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GRAVITATIONAL INSTABILITY OF A COMPRESSIBLE POLYTROPIC CYLINDER SURROUNDED BY HALO GAS

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ABSTRACT

The critical wavelength of the gravitational varicose instability in a compressible polytropic cylinder is found for various halo pressures. It is found that the halo pressure stabilizes the varicose instability.

1. Introduction

Chandrasekhar (1961) has found that an infinite cylinder of an incompressible fluid is gravitationally unstable for all axisymmetric varicose deformations with wavelengths exceeding six times as large as its radius. In the case of a homogeneous compressible cylinder, the varicose instability is not manifested as a separate mode but appears to enhance the instability of the convective modes (Ostriker 1964). However, in a polytropic compressible cylinder the varicose instability is isolated when the cylinder is convective stable (Robe 1967).

In this paper we consider a compressible and polytropic cylinder surrounded by the halo gas with a finite pressure, which is light and has a large sound velocity so that it will continue to exert a constant pressure on the perturbed surface of the cylinder, and study the effect of the halo pressure on the critical wavelength of the varicose instability. We confine ourselves to a polytropic configuration of equilibrium with $\gamma=2$ and suppose that the perturbations also obey a polytropic equation of state with the same γ. The convective modes are then critically stable according to Schwarzshild's criterion and so the varicose instability is separated from them. Overstability never occurs in our case. In fact, the proof that ω^2 is real can be easily made in a way similar to Goldreich and Lynden-Bell (1965). Here exp $(i\omega t)$ is the time dependence of the perturbations. Therefore we may discover the critical wavelength by examining only the behavior of the solution which has $\omega=0$.

2. Basic Equations

The equilibrium distribution of density for $\gamma=2$ is given by

$$\frac{\rho}{\rho_a}=J_o\,(k_J r), \tag{1}$$

where ρ_a is the density on the axis and k_J is the Jeans wavenumber

$$k_J=\left(\frac{4\pi G\rho}{c^2}\right)^{1/2}, \tag{2}$$

which is constant across the cylinder. The cylinder ends at the surface $r=R$ where the halo pressure p_h is attained. Thus the radius of the cylinder is given by

$$J_o(k_J R)=\left(\frac{p_h}{p_a}\right)^{1/2}, \tag{3}$$

where p_a is the pressure on the axis.

The method adopted in the sequel will in main be that of Goldreich and Lynden-Bell (1965).

The perturbed quantities of density, pressure, velocity and potential are shown by ρ_1, $\boldsymbol{u}$, p_1 and ϕ_1. The r and z components of $\boldsymbol{u}$ are denoted by u_r and u_z. Using the assumption that both the perturbations and the equilibrium configuration obey the same polytropic law and the relation $p_1=c^2\rho_1$, the equation govering the perturbed quantities are

$$\frac{\partial p_1}{\partial t}+\text{div}\,(\rho\boldsymbol{u})=0, \tag{4}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}=-\text{grad}\left(\phi_1+c^2\frac{\rho_1}{\rho}\right), \tag{5}$$

$$\nabla^2\phi_1=4\pi G\rho_1. \tag{6}$$

We consider only the normal modes in the form exp $(ikz+i\omega t)$. We shall introduce a new variable Θ defined by

$$\Theta=c^2\frac{\rho_1}{\rho}. \tag{7}$$

Equations (4), (5) and (6) are reduced to

$$i\omega\rho_1+ik\rho u_z+\frac{1}{r}\frac{d}{dr}(r\rho u_r)=0, \tag{8}$$

$$i\omega u_r=-\frac{d}{dr}(\phi_1+\Theta), \tag{9}$$

$$i\omega u_z=-ik(\phi_1+\Theta), \tag{10}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_1}{dr}\right)-k^2\phi_1=\frac{4\pi G\rho}{c^2}\Theta. \tag{11}$$

One of the boundary conditions is derived from the fact that the pressure on the surface is always equal to the halo pressure, i.e., the Lagrangian change in pressure vanishes on the surface. Let the radial Lagrangian displacement of the surface element be ξ_r, which is related to u_r as $u_r=i\omega\xi_r$. We have

$$\Theta\Big|_R=\xi_r\frac{d\phi}{dr}\Big|_R. \tag{12}$$

Another boundary condition is determined by continuity of potential and Gauss flux theorem. Because the halo gas is light, the perturbed potential outside the cylinder is expressed by $\phi_1(r)\sim K_0(kr)$. Thus we have

$$-k\frac{K_1(kR)}{K_0(kR)}\phi_1\Big|_R-\frac{d\phi_1}{dr}\Big|_R=4\pi G\xi_r\rho\Big|_R. \tag{13}$$

3. Critical Wavelengths

As stated in Introduction, we consider only the critical modes which have $\omega=0$. When $\omega=0$, we found from equation (10) that

$$\phi_1=-\Theta. \tag{14}$$

The substitution of equation (14) into equation (11) gives us

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Theta}{dr}\right)+(k_J{}^2-k^2)\,\Theta=0. \tag{15}$$

The solution finite at the origin is given by

$$\Theta(r)=\begin{cases}J_0(\sqrt{k_J{}^2-k^2}\,r), & k\leq k_J,\\ I_0(\sqrt{k^2-k_J{}^2}\,r), & k>k_J,\end{cases} \tag{16}$$

where we take $\Theta(0)=1$.

The critical wavenumber may be determined from the requirement that the solutions (16) satisfy the boundary conditions (12) and (13). The combination of the two boundary conditions and equation (14) gives

$$k\frac{K_1(kR)}{K_0(kR)}\Theta\Big|_R+\frac{d\Theta}{dr}\Big|_R=\frac{4\pi G\rho}{\dfrac{d\phi}{dr}}\Theta\Big|_R. \tag{17}$$

From the equation of hydrostatic equilibrium we obtain

$$\frac{d\phi}{dr}=-\frac{c^2}{\rho}\frac{d\rho}{dr} \tag{18}$$

Making use of equations (16), (17), (18), (1) and (2) and defining

$$Z=k_JR,\qquad K=kR, \tag{19}$$

we obtain the equation determining the critical wavenumber K,

$$K\frac{K_1(K)}{K_0(K)}-\sqrt{Z^2-K^2}\frac{J_1(\sqrt{Z^2-K^2})}{J_0(\sqrt{Z^2-K^2})}=Z\frac{J_0(Z)}{J_1(Z)},\quad K\leq Z, \tag{20}$$

$$K\frac{K_1(K)}{K_0(K)}+\sqrt{K^2-Z^2}\frac{I_1(\sqrt{K^2-Z^2})}{I_0(\sqrt{K^2-Z^2})}=Z\frac{J_0(Z)}{J_1(Z)},\quad K>Z. \tag{21}$$

In Figure 1 we plot the critical wavenumber K against the radius Z. Notice that $0\leq Z\leq 2.405$, as judged from equation (3). The case $Z=2.405$ corresponds to the limit of zero halo pressure. The critical wavenumber in this case is $K=1.7527$, whose wavelength is $3.6R$. The case $Z=0$ means an infinitesimal thin cylinder with $p_a=p_h$. In this case, equation (20) is reduced to

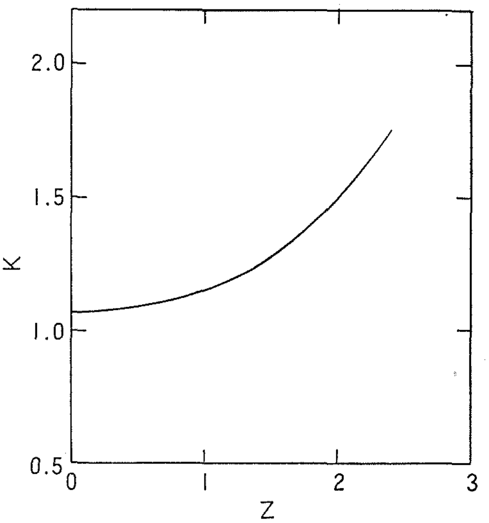

Fig. 1. The critical wavenumber K plotted against the radius Z.

$$K_0(K)\,I_0(K) = \frac{1}{2}, \tag{22}$$

which offers Chandrasekhar's critical wavenumber $K=1.0667$ for an incompressible cylinder. As Z decreases, i.e., as the halo pressure increases, the critical wavenumber decreases. Thus the halo pressure stabilizes the gravitational varicose instability.

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