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THE OSCILLATIONS OF AN INCOMPRESSIBLE VISCOUS CYLINDER

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ABSTRACT

The characteristic frequencies for non-axisymmetric oscillations of an incompressible viscous cylinder are determined. The viscous damping of the oscillations is discussed.

1. Introduction

The viscous damping is one of the important elementary processes for energy dissipation in an oscillating body. The oscillation of a viscous liquid sphere has been solved by LAMB (1932) and CANDRASEKHAR (1961). CHANDRASEKHAR (1961) has also studied the effect of viscosity on the gravitational instability of an incompressible viscous cylinder. In this study he has restricted himself to axisymmetric perturbations because the cylinder is supposed to be stable for all non-axisymmetric perturbations. The non-axisymmetric deformations are important in the study of the stability of a rotating fluid mass (see ISHIZAWA 1974). In view of such possible applications, we shall study the non-axisymmetric oscillations of a non-rotating viscous cylinder.

2. Characteristic equations and frequencies

We consider non-axisymmetric departures from an equilibrium cylindrical shape of an incompressible fluid. A normal mode can be expressed uniquely in terms of the deformed surface. The deformed surface is described by the equation

$$r=R+\epsilon e^{im\varphi}, \tag{1}$$

$$\epsilon=\epsilon_0 e^{-\sigma t}, \tag{2}$$

where R is the radius of the unperturbed cylinder, m is an integer, and σ is a characteristic frequency to be determined.

The characteristic frequencies of an inviscid fluid, namely, the cylindrical Kelvin frequencies are obtained by OSTRIKER (1964) as

$$\sigma^2=2\pi G\rho(m-1), \qquad m=2, 3, 4, \cdots. \tag{3}$$

The characteristic frequencies in the viscous case are determined so that the velocities, the solutions of the perturbation equations governing the departures from an equilibrium state, satisfy the boundary conditions: (a) the radial component must be compatible with the assumed form of the deformed surface given by equation (1); (b) the tangential viscous stresses must vanish at $r=R$; and (c) the (r, r)-component of the total stress tensor must vanish on the deformed surface. The method is in main that of CHANDRASEKHAR (1961).

Putting

$$\beta=(2\pi G\rho)^{1/2}R^2/\nu, \tag{4}$$

$$x=\sqrt{\sigma R^2/\nu}, \tag{5}$$

$$Q_m(x)=xJ_{m+1}(x)/J_m(x), \tag{6}$$

we obtain the characteristic equation

$$(m-1)\beta^2=2m(m-1)x^2\left[1+\frac{x^2-2mQ_m(x)}{x^2-2Q_m(x)}\right]-x^4\equiv\Phi_m(x). \tag{7}$$

This characteristic equation is closely analogous to that in the case of a viscous liquid sphere (see CHANDRASEKHAR 1961).

The curve of $\Phi_m(x)$ is divided into an infinite number of separate sections by its sigular points at the roots of $x-2Q_m(x)=0$, $q_{m,s}$. The root $q_{m,s}$ is just before the root of $J_m(x)$, $j_{m,s}$. In the first section between the origin and $q_{m,1}$, the value of $\Phi_m(x)$ gradually increases with x and reaches a maximum and then decreases, as shown in Figure 1. Between $q_{m,s}$ and $j_{m,s}$, $\Phi_m(x)$ rapidly falls down from a positive infinity to a negative value. In such a singular part, we can always find an aperiodic mode, decaying without oscillation. However, we restrict the following discussions to the lowest modes found in the first section because these modes can survive the longest.

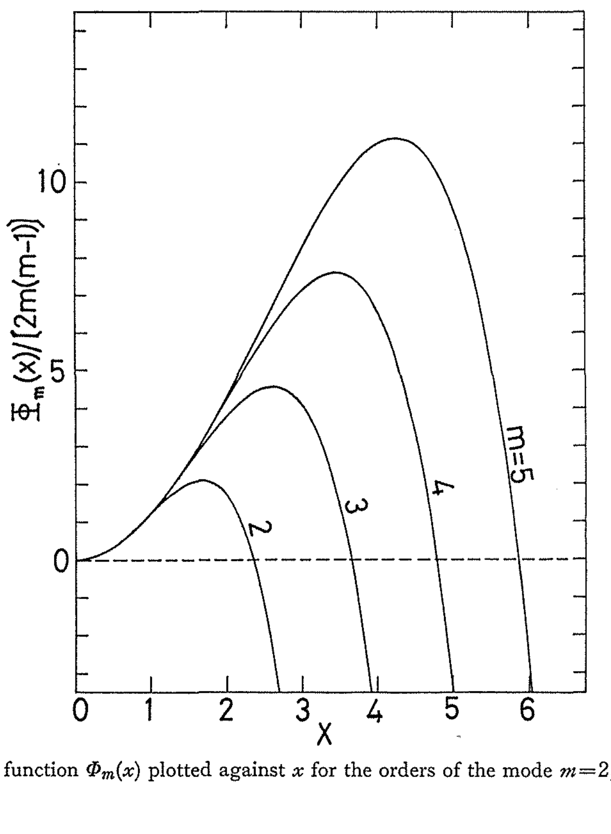

Fig. 1. The function $\Phi_m(x)$ plotted against x for the orders of the mode $m=2, 3, 4$ and 5.

From Figure 1, we observe that for any $\beta<\beta_{\max}$ there are two real roots of equation (7) which lead to the two aperiodic modes of decay: a creeping mode decaying very slowly for large viscosity and a viscous mode decaying ver rapidly for large viscosity. When $x\to 0$, we have

$$\Phi_m(x)=2(m-1)(m+1)x^2+O(x^4). \tag{8}$$

From equation (7), we find the solution satisfying $x\ll 1$,

$$x^2\approx\frac{\beta^2}{2(m+1)}. \tag{9}$$

From equations (4), (5) and (9), we obtain the characteristic frequency for the creeping mode

$$\frac{\sigma}{\sqrt{2\pi G\rho}}=\frac{\beta}{2(m+1)}. \tag{10}$$

The characteristic frequencies of the lowest aperiodic modes of decay for $m=2$, 3, 4 and 5 are given in Table 1 and plotted in Figure 2.

For $\beta>\beta_{\max}$, the characteristic frequencies of the lowest modes of decay are complex. If x is the root of equation (7), the complex conjugate $\bar{x}$ also is the root because $J_m(\bar{x})=\overline{J_m(x)}$ and $Q_m(\bar{x})=\overline{Q_m(x)}$. When $|x|\to\infty$ and $|x|\gg|Q_m(x)|$, we obtain from equations (4), (5) and (7)

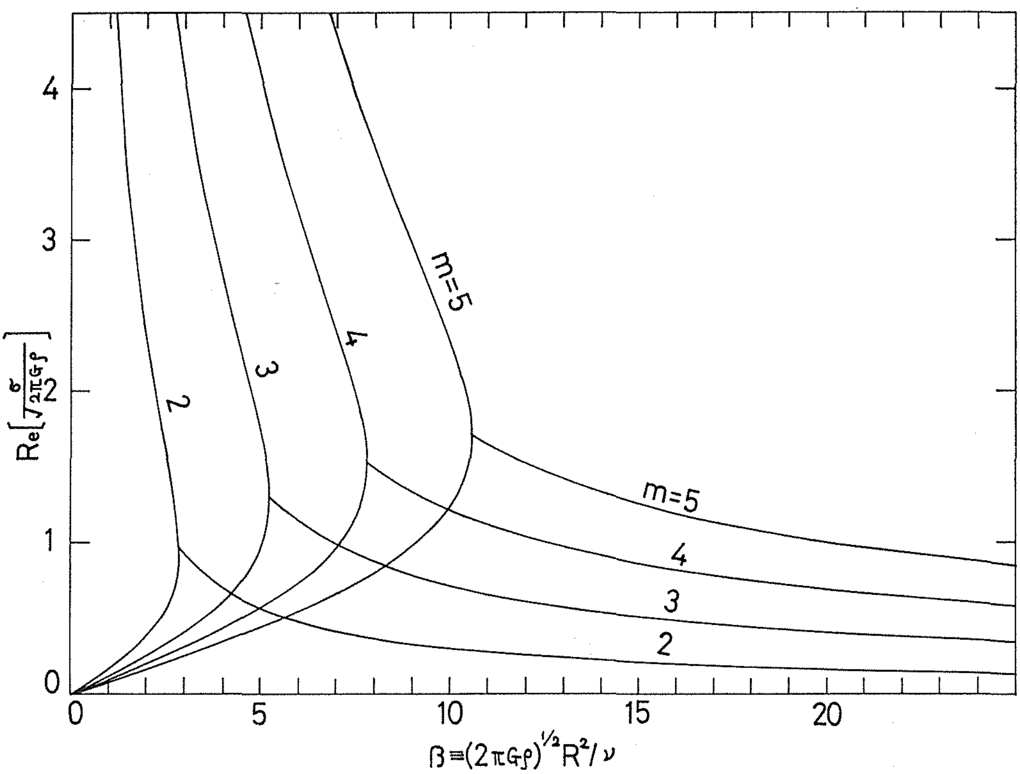

Fig. 2. The real parts of the characteristic frequencies of the lowest modes of decay in the oscillations of a viscous cylinder. The curves are labelled by the order of the mode m.

Table 1. The Characteristic Frequencies of the Lowest Aperiodic Modes of Decay in the Oscillations of a Viscous Clyinder

$m=2$			$m=3$			$m=4$			$m=5$		
β	$\sigma/\sqrt{2\pi G\rho}$		β	$\sigma/\sqrt{2\pi G\rho}$		β	$\sigma/\sqrt{2\pi G\rho}$		β	$\sigma/\sqrt{2\pi G\rho}$	
0.0	0.0	∞	0.0	0.0	∞	0.0	0.0	∞	0.0	0.0	∞
0.5	0.083598	11.000	1.0	0.12613	13.186	2.0	0.20327	11.287	2.0	0.16811	17.055
1.0	0.17190	5.3730	2.0	0.25956	6.4120	3.0	0.31155	7.3714	4.0	0.34560	8.3112
1.5	0.26928	3.4307	3.0	0.41104	4.0538	4.0	0.42932	5.3569	6.0	0.54614	5.2751
2.0	0.38629	2.3923	4.0	0.60423	2.7622	5.0	0.56328	4.0901	8.0	0.79953	3.6183
2.5	0.55214	1.6743	4.5	0.73794	2.2640	6.0	0.72710	3.1754	9.0	0.97252	2.9821
2.6	0.59955	1.5421	5.0	0.94923	1.7620	7.0	0.96090	2.4088	10.0	1.2384	2.3485
2.7	0.65811	1.4050	5.1	1.0202	1.6398	7.5	1.1605	1.9973	10.4	1.4445	2.0156
2.8	0.73918	1.2510	5.2	1.1299	1.4809	7.7	1.3084	1.7726	10.5	1.5453	1.8848
2.8985	0.96167	0.96167	5.2493	1.2936	1.2936	7.7941	1.5231	1.5231	10.556	1.7068	1.7068

Table 2. The Characteristic Frequencies of the Lowest Periodic Modes of Decay in the Oscillations of a Viscons Cylinder

$m=2$			$m=3$			$m=4$			$m=5$		
β	$\mathrm{Re}(\sigma)/\sqrt{2\pi G\rho}$	$\mathrm{Im}(\sigma)/\sqrt{2\pi G\rho}$	β	$\mathrm{Re}(\sigma)/\sqrt{2\pi G\rho}$	$\mathrm{Im}(\sigma)/\sqrt{2\pi G\rho}$	β	$\mathrm{Re}(\sigma)/\sqrt{2\pi G\rho}$	$\mathrm{Im}(\sigma)/\sqrt{2\pi G\rho}$	β	$\mathrm{Re}(\sigma)/\sqrt{2\pi G\rho}$	$\mathrm{Im}(\sigma)/\sqrt{2\pi G\rho}$
2.8985	0.96167	0.0	5.2493	1.2936	0.0	7.7941	1.5231	0.0	10.556	1.7068	0.0
3.0	0.92953	±0.24675	5.3	1.2818	±0.17552	8.0	1.4866	±0.33380	10.7	1.6858	±0.26888
3.5	0.79857	±0.53638	5.5	1.2371	±0.37969	8.5	1.4054	±0.59137	11.0	1.6441	±0.46262
4.0	0.70057	±0.65970	6.0	1.1387	±0.61655	9.0	1.3336	±0.74180	12.0	1.5207	±0.78366
5.0	0.56387	±0.78065	7.0	0.98491	±0.84352	10.0	1.2123	±0.93129	13.0	1.4173	±0.96361
6.0	0.47323	±0.83966	8.0	0.87054	±0.96397	12.0	1.0328	±1.1353	15.0	1.2539	±1.1781
8.0	0.36083	±0.89587	10.0	0.71237	±1.0919	15.0	0.85735	±1.2858	17.0	1.1314	±1.3061
10.0	0.29404	±0.92232	15.0	0.50626	±1.2172	20.0	0.68676	±1.4072	20.0	0.99655	±1.4269
15.0	0.20546	±0.95212	20.0	0.40401	±1.2710	25.0	0.58510	±1.4753	25.0	0.84793	±1.5462
20.0	0.16039	±0.96615	25.0	0.34045	±1.3037	30.0	0.51494	±1.5223	30.0	0.74990	±1.6228

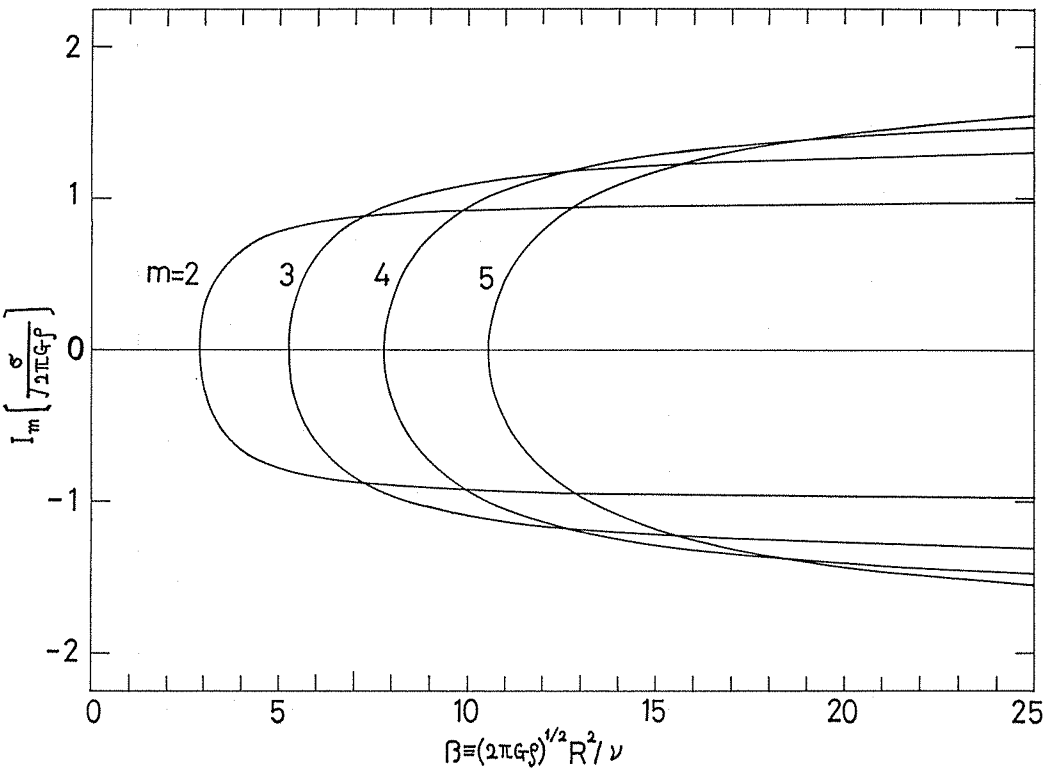

Fig. 3. The imaginary parts of the characteristic frequencies of the lowest periodic modes of decay in the oscillations of a viscous cylinder. The curves are labelled by the order of the mode m.

$$\frac{\sigma}{\sqrt{2\pi G\rho}}=\pm i\sqrt{m-1}+\frac{2m(m-1)}{\beta}. \tag{11}$$

The characteristic frequencies of the lowest periodic modes of decay for $m=2$, 3, 4 and 5 are given in Table 2 and plotted in Figures 2 and 3.

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