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ON FREEMAN'S COLLISIONLESS STELLAR SYSTEMS

BY

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ABSTRACT

The structure of Freeman's two dimensional elliptical collisionless stellar systems is re-studied in the revised range of the parameters b/a and $\Omega^2/2\pi G\rho$(the axial ratio and the angular velocity of the cylinder). It is found that Freeman's systems contain rigidly rotating circular (Maclaurin) and elliptical (Jacobi) cylinders and that, if the angular velocity is less than that of the Jacobi elliptical cylinder, the mean circulation is in the direction of rotation and otherwise it is counter to the direction of rotation.

1. Introduction

FREEMAN (1966) has constructed self-gravitating infinite homogeneous rotating elliptical cylinders of stars under the condition that every stellar orbit touches the surface of the cylinder. However, he misunderstands this condition and imposes an unnecessray restriction upon the parameters(the axial ratio and the angular velocity of the cylinder) to obtain a uniform density. The aim of this paper is to show that, if this restriction is removed, the Freeman's cylinder can be a unified model of homogeneous cylinders including particulars so far known.

In the following sections, we use the same notations as FREEMAN. Equation numbers from FREEMAN (1966) appear in square breackets such as equation [2].

2. The restrictions upon the parameters b/a and $\Omega^2/2\pi G\rho$.

In a rotating uniform elliptical cylinder of axial lengths a, b ($a>b$), density ρ and angular velocity Ω, the orbit of a star, referred to axes rotating with the cylinder, is given by equation [9]:

$$x=A_\alpha \sin(\alpha t+\epsilon_\alpha)+A_\beta \sin(\beta t+\epsilon_\beta),$$
$$y=k_\alpha A_\alpha \cos(\alpha t+\epsilon_\alpha)-k_\beta A_\beta \cos(\beta t+\epsilon_\beta). \tag{1}$$

This orbit represents the superposition of a guiding center drift around an ellipse $x^2+y^2/k_\alpha^2=A_\alpha^2$ with frequency α in the sense of rotation and an epicyclic motion around an ellipse $x^2+y^2/k^2_\beta=A_\beta^2$ about the guiding center with frequency $\beta(\beta>\alpha)$ in the sense counter to the rotation. In general, the orbit occupies the doughnut-like area along the ellipse of the guiding center.

The condition that the orbit (1) touches the surface of the cylinder is given by equation [40]:

$$J \equiv (k_\beta^2 - k_\alpha^2)\left\{\frac{A_\alpha^2}{a^2 k_\beta^2 - b^2} + \frac{A_\beta^2}{b^2 - a^2 k_\alpha^2}\right\} = 1. \tag{2}$$

FREEMAN (1966) has found that the distribution function

$$f(J) = \frac{\rho}{\pi}\frac{ab\Lambda^2}{k_\alpha k_\beta}\delta(J-1), \tag{3}$$

$$\Lambda^2 = \frac{(k_\beta^2 - k_\alpha^2)\sigma^2}{(b^2 - a^2 k_\alpha^2)(a^2 k_\beta^2 - b^2)}, \tag{4}$$

satisfies both the time-independent collisionless Boltzman equation and the Poisson equation and produces a cylinder of uniform density ρ. However, he has considered that the cylinder should be formed only from stars really touching the surface of the cylinder. This is not true. The condition $J=1$ is the necessary condition for a stellar orbit to touch the surface of the cylinder, and not the sufficient condition. It also contains the non-touching cases when the x- or y-coordinate of the touching point is purely imaginary(that is, when γ^2 or μ^2 given by equation [38] becomes negative). Even in such cases, the whole orbit is inside the surface fo the cylinder. Because, when the orbit is inside the cylinder, the velocity satisfing the condition $J=1$ always exists as seen from equation [43], stars moving in the non-touching orbits contribute to the integral

$$\int f(J)\, d^2c = \rho. \tag{5}$$

FREEMAN's restriction given by equation [52], meaning that the cylinder is formed only from stars really touching the surface and some of them necessarily pass the center of the cylinder,

$$\frac{1}{k_\alpha^2} + \frac{1}{k_\beta^2} \geqslant \frac{2a^2}{b^2}, \tag{6}$$

therefore becomes unnecessary. There are always orbits which pass the center if the non-touching orbits are included. As an illustration, we show in figure 1 the areas occupied by the orbits satisfying $J=1$ for $a=b$ and $\Omega/\sqrt{2\pi G\rho} = \cos\frac{\pi}{9}$, in which case the condition (6) is satisfied. The condition $J=1$ then leads to

$$4(A_\alpha/a)^2 + \frac{4}{3}(A_\alpha/a)^2 = 1, \tag{7}$$

(see Section 3). It is found from figure 1 that the orbit for $(A_\alpha/a,\ A_\beta/a) = (1/4,\ 3/4)$, which is the only one touching case, does not pass the center and the orbit for $(A_\alpha/a,\ A_\beta/a) = (\sqrt{3}/4,\ \sqrt{3}/4)$ does not touch the surface though it passes the center. It is clear from this that the cylinder of uniform density can not be constructed only from stars touching the surface. The Freeman's second restriction that the gravitational force along the x-axis is equal or superior to the centrifugal force

$$\Omega^2 \leqslant \frac{4\pi G\rho b}{a+b}, \tag{8}$$

is assumed again. This restriction ensures the zero or positive velocity dispersion as shown later. Without loss of generality, we can restrict ourselves to

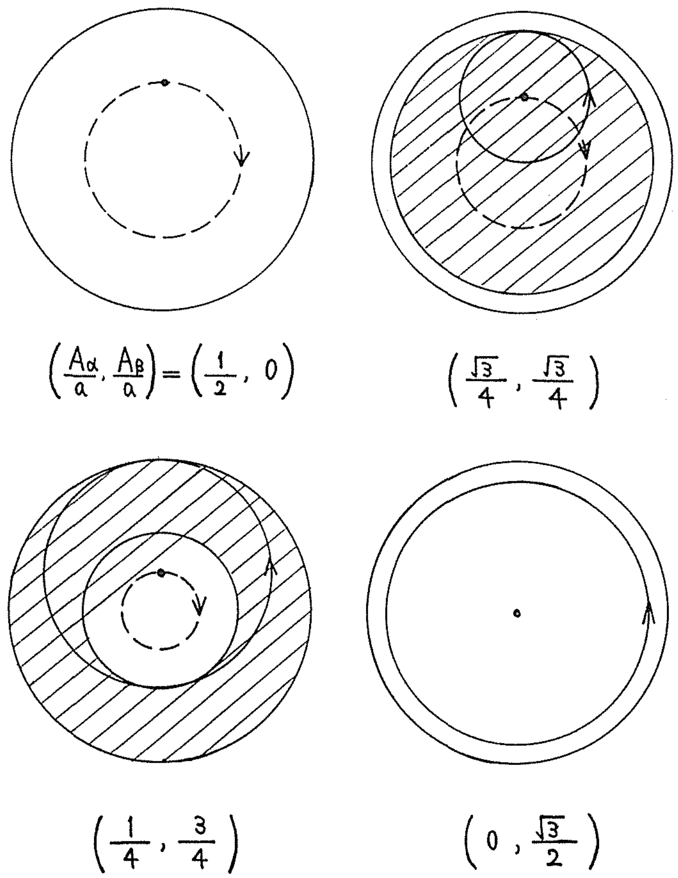

Fig. 1. The areas occupied by the orbits for $a=b$ and $\Omega/\sqrt{2\pi G\rho} = \cos\frac{\pi}{9}$. The epicycle gyration and the guiding center orbit are shown by the solid line (——) and the broken line (----), respectively.

$$b/a \leqslant 1 \quad \text{and} \quad \Omega \geqslant 0. \tag{9}$$

Except these restrictions, the parameters b/a and $\Omega_* \equiv \Omega/\sqrt{2\pi G\rho}$ are arbitrary.

3. The structure of Freeman's cylinders

Let us study the macroscopic dynamical structure of the cylinder in the revised range of the parameters. Using equations [49] and [50] with the identities for α, β, k_α and k_β [12]~[19], we can express the mean velocity and the velocity dispersion in terms of b/a and Ω_*. The mean velocity in the rotating frame is

$$\bar{c}_x = \frac{a^2}{a^2+b^2}\zeta y, \tag{10}$$

$$\bar{c}_y = -\frac{b^2}{a^2+b^2}\zeta x, \tag{11}$$

where ζ is the vorticity in the rotating frame given by

$$\zeta=-\frac{(a^2+b^2)(a+b)^2}{a^2b^2}\Omega\left\{\Omega_*{}^2-\frac{2ab}{(a+b)^2}\right\}. \tag{12}$$

The velocity dispersion is given by

$$\sigma_{xx}=\pi G\rho\frac{(a+b)^4}{b^2}\left\{\frac{2b}{a+b}-\Omega_*{}^2\right\}\left\{\Omega_*{}^2-\frac{ab}{(a+b)^2}\right\}^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right), \tag{13}$$

$$\sigma_{yy}=\pi G\rho\frac{(a+b)^4}{a^2}\left\{\frac{a+b}{2a}-\Omega_*{}^2\right\}\left\{\Omega_*{}^2-\frac{ab}{(a+b)^2}\right\}^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right), \tag{14}$$

$$\sigma_{xy}=0. \tag{15}$$

If $a=b$ or $\Omega_*{}^2=2ab/(a+b)^2$*), we find rigidly rotating cylinders with isotropic velocity dispersion: the Maclaurin circular cylinder and the Jacobi elliptical cylinder as follows.

(1) The Maclaurin circular cylinder

In the limit $a\to b$, we find from equation (2) and equation [10] and the identities [12]~[19] that the condition $J=1$ becomes

$$\frac{1}{2\left(\Omega_*+\frac{1}{2}\right)^2(1-\Omega_*)}\left(\frac{A_\alpha}{a}\right)^2+\frac{1}{2\left(\Omega_*-\frac{1}{2}\right)^2(1+\Omega_*)}\left(\frac{A_\beta}{a}\right)^2=1. \tag{16}$$

The mean velocity leads to

$$\bar{c}_x=-4\Omega\left(\Omega_*{}^2-\frac{1}{2}\right)y, \tag{17}$$

$$\bar{c}_y=4\Omega\left(\Omega_*{}^2-\frac{1}{2}\right)x. \tag{18}$$

The velocity dispersion is reduced to

$$\sigma_{xx}=\sigma_{yy}=16\pi G\rho(1-\Omega_*{}^2)\left(\Omega_*{}^2-\frac{1}{4}\right)^2(a^2-x^2-y^2). \tag{19}$$

If the angular velocity Ω_* is transformed into $\Omega_{*\mathrm{MAC}}$ defined by

$$\Omega_{*\mathrm{MAC}}=-4\Omega_*\left(\Omega_*{}^2-\frac{3}{4}\right), \tag{20}$$

equation (16) leads to

$$\frac{2}{1+\Omega_{*\mathrm{MAC}}}\left(\frac{A_\alpha}{a}\right)^2+\frac{2}{1-\Omega_{*\mathrm{MAC}}}\left(\frac{A_\beta}{a}\right)^2=1. \tag{21}$$

It can be easily shown that this is the mono-energy condition for a uniform circular cylinder of stars rigidly rotating with an angular veloctiy Ω_{MAC} that every star has one and the same energy referred to the rotating frame for which the surface of zero velocity coincides with the surface of the cylinder:

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*) These are deduced from the condition $b^2-a^2k_\alpha{}^2k_\beta{}^2=0$ (see equation [50]).

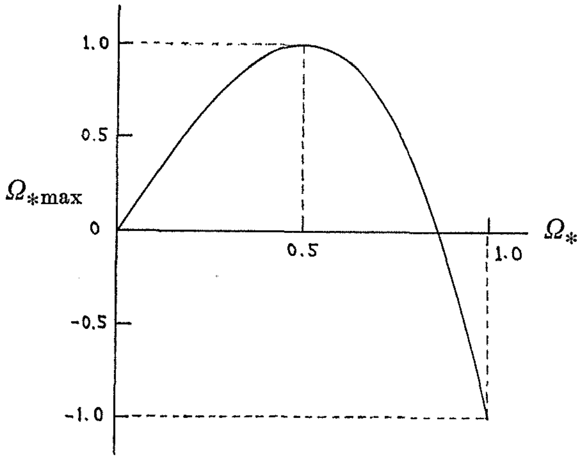

Fig. 2. The relation between Ω_* and $\Omega_{*\mathrm{MAC}}$.

$$\frac{c_x^2+c_y^2}{a^2(2\pi G\rho-\Omega_{\mathrm{MAC}}^2)}+\frac{x^2+y^2}{a^2}-1=0. \tag{22}$$

The mean velocity, referred to an inertial frame, is

$$\bar{c}_x+\Omega y=\Omega_{\mathrm{MAC}}y, \tag{23}$$

$$\bar{c}_y-\Omega x=-\Omega_{\mathrm{MAC}}x. \tag{24}$$

Also, the velocity dispersion becomes

$$\sigma_{xx}=\sigma_{yy}=G\rho(1-\Omega_{*\mathrm{MAC}}^2)(a^2-x^2-y^2). \tag{25}$$

Thus we find a rigidly rotating uniform circular cylinder (Maclaurin circular cylinder) found by BISNOVATYI-KOGAN (1971). As Ω_* increases from 0 to 1, $\Omega_{*\mathrm{MAC}}$ starts increasing from 0, attains a maximum of 1 at $\Omega_*=1/2$, and then decreases to -1 at $\Omega_*=1$ as shown in figure 2. At $\Omega_{*\mathrm{MAC}}=\pm1$, the gravitational force is equal to the centrifugal force and every star moves in a circular orbit so that the velocity dispersion vanishes, as seen from equation (25).

(2) The Jacobi elliptical cylinder

When $\Omega_*^2=2ab/(a+b)^2$, the condition $J=1$ is reduced to

$$\frac{\alpha k_\alpha}{2\sigma}A_\alpha^2+\frac{\beta k_\beta}{2\sigma}A_\beta^2=\frac{1}{2}a^2A_1^2=\frac{1}{2}b^2B_1^2.^{*)} \tag{26}$$

The left-hand side of this equation is the Hamiltonian referred to the rotating frame given by equation [27]. This condition is in agreement with the mono-energy condition for a rigidly rotating uniform elliptical cylinder of stars. From equations (10)~(14), we then find that

*) Here we make use of the equality

$$\beta k_\beta(b^2-a^2k_\alpha^2)-\alpha k_\alpha(a^2k_\beta^2-b^2)=\frac{b^2-a^2k_\alpha^2k_\beta^2}{\sigma}=0.$$

$$c_x=c_y=0, \tag{27}$$

$$\sigma_{xx}=\sigma_{yy}=\frac{2\pi G\rho a^2b^2}{(a+b)^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right). \tag{28}$$

Thus, in this case, we have a rigidly rotating uniform elliptical cylinder (Jacobi elliptical cylinder). This elliptical cylinder bifurcates from the Maclaurin circular cylinder with $\Omega_*=\Omega_{*\mathrm{MAC}}=1/\sqrt{2}$ as an incompressible elliptical cylinder (Jeans 1919).

There are other interesting particular cylinders. First, at $\Omega_*^2=2b/(a+b)$, the gravitational force in the x-direction is balanced with the centrifugal force and the velocity dispersion σ_{xx} vanishes as found from equation (13). Even then, σ_{yy} remains positive as $a>b$. Remember that we have before assumed that $\Omega_*^2\leqslant 2b/(a+b)$ in order to assure that the velocity dispersion is zero or positive. Second, at $\Omega_*^2=ab/(a+b)^2$, which is half the angular velocity of the Jacobi elliptical cylinder, we have a cold cylinder in which $\sigma_{xx}=\sigma_{yy}=0$. It is found from equations (13) and (14) that, for angular velocities larger as well as smaller than this angular velocity, the velocity dispersion is positive. Third, in the case of no rotation, we find that Freeman's cylinder is reduced to a stationary elliptical cylinder found by Bisnvatyi-kogan and Zeldovich (1970).

The angular momentum per unit length of an elliptical cylindêr L is

$$\frac{L}{\sqrt{GM^3ab}}=\frac{1}{\sqrt{2}}\Omega_*\frac{(a+b)^2}{ab}\left\{\frac{1}{2}+\frac{ab}{(a+b)^2}-\Omega_*^2\right\}, \tag{29}$$

where

$$M=\pi ab\rho. \tag{30}$$

At $\Omega_*^2=1/2+ab/(a+b)^2$, the angular momentum is zero although the elliptical form reminds us of non-zero angular momentum. For $\Omega_*^2>1/2+ab/(a+b)^2$, the angular momentum is in the opposite direction of the angular velocity. For a fixed value of b/a, the angular momentum attains a maximum at $\Omega_*^2=\{1/2+ab/(a+b)^2\}/3$.

In figure 3, we show the revised range of the parameters b/a and Ω_*^2 and plot the loci of particular cylinders discussed above:

Maclaurin circular cylinder	$a=b$,
Jacobi elliptical cylinder	$\Omega_*^2=2ab/(a+b)^2$,
cylinder of balance in the x-direction	$\Omega_*^2=2b/(a+b)$,
cold cylinder	$\Omega_*^2=ab/(a+b)^2$,
cylinder of zero angular momentum	$\Omega_*^2=1/2+ab/(a+b)^2$,
cylinder of maximum angular momentum	$\Omega_*^2=\{1/2+ab/(a+b)^2\}/3$.

If the angular velocity is smaller than that of the Jacobi elliptical cylinder, that is, if $\Omega_*^2<2ab/(a+b)^2$, the mean circulation is in the direction of rotation and, if $\Omega_*^2>2ab/(a+b)^2$, it is counter to the direction of rotation, although Freeman have considered that the mean motion in his cylinder is a circulation counter to the direction of rotation. In the case of an incompressible elliptical cylinder, the angular velocity and the vorticity in the rotating frame ζ' satisfy the relation

$$\Omega_*^2+\frac{a^2b^2}{(a^2+b^2)^2}\zeta_*'^2=\frac{2ab}{(a+b)^2}, \tag{31}$$

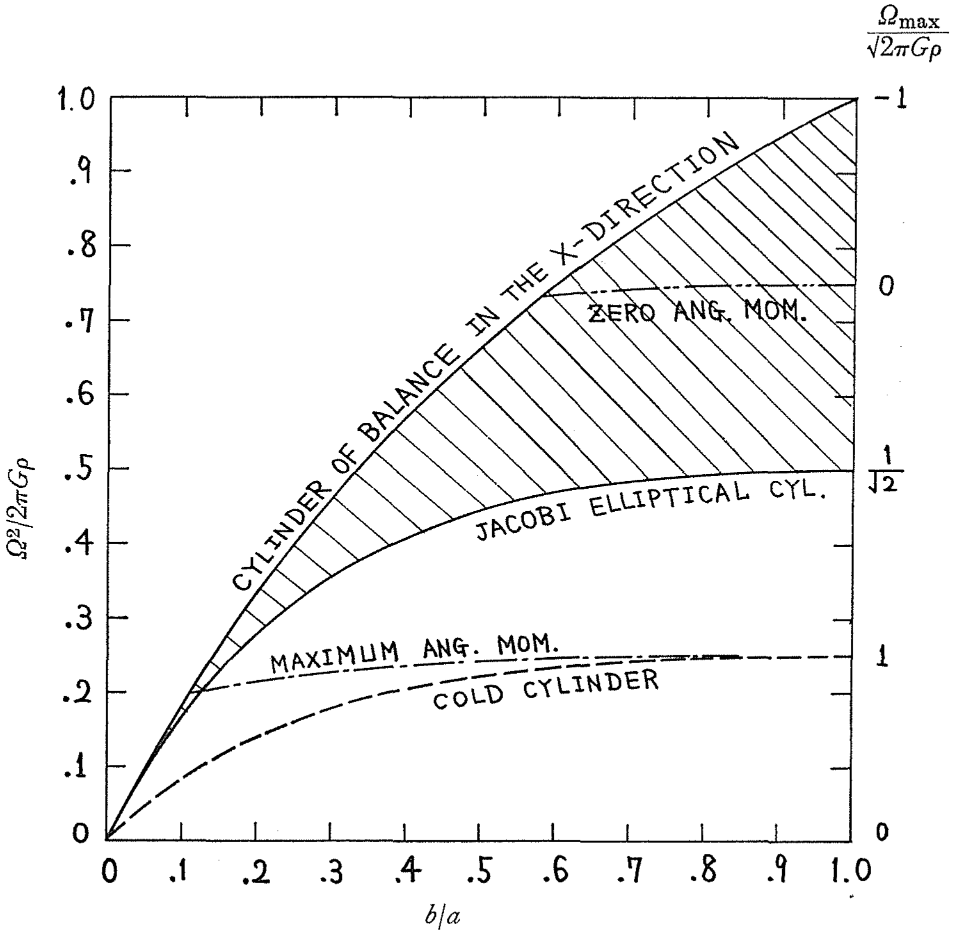

Fig. 3. The revised range of b/a and $\Omega^2/2\pi G\rho$ in which the velocity dispersion is zero or positive and the loci of particular cylinders. The angular velocities of rigid rotation for the Maclaurin circular cylinder $\Omega_{MAC}/\sqrt{2\pi G\rho}$ are given for $\Omega^2/2\pi G\rho=0$, 1/4, 1/2, 3/4, 1. In the shaded area, the mean cinculation is counter to the rotation.

where

$$\zeta_*'=\zeta'/\sqrt{2\pi G\rho}\ , \tag{32}$$

(Lamb 1932). Its angular velocity is always less than that of the Jacobi elliptical cylinder. This difference from collisionless stellar systems is caused from the characteristic of the fluid that the pressure is isotropic.

The total kinetic energy referred to an intertial frame per unit length of the cylinder K is

$$K=\frac{1}{2}\iint[(c_x+\Omega y)^2+(c_y-\Omega x)^2]fd^2rd^2c, \tag{33}$$

$$=\frac{1}{2}GM^2, \tag{34}$$

which is constant for a fixed value of M. The potential energy per unit length, normalized to that of the circular cylinder with the same M, U is

$$\frac{U}{GM^2}=\ln\frac{a+b}{2\sqrt{ab}}, \tag{35}$$

(ISHIZAWA 1974). The total energy per unit length, normalized as above, is

$$\frac{K+U}{GM^2}=\ln\frac{a+b}{2\sqrt{ab}}, \tag{36}$$

which depends only on the axial ratio. It is found from equations (34) and (35) that the virial theorem does not hold for the two-dimensional infinite body.

Note added in proof : After this paper was submitted, it is found that the same results have been obtained independently by C. Hunter (1974, Mon. Not. R. Astr. Soc. **166**, 633).

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