigenfunctions, this expansion is unique, but the completeness of this system and the convergence of the series should be carefully examined. The above remark will be applied also to all investigations of stellar convection, if we try to take into account the variation and the changes of sign of the superadiabatic gradient.

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Note added in proof; After sending the manuscript, we have found that there is another eigensolution belonging to negative eigenvalues of $A^{-1}\alpha\sigma(\sigma+q)$. This mode gives an oscillation damping in time and shows the character of gravity wave.

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