Irrationality of certain Lambert series

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1 Introduction and the results

For any fixed \( q \in \mathbb{C} \) with \(|q| > 1\) and \( z \in \mathbb{C} \), the \( q\)-logarithmic function \( L_q(z) \) and the \( q\)-exponential \( E_q(z) \) are defined by

\[
L_q(z) := \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{z}{q^n - z} \quad (|z| < |q|),
\]

\[
E_q(z) := 1 + \sum_{n=1}^{\infty} \frac{z^n}{(q-1) \cdots (q^n-1)} = \prod_{n=1}^{\infty} \left(1 + \frac{z}{q^n}\right),
\]

respectively. Bézivin [2] showed that the numbers \( 1, E_q^{(k)}(\alpha_i) (i = 1, \ldots, m, k = 0, 1, \ldots, l) \) are linearly independent over \( \mathbb{Q} \), where \( q \in \mathbb{Z} \setminus \{0, \pm 1\} \) and \( \alpha_i \in \mathbb{Q}^\times \) satisfy \( \alpha_i \neq -q^\mu \) and \( \alpha_i \neq \alpha_j q^\nu \) for all \( \mu, \nu \in \mathbb{Z} \) with \( \mu \geq 1 \) and \( i \neq j \). This implies that

\[
\sum_{n=1}^{\infty} \frac{1}{q^n + \alpha} \notin \mathbb{Q},
\]

where \( q \in \mathbb{Z} \setminus \{0, \pm 1\} \) and \( \alpha \in \mathbb{Q}^\times \) with \( \alpha \neq -q^i (i \geq 1) \). Under the same conditions on \( q \) and \( \alpha \), Borwein [3], [4] obtained irrationality measures for the numbers \( \sum_{n=1}^{\infty} 1/(q^n + \alpha) \) and \( \sum_{n=1}^{\infty} (-1)^n/(q^n + \alpha) \). These results include the irrationality of \( L_2(1) = \sum_{n=1}^{\infty} 1/(2^n - 1) \) proved by Erdős [10]. Furthermore, Bundschuh and Väänänen [6], and Matala-Aho and Väänänen [11] obtained quantitative irrationality results for the values of the \( q \)-logarithm both in the Archimedean and \( p \)-adic cases. In [7], Duverney generalized certain results obtained by Borwein [3], [4], and Bundschuh and Väänänen [6]. Recently, Van Assche [15] gave irrationality measures for the numbers \( L_q(1) \) and \( L_q(-1) \) by using little \( q \)-Legendre polynomials. In this paper, we prove irrationality results for certain Lambert series, which in particular implies the linear independence of the numbers \( L_q(1), L_q(-1) \) with \( q \in \mathbb{Z} \setminus \{0, \pm 1\} \) by developing Borwein's idea in [4].

Let \( R_n \) be a binary recurrence defined by

\[
R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0), \quad A_1, A_2 \in \mathbb{Q}^\times, \quad R_0, R_1 \in \mathbb{Q}.
\]
André-Jeannin [1] proved for some $R_n$ the irrationality of the value of the function $f(x) = \sum_{n=1}^{\infty} x^n / R_n$ at a nonzero rational integer $x$ in the disk of convergence of $f$, which gave the first proof of the irrationality of the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\sum_{n=1}^{\infty} 1/L_n$, where $F_n$ and $L_n$ are Fibonacci numbers and Lucas numbers, respectively. Prévost [13] extended this result to any rational $x$ in the domain of meromorphy of $f$. Recently, Matala-aho and Prévost [12] obtained for some type of $R_n$ irrationality measures for the number $\sum_{n=1}^{\infty} \gamma^n / R_{an}$, where $\gamma$ belongs to an imaginary quadratic field, and $a > 0$ is an integer. We will prove for some $R_n$ the irrationality of the numbers $\sum_{n=1}^{\infty} \gamma^n / R_{an+b}$ and $\sum_{n=1}^{\infty} \gamma^n / R_{an+b} R_{n+1+b}$, where $a > 0, b \geq 0$ are integers and $\gamma$ is a certain number in a real quadratic field (see Corollaries 2 and 3, below).

For an algebraic number $\alpha$, we denote by $|\alpha|$ the maximum of absolute values of its conjugates and by $\text{den} \alpha$ the least positive integer such that $\alpha - \text{den} \alpha$ is an algebraic integer. We define generalized Pisot number $\alpha$ by algebraic integer $\alpha$ satisfying $|\alpha| > 1$ and $|\alpha^\sigma| < 1$ for any $\sigma \in \text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\alpha^\sigma \neq \alpha$. We put $\mathbb{N} = \{0, 1, 2, \ldots \}$.

**Theorem 1.** Let $K$ be either $\mathbb{Q}$ or an imaginary quadratic field. Assume that $q$ is an integer in $K$ with $|q| > 1$ and $\{a_n\}$ a periodic sequence in $K$ of period two, not identically zero. Then

$$\theta = \sum_{n=1}^{\infty} \frac{a_n}{1 - q^n} \notin K.$$

**Corollary 1.** Let $q \in \mathbb{Z}$ with $|q| \geq 2$ and $\{a_n\}, \{b_n\}$ be periodic sequences in $\mathbb{Q}$ of period two, not identically zero. Then the numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{a_n}{q^n - 1}, \quad \sum_{n=1}^{\infty} \frac{b_n}{q^n - 1}$$

are linearly independent over $\mathbb{Q}$ if and only if $\{a_n\}$ and $\{b_n\}$ are linearly independent over $\mathbb{Q}$.

**Proof.** This follows immediately from Theorem 1.

**Example 1.** Let $q \in \mathbb{Z}$ with $|q| \geq 2$. Then

$$1, \quad L_q(1) = \sum_{n=1}^{\infty} \frac{1}{q^n - 1}, \quad L_q(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{-1}{q^n + 1}$$

are linearly independent over $\mathbb{Q}$.

**Theorem 2.** Let $q$ be a quadratic generalized Pisot number, $\gamma$ a unit in $\mathbb{Q}(q)$ with $|\gamma| \leq 1$, and $\alpha \in \mathbb{Q}(q)^{\times}$ with $(\text{den}(q^l \alpha))^4 < |q|$ for some $l \in \mathbb{N}$. Then

$$\xi = \sum_{n=1}^{\infty} \frac{\gamma^n}{1 - \alpha q^n} \notin \mathbb{Q}(q),$$

provided that $\alpha q^n \neq 1$ for all $n \geq 1$. 
In the following Corollaries 2 and 3, we consider the binary recurrences \{R_n\}_{n \geq 0} defined by
\[ R_{n+2} = A_1 R_{n+1} + A_2 R_n, \quad A_1, A_2 \in \mathbb{Z} \setminus \{0\}, \quad R_0, R_1 \in \mathbb{Z}. \]
We suppose that \( R_n \neq 0 \) for all \( n \geq 1 \), the corresponding polynomial \( \Phi(X) = X^2 - A_1 X - A_2 \) is irreducible in \( \mathbb{Q}[X] \), and \( \Delta = A_1^2 + 4A_2 > 0 \). We can write \( R_n \) as
\[ R_n = g_1 \rho_1^n + g_2 \rho_2^n \quad (n \geq 0), \quad g_1, g_2 \in \mathbb{Q}(\rho_1)^x, \]
where \( \rho_1 \) and \( \rho_2 \) are the roots of \( \Phi(X) \). We may assume \( |\rho_1| > |\rho_2| \), since \( \Delta > 0 \).

For \( a, b \in \mathbb{N} \) with \( a \neq 0 \), we define
\[ R(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} \quad (|z| < |\rho_1|^a). \]
This function can be extended to a meromorphic function on the whole complex plane \( \mathbb{C} \) with poles \( \{((\rho_1^{n+1}/\rho_2^n)^a | n \geq 0\} \), since
\[ \sum_{n=1}^{\infty} \frac{z^n}{1 - \alpha q^n} = \sum_{m=1}^{\infty} \frac{\alpha^{-m}z}{z - q^m} \quad (|z| < |q|) \]
for any complex numbers \( q \) and \( \alpha \) with \( |q| > 1 \) and \( |\alpha| \geq 1 \), and so
\[ \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} = \sum_{n=1}^{i} \frac{z^n}{R_{an+b}} - \frac{z^{i+1}}{g_1 \rho_1^{ai+b}} \sum_{n=0}^{\infty} \frac{(-(g_2/g_1)(\rho_2/\rho_1)^{ai+b})^n}{z - \rho_1^a(\rho_1/\rho_2)^{an}}, \tag{2} \]
where \( i \) is chosen as \( |(g_2/g_1)(\rho_2/\rho_1)^{ai+b}| < 1 \). We denote the function again by \( R(z) \).

**Corollary 2.** Let \( R_n \) be a binary recurrence given by (1) and \( a, b \in \mathbb{N} \) with \( a \neq 0 \). Assume that \( g_1/g_2 \) and \( \rho_1/\rho_2 \) are units in \( \mathbb{Q}(\rho_1) \) and \( \gamma \in \mathbb{Q}(\rho_1)^x \) is not a pole of \( R(z) \) with \((\text{den}(\rho_1^a/\gamma))^4 < |\rho_1/\rho_2|^a \). Then we have \( R(\gamma) \notin \mathbb{Q}(\rho_1) \).

**Proof.** Apply Theorem 2 to the last sum in (2).

**Example 2.** Let \( F_n \) and \( L_n \) be Fibonacci numbers and Lucas numbers defined by \( F_{n+2} = F_{n+1} + F_n \ (n \geq 0), \ F_0 = 0, \ F_1 = 1 \) and \( L_{n+2} = L_{n+1} + L_n \ (n \geq 0), \ L_0 = 2, \ L_1 = 1 \), respectively. Then for every \( a, b \in \mathbb{N} \) with \( a \neq 0 \),
\[ \sum_{n=1}^{\infty} \frac{1}{F_{an+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{an+b}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{an+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{an+b}} \notin \mathbb{Q}(\sqrt{5}). \]
André-Jeannin[1] proved that each of these numbers is irrational. We remark that the numbers \( \sum_{n=1}^{\infty} 1/F_{2n+1} \) and \( \sum_{n=1}^{\infty} 1/L_{2n} \) are transcendental (cf. [8], [9]).
Example 3. Let $F_n$ be Fibonacci numbers. Then for every $a, b \in \mathbb{N}$ with $a \neq 0$,
\[
\sum_{n=1}^{\infty} \frac{1}{F_{(2a-1)n+b}F_{(2a-1)(n+1)+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{2an+b}F_{2a(n+1)+b}} \notin \mathbb{Q}(\sqrt{5}).
\]
The same holds for Lucas numbers. We put
\[
T_l := \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+l}}, \quad T_l^{*} := \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+l}} \quad (l \geq 1).
\]
Then Brousseau [5] and Rabinowitz [14] proved that
\[
T_{2l} = \frac{1}{F_{2l}} \sum_{n=1}^{l} \frac{1}{F_{2n-1}F_{2n}}, \quad T_{2l+1} = \frac{1}{F_{2l+1}} \left( T_1 - \sum_{n=1}^{l} \frac{1}{F_{2n}F_{2n+1}} \right),
\]
so that $T_{2l} \in \mathbb{Q}$ and $T_l^{*} \in \mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q}$ for all $l \geq 1$. We see that $T_{2l+1} \notin \mathbb{Q}(\sqrt{5})$ for all $l \geq 0$, since the first sum in this example with $a = 1, b = 0$ implies
\[
T_1 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} \notin \mathbb{Q}(\sqrt{5}).
\]

2 Lemmas

For the proof of theorems, we prepare some lemmas. Let $\{a_m\}_{m \geq 1}$ be a periodic sequence of complex numbers of period two, not identically zero. We put
\[
\theta = \sum_{m=1}^{\infty} \frac{a_m}{1 - q^m},
\]
where $q \in \mathbb{C}$ with $|q| > 1$. We start with the integral
\[
F_n(q) = \frac{1}{2\pi i} \int_{|t|=1} \frac{(-1/t)^{2n}}{\prod_{k=1}^{\infty} (1 - q^k/t)} \sum_{m=1}^{\infty} \frac{a_m}{1 - q^m/t} dt,
\]
which is a variant of that used by Borwein [4]. We note that the integrand is meromorphic in $t$ provided $|q| > 1$. We use the notations
\[
[n]_q! := \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q)}{(1 - q)^n}, \quad [0]_q := 1,
\]
$\binom{n}{i}_q := \frac{[n]_q!}{[i]_q![n-i]_q!} \in \mathbb{Z}[q].$

In what follows, we denote $c_1, c_2, \ldots$ positive constants independent of $n$.

**Lemma 1.**

\[ F_n(q) = \sum_{i=1}^{n} \prod_{k=1}^{2n} (1-q^{k+2i}) \left( \theta - \sum_{m=1}^{2i} \frac{a_m}{1-q^m} \right) \]

Proof. This can be proved by using the residue theorem similarly as the proof of Lemma 1 in [4].

We put $D_n(q) := \prod_{k=n+1}^{2n} (1-q^{2k})$. Then we have

\[ |D_n(q)| \leq c_1 |q|^{3n^2+n}. \]

**Lemma 2.**

\[ D_n(q)F_n(q) = A_n(q)\theta + B_n(q), \]

where $A_n(q), B_n(q) \in \mathbb{Z}[a_1, a_2, q]$.

Proof. Since

\[ \frac{1}{\prod_{k=1}^{n} (1-q^{2k})} = \prod_{k=1}^{n-1} \frac{1}{(q^{2k}-1) \prod_{k=n+1}^{2n} (1-q^{2k})}, \]

we have by (4)

\[ F_n(q) = \frac{1}{\prod_{k=1}^{n} (1-q^{2k})} \sum_{i=1}^{n} (-1)^{i-1} q^{i(i-1)} \left[ \binom{n-1}{i-1} \prod_{k=1}^{2n} (1-q^{k+2i}) \right] \left( \theta - \sum_{m=1}^{2i} \frac{a_m}{1-q^m} \right) \]

\[ - \sum_{\lambda, \mu, \nu \geq 0, \lambda + \mu + \nu = 2n-1} \frac{1}{\lambda!\mu!\nu!} \left( \prod_{k=1}^{2n} (t-q^k) \right)^{(\lambda)} \left( \prod_{k=1}^{n} (1-q^{2k})^{-1} \right)^{(\mu)} \left( \sum_{m=1}^{\infty} \frac{a_m}{t-q^m} \right)^{(\nu)} \Rightarrow \theta \]
with
\[
\left( \prod_{k=1}^{2n} (t - q^k) \right)^{(\lambda)} \bigg|_{t=0} = \lambda! (-1)^{2n-\lambda} \sum_{\lambda_1 + \cdots + \lambda_{2n} = 2n-\lambda} q^{\lambda_1 + 2\lambda_2 + \cdots + 2n\lambda_{2n}},
\]
\[
\left( \prod_{k=1}^{n} (1 - q^{2k} t)^{-1} \right)^{(\mu)} \bigg|_{t=0} = \mu! \sum_{\mu_1 + \cdots + \mu_n = \mu} q^{2(\mu_1 + 2\mu_2 + \cdots + n\mu_n)},
\]
\[
\left( \sum_{m=1}^{\infty} \frac{a_m}{t - q^m} \right)^{(\nu)} \bigg|_{t=0} = -\nu! \sum_{m=1}^{\infty} \frac{a_m}{(q^{\nu+1})^m} = \nu! (a_1 q^{\nu+1} + a_2) \frac{1}{1-q^{2(\nu+1)}}.
\]

Hence we get
\[
F_n(q) = \frac{1}{\prod_{k=1}^{n-1} (1 - q^{2k})} \sum_{i=1}^{n} (-1)^{i-1} q^{i(i-1)} \left[ \frac{n-1}{i-1} \right] \prod_{k=1}^{2n} (1 - q^{k+2i}) \left( \frac{2i}{q^2} \prod_{k=1}^{i} \frac{a_m}{1-q^m} \right)
\]
\[
+ \sum_{\lambda, \mu, \nu \geq 0} Q_{\lambda\mu\nu}(q) \frac{1}{1-q^{2(\nu+1)}}
\]
(7)

with \(Q_{\lambda\mu\nu}(q)\) a polynomial in \(\mathbb{Z}[a_1, a_2, q]\) for all \(\lambda, \mu, \nu \geq 0\). Here we note that
\[
\prod_{k=1}^{2n} (1 - q^{k+2i}) \sum_{m=1}^{\infty} \frac{a_m}{1-q^m} \in \mathbb{Z}[a_1, a_2, q], \quad i = 1, 2, \ldots, n,
\]
and each of \(\prod_{k=1}^{n-1} (1 - q^{2k})\) and \(1 - q^{2l} (l = 1, \ldots, 2n)\) divides \(D_n(q)\) in \(\mathbb{Z}[q]\). Therefore the lemma follows from (7).

Lemma 3. For large \(n\), we have
\[
0 < |F_n(q)| \leq c_3 |q|^{-3n^2-2n}.
\]

Proof. Similarly to the proof of Lemma 4 in [4], the residue theorem applied exterior to the circle \(|t| = 1\) shows that
\[
F_n(q) = \sum_{m=2m+1}^{\infty} I_m, \quad I_m = a_m \frac{\prod_{k=1}^{2n} (1 - q^{k-m})}{\prod_{k=1}^{n} (1 - q^{2k+m})}
\]
for large \(n\). Since \(|I_m| \leq c_2 |q|^{-n^2-n(m+1)}\), we get the upper bound for \(|F_n(q)|\). Furthermore, if \(a_1 \neq 0\), it follows that,
\[
F_n(q) = a_1 \frac{\prod_{k=1}^{2n} (1 - q^{k-2m+1})}{\prod_{k=1}^{n} (1 - q^{2k+2m+1})} \left( 1 + \sum_{i=1}^{\infty} b_{ni} \right)
\]
with
\[ b_{nl} = \frac{a_{l+1}}{a_{1}} \prod_{k=1}^{n} \left( \frac{1 - q^{2k+2n+1}}{1 - q^{2k+2n+l+1}} \right) \prod_{k=1}^{2n} \left( \frac{1 - q^{k-2n-l-1}}{1 - q^{k-2n-1}} \right), \]
where \( |b_{nl}| \leq c_{4}|q^{-n}|^{l} \). Hence we have \( F_n(q) \neq 0 \), since \( \sum_{l=1}^{\infty} |b_{nl}| < 1 \) for large \( n \). The proof is similar in the case of \( a_{1} = 0, a_{2} \neq 0 \).

3 Proofs of Theorems

Proof of Theorem 1. Let \( K, q, \) and \( \{a_{m}\} \) be as in Theorem 1. We may suppose that \( a_{1} \) and \( a_{2} \) are integers in \( K \). Assume that \( \theta \in K \) and let \( d = \text{den}\theta \). Then by (5), (6), and (8), we have
\[ 0 < d |A_n(q)\theta + B_n(q)| \leq dc_5 |q|^{-n} \]
for large \( n \); which is a contradiction.

Proof of Theorem 2. Let \( q, \alpha, \) and \( \gamma \) be as in Theorem 2. Since
\[ \sum_{m=1}^{\infty} \frac{\gamma^{m}}{1 - \alpha q^{m}} = \gamma^{-l} \left( \sum_{m=1}^{\infty} \frac{\gamma^{m}}{1 - \alpha q^{m}} - \sum_{m=1}^{l} \frac{\gamma^{m}}{1 - \alpha q^{m}} \right) \quad (l \geq 1), \]
we can assume that \( \alpha \) is a generalized Pisot number, by replacing \( \alpha \) by \( q^{l} \alpha \) with suitable \( l \). We modify Borwein’s integral in [4] as follows:
\[ G_n(q, \alpha, \gamma) = \frac{1}{2\pi i} \int_{|t|=1} \prod_{k=1}^{n} \left( \frac{1 - \alpha q^{k}/t}{1 - q^{k}t} \right)^{-1/t} \sum_{m=1}^{\infty} \frac{\gamma^{m}}{1 - \alpha q^{m}/t} dt. \]

Theorem 2 can be proved by replacing \( F_n(q) \) by \( G_n(q, \alpha, \gamma) \) in Lemmas.

References


[7] D. Duverney, A propos de la série \( \sum_{n=1}^{+\infty} \frac{x^n}{q^n-1} \), J. Théor. Nombres Bordeaux 8 (1996), 173–181.


