Irrationality of certain Lambert series

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1 Introduction and the results

For any fixed $q \in \mathbb{C}$ with $|q| > 1$ and $z \in \mathbb{C}$, the $q$–logarithmic function $L_q(z)$ and the $q$–exponential $E_q(z)$ are defined by

\[
L_q(z) := \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{z}{q^n - z} \quad (|z| < |q|),
\]

\[
E_q(z) := 1 + \sum_{n=1}^{\infty} \frac{z^n}{(q-1) \cdots (q^n-1)} = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{q^n} \right),
\]

respectively. Bézivin [2] showed that the numbers $1$, $E_q^{(k)}(\alpha_i)$ ($i = 1, \ldots, m$, $k = 0, 1, \ldots, l$) are linearly independent over $\mathbb{Q}$, where $q \in \mathbb{Z}\setminus\{0, \pm 1\}$ and $\alpha_i \in \mathbb{Q}^\times$ satisfy $\alpha_i \neq -q^\mu$ and $\alpha_i \neq \alpha_j q^\nu$ for all $\mu, \nu \in \mathbb{Z}$ with $\mu \geq 1$ and $i \neq j$. This implies that

\[
\sum_{n=1}^{\infty} \frac{1}{q^n + \alpha} \notin \mathbb{Q},
\]

where $q \in \mathbb{Z}\setminus\{0, \pm 1\}$ and $\alpha \in \mathbb{Q}^\times$ with $\alpha \neq -q^i$ ($i \geq 1$). Under the same conditions on $q$ and $\alpha$, Borwein [3], [4] obtained irrationality measures for the numbers $\sum_{n=1}^{\infty} 1/(q^n + \alpha)$ and $\sum_{n=1}^{\infty} (-1)^n/(q^n + \alpha)$. These results include the irrationality of $L_2(1) = \sum_{n=1}^{\infty} 1/(2^n - 1)$ proved by Erdős [10]. Furthermore, Bundschuh and Väänänen [6], and Matala-Aho and Väänänen [11] obtained quantitative irrationality results for the values of the $q$-logarithm both in the Archimedean and $p$–adic cases. In [7], Duverney generalized certain results obtained by Borwein [3], [4], and Bundschuh and Väänänen [6]. Recently, Van Assche [15] gave irrationality measures for the numbers $L_q(1)$ and $L_q(-1)$ by using little $q$–Legendre polynomials. In this paper, we prove irrationality results for certain Lambert series, which in particular implies the linear independence of the numbers $1$, $L_q(1)$, $L_q(-1)$ with $q \in \mathbb{Z}\setminus\{0, \pm 1\}$ by developing Borwein's idea in [4].

Let $R_n$ be a binary recurrence defined by

\[
R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0), \quad A_1, A_2 \in \mathbb{Q}^\times, \quad R_0, R_1 \in \mathbb{Q}.
\]
André-Jeannin [1] proved for some \( R_n \) the irrationality of the value of the function \( f(x) = \sum_{n=1}^{\infty} x^n / R_n \) at a nonzero rational integer \( x \) in the disk of convergence of \( f \), which gave the first proof of the irrationality of the numbers \( \sum_{n=1}^{\infty} 1 / F_n \) and \( \sum_{n=1}^{\infty} 1 / L_n \), where \( F_n \) and \( L_n \) are Fibonacci numbers and Lucas numbers, respectively. Prévost [13] extended this result to any rational \( x \) in the domain of meromorphy of \( f \). Recently, Matala-aho and Prévost [12] obtained for some type of \( R_n \) irrationality measures for the number \( \sum_{n=1}^{\infty} \gamma^n / R_{an} \), where \( \gamma \) belongs to an imaginary quadratic field, and \( a > 0 \) is an integer. We will prove for some \( R_n \) the irrationality of the numbers \( \sum_{n=1}^{\infty} \gamma^n / R_{an+b} \) and \( \sum_{n=1}^{\infty} \gamma^n / R_{an+b} R_{a(n+1)+b} \), where \( a > 0, b \geq 0 \) are integers and \( \gamma \) is a certain number in a real quadratic field (see Corollaries 2 and 3, below).

For an algebraic number \( \alpha \), we denote by \( \overline{\alpha} \) the maximum of absolute values of its conjugates and by \( \text{den} \alpha \) the least positive integer such that \( \alpha - \text{den} \alpha \) is an algebraic integer. We define generalized Pisot number \( \alpha \) by algebraic integer \( \alpha \) satisfying \( \overline{\alpha} > 1 \) and \( \alpha^\sigma \neq \alpha \) for any \( \sigma \in \text{Aut}(\overline{\mathbb{Q}}) \) with \( \alpha^\sigma \neq \alpha \). We put \( \mathbb{N} = \{0, 1, 2, \ldots\} \).

**Theorem 1.** Let \( K \) be either \( \mathbb{Q} \) or an imaginary quadratic field. Assume that \( q \) is an integer in \( K \) with \( |q| > 1 \) and \( \{a_n\} \) a periodic sequence in \( K \) of period two, not identically zero. Then

\[
\theta = \sum_{n=1}^{\infty} \frac{a_n}{1 - q^n} \notin K.
\]

**Corollary 1.** Let \( q \in \mathbb{Z} \) with \( |q| \geq 2 \) and \( \{a_n\}, \{b_n\} \) be periodic sequences in \( \mathbb{Q} \) of period two, not identically zero. Then the numbers

\[
1, \quad \sum_{n=1}^{\infty} \frac{a_n}{q^n - 1}, \quad \sum_{n=1}^{\infty} \frac{b_n}{q^n - 1}
\]

are linearly independent over \( \mathbb{Q} \) if and only if \( \{a_n\} \) and \( \{b_n\} \) are linearly independent over \( \mathbb{Q} \).

**Proof.** This follows immediately from Theorem 1.

**Example 1.** Let \( q \in \mathbb{Z} \) with \( |q| \geq 2 \). Then

\[
1, \quad L_q(1) = \sum_{n=1}^{\infty} \frac{1}{q^n - 1}, \quad L_q(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{-1}{q^n + 1}
\]

are linearly independent over \( \mathbb{Q} \).

**Theorem 2.** Let \( q \) be a quadratic generalized Pisot number, \( \gamma \) a unit in \( \mathbb{Q}(q) \) with \( |\gamma| \leq 1 \), and \( \alpha \in \mathbb{Q}(q)^\times \) with \( (\text{den}(q^l \alpha))^4 < |q| \) for some \( l \in \mathbb{N} \). Then

\[
\xi = \sum_{n=1}^{\infty} \frac{\gamma^n}{1 - \alpha q^n} \notin \mathbb{Q}(q),
\]

provided that \( \alpha q^n \neq 1 \) for all \( n \geq 1 \).
In the following Corollaries 2 and 3, we consider the binary recurrences \( \{ R_n \}_{n \geq 0} \) defined by
\[
R_{n+2} = A_1 R_{n+1} + A_2 R_n, \quad A_1, A_2 \in \mathbb{Z} \setminus \{0\}, \quad R_0, R_1 \in \mathbb{Z}.
\]
We suppose that \( R_n \neq 0 \) for all \( n \geq 1 \), the corresponding polynomial \( \Phi(X) = X^2 - A_1 X - A_2 \) is irreducible in \( \mathbb{Q}[X] \), and \( \Delta = A_1^2 + 4A_2 > 0 \). We can write \( R_n \) as
\[
R_n = g_1 \rho_1^n + g_2 \rho_2^n (n \geq 0), \quad g_1, g_2 \in \mathbb{Q}(\rho_1),
\]
where \( \rho_1 \) and \( \rho_2 \) are the roots of \( \Phi(X) \). We may assume \( |\rho_1| > |\rho_2| \), since \( \Delta > 0 \).

For \( a, b \in \mathbb{N} \) with \( a \neq 0 \), we define
\[
R(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} (|z| < |\rho_1|^a).
\]
This function can be extended to a meromorphic function on the whole complex plane \( \mathbb{C} \) with poles \( \{ (\rho_1^{n+1}/\rho_2^n)^a | n \geq 0 \} \) since
\[
\sum_{n=1}^{\infty} \frac{z^n}{1-\alpha q^n} = \sum_{m=1}^{\infty} \frac{\alpha^{-m}z}{z-\alpha^n} (|z| < |q|)
\]
for any complex numbers \( q \) and \( \alpha \) with \( |q| > 1 \) and \( |\alpha| \geq 1 \), and so
\[
\sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} = \sum_{n=1}^{\infty} \frac{z^n}{R_{an+b}} - \sum_{n=0}^{\infty} \frac{(-q_n/\gamma)^{an}}{z - \rho_1^{an} \gamma},
\]
where \( i \) is chosen as \( |(g_n/\gamma)^{\rho_1^{an} \gamma}| < 1 \). We denote the function again by \( R(z) \).

**Corollary 2.** Let \( R_n \) be a binary recurrence given by (1) and \( a, b \in \mathbb{N} \) with \( a \neq 0 \). Assume that \( g_1/g_2 \) and \( \rho_1/\rho_2 \) are units in \( \mathbb{Q}(\rho_1) \) and \( \gamma \in \mathbb{Q}(\rho_1)^* \) is not a pole of \( R(z) \) with \( (\text{deg}(\rho_1^{\alpha} \gamma))^4 < |\rho_1/\rho_2|^a \). Then we have \( R(\gamma) \notin \mathbb{Q}(\rho_1) \).

**Proof.** Apply Theorem 2 to the last sum in (2).

**Example 2.** Let \( F_n \) and \( L_n \) be Fibonacci numbers and Lucas numbers defined by \( F_{n+2} = F_{n+1} + F_n (n \geq 0), F_0 = 0, F_1 = 1 \) and \( L_{n+2} = L_{n+1} + L_n (n \geq 0), L_0 = 2, L_1 = 1 \), respectively. Then for every \( a, b \in \mathbb{N} \) with \( a \neq 0 \),
\[
\sum_{n=1}^{\infty} \frac{1}{F_{an+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{an+b}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{an+b}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{L_{an+b}} \notin \mathbb{Q}(\sqrt{5}).
\]
André-Jeannin[1] proved that each of these numbers is irrational. We remark that the numbers \( \sum_{n=1}^{\infty} 1/F_{2n+1} \) and \( \sum_{n=1}^{\infty} 1/L_{2n} \) are transcendental (cf. [8], [9]).
Example 3. Let $F_n$ be Fibonacci numbers. Then for every $a, b \in \mathbb{N}$ with $a \neq 0$,
\[
\sum_{n=1}^{\infty} \frac{1}{F_{(2a-1)n+b}F_{(2a-1)(n+1)+b}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{2an+b}F_{2a(n+1)+b}} \notin \mathbb{Q}(\sqrt{5}).
\]
The same holds for Lucas numbers. We put
\[
T_l := \sum_{n=1}^{\infty} \frac{1}{F_nF_{n+l}}, \quad T_l^* := \sum_{n=1}^{\infty} \frac{(-1)^n}{F_nF_{n+l}} \quad (l \geq 1).
\]
Then Brousseau [5] and Rabinowitz [14] proved that
\[
T_{2l} = \frac{1}{F_{2l}} \sum_{n=1}^{l} \frac{1}{F_{2n-1}F_{2n}}, \quad T_{2l+1} = \frac{1}{F_{2l+1}} \left( T_1 - \sum_{n=1}^{l} \frac{1}{F_{2n}F_{2n+1}} \right),
\]
\[
T_l^* = \frac{1}{F_l} \left( \frac{1-\sqrt{5}}{2} l + \sum_{n=1}^{l} \frac{F_{n-1}}{F_n} \right),
\]
so that $T_{2l} \in \mathbb{Q}$ and $T_l^* \in \mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q}$ for all $l \geq 1$. We see that $T_{2l+1} \notin \mathbb{Q}(\sqrt{5})$ for all $l \geq 0$, since the first sum in this example with $a = 1, b = 0$ implies
\[
T_1 = \sum_{n=1}^{\infty} \frac{1}{F_nF_{n+1}} \notin \mathbb{Q}(\sqrt{5}).
\]

2 Lemmas

For the proof of theorems, we prepare some lemmas. Let $\{a_m\}_{m \geq 1}$ be a periodic sequence of complex numbers of period two, not identically zero. We put
\[
\theta = \sum_{m=1}^{\infty} \frac{a_m}{1-q^m},
\]
where $q \in \mathbb{C}$ with $|q| > 1$. We start with the integral
\[
F_n(q) = \frac{1}{2\pi i} \int_{|t|=1} \frac{(-1/t)^{2n} (1-q^b/t)}{\prod_{k=1}^{2n} (1-q^{2k}t)} \sum_{m=1}^{\infty} \frac{a_m}{1-q^m/t} dt, \quad (3)
\]
which is a variant of that used by Borwein [4]. We note that the integrand is meromorphic in $t$ provided $|q| > 1$. We use the notations
\[
[n]_q! := \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q)}{(1-q)^n}, \quad [0]_q := 1,
\]
\[
[n]_q := \frac{[n]_q!}{[n][n]_q!} \in \mathbb{Z}[q].
\]

In what follows, we denote \(c_1, c_2, \ldots\) positive constants independent of \(n\).

**Lemma 1.**

\[
F_n(q) = \sum_{i=1}^{n} \frac{\prod_{k=1}^{2n}(1-q^{k+2i})}{\prod_{k=1}^{n}(1-q^{2k-2i})} \left( \theta - \sum_{m=1}^{2i-1} \frac{a_m}{1-q^m} \right)
\]

\[-\frac{1}{(2n-1)!} \left( \prod_{k=1}^{2n}(t-q^k) \prod_{k=1}^{n}(1-q^{2k}t)^{-1} \sum_{m=1}^{\infty} \frac{a_m}{t-q^m} \right)^{(2n-1)} \bigg|_{t=0}
\]

**Proof.** This can be proved by using the residue theorem similarly as the proof of Lemma 1 in [4].

We put \(D_n(q) := \prod_{k=n+1}^{2n}(1-q^{2k})\). Then we have

\[|D_n(q)| \leq c_1 |q|^{3n^2+n}.\]  

**Lemma 2.**

\[D_n(q)F_n(q) = A_n(q)\theta + B_n(q),\]

where \(A_n(q), B_n(q) \in \mathbb{Z}[a_1, a_2, q]\).

**Proof.** Since

\[\frac{1}{\prod_{k=1, k \neq i}^{n}(1-q^{2k-2i})} = \frac{q^{i(i-1)}}{\prod_{k=1}^{i-1}(q^{2k}-1) \prod_{k=i}^{n}(1-q^{2k})},\]

we have by (4)

\[F_n(q) = \frac{1}{\prod_{k=1}^{n-1}(1-q^{2k})} \sum_{i=1}^{n} (-1)^{i-1} q^{i(i-1)} \left( \prod_{k=1}^{n-1} (1-q^{k+2i}) \right) \left( \theta - \sum_{m=1}^{2i} \frac{a_m}{1-q^m} \right)
\]

\[-\frac{1}{\lambda! \mu! \nu!} \left| \sum_{\lambda, \mu, \nu \geq 0, \lambda + \mu + \nu = 2n-1} \left( \prod_{k=1}^{2n}(t-q^k) \right)^{(\lambda)} \left( \prod_{k=1}^{n}(1-q^{2k})^{-1} \right)^{(\mu)} \left( \sum_{m=1}^{\infty} \frac{a_m}{t-q^m} \right)^{(\nu)} \bigg|_{t=0}
\]
with
\[
\left( \prod_{k=1}^{2n}(t-q^k) \right)^{(\lambda)}\right|_{t=0} = \lambda!(-1)^{2n-\lambda} \sum_{\lambda_1+\cdots+\lambda_{2n}=2n-\lambda,\lambda_i=0,1} q^{\lambda_1+2\lambda_2+\cdots+2n\lambda_{2n}},
\]
\[
\left( \prod_{k=1}^{n}(1-q^{2k}t)^{-1} \right)^{(\mu)}\right|_{t=0} = \mu! \sum_{\mu_1+\cdots+\mu_{2n}=\mu,\mu_i\geq 0} q^{2(\mu_1+2\mu_2+\cdots+n\mu_n)},
\]
\[
\left( \sum_{m=1}^{\infty} \frac{a_m}{t-q^m} \right)^{(\nu)}\right|_{t=0} = -\nu! \sum_{m=1}^{\infty} \frac{a_m}{(q^\nu+1)^m} = \nu!(a_{\nu+1} + a_2) \frac{1}{1-q^{2(\nu+1)}}.
\]

Hence we get
\[
F_n(q) = \frac{1}{(1-q^{2k})} \sum_{i=1}^{n} (-1)^{i-1} q^{i(1-i)} \left[ \frac{n-1}{i-1} \right] \prod_{k=1}^{2n}(1-q^{k+2i}) \left( \theta - \sum_{m=1}^{2i} \frac{a_m}{1-q^m} \right) + \sum_{\lambda+\mu+\nu=2n-1,\lambda,\mu,\nu\geq 0} Q_{\lambda\mu\nu}(q) \frac{1}{1-q^{2(\nu+1)}}
\]
(7)

with \(Q_{\lambda\mu\nu}(q)\) a polynomial in \(\mathbb{Z}[a_1, a_2, q]\) for all \(\lambda, \mu, \nu \geq 0\). Here we note that
\[
\prod_{k=1}^{2n}(1-q^{k+2i}) \sum_{m=1}^{2i} \frac{a_m}{1-q^m} \in \mathbb{Z}[a_1, a_2, q], \quad i = 1, 2, \ldots, n,
\]
and each of \(\prod_{k=1}^{n-1}(1-q^{2k})\) and \(1-q^{2l} (l = 1, \ldots, 2n)\) divides \(D_n(q)\) in \(\mathbb{Z}[q]\). Therefore the lemma follows from (7).

**Lemma 3.** For large \(n\), we have
\[
0 < |F_n(q)| \leq c_3 |q|^{-3n^2-2n}.
\]
(8)

**Proof.** Similarly to the proof of Lemma 4 in [4], the residue theorem applied exterior to the circle \(|t|=1\) shows that
\[
F_n(q) = \sum_{m=2n+1}^{\infty} I_m, \quad I_m = a_m \prod_{k=1}^{2n}(1-q^{k-m}) \prod_{k=1}^{n}(1-q^{2k+m})
\]
for large \(n\). Since \(|I_m| \leq c_2|q|^{-n^2-n(m+1)}\), we get the upper bound for \(|F_n(q)|\). Furthermore, if \(a_1 \neq 0\), it follows that,
\[
F_n(q) = a_1 \prod_{k=1}^{2n}(1-q^{k-2n-1}) \prod_{k=1}^{n}(1-q^{2k+2n+1}) \left( 1 + \sum_{i=1}^{\infty} b_{ni} \right)
\]
with
\[ b_{nl} = \frac{a_{l+1}}{a_{1}} \prod_{k=1}^{n} \left( \frac{1-q^{2k+2n+1}}{1-q^{2k+2n+l+1}} \right) \prod_{k=1}^{2n} \left( \frac{1-q^{k-2n-l-1}}{1-q^{k-2n-1}} \right), \]
where \(|b_{nl}| \leq c_{4}|q^{-n}|^l|\). Hence we have \( F_n(q) \neq 0 \), since \( \sum_{l=1}^{\infty} |b_{nl}| < 1 \) for large \( n \). The proof is similar in the case of \( a_1 = 0, a_2 \neq 0 \).

3 Proofs of Theorems

Proof of Theorem 1. Let \( K, q, \) and \( \{a_m\} \) be as in Theorem 1. We may suppose that \( a_1 \) and \( a_2 \) are integers in \( K \). Assume that \( \theta \in K \) and let \( d = \text{den} \theta \). Then by (5),(6), and (8), we have
\[ 0 < d |A_n(q)\theta + B_n(q)| \leq dc_5|q|^{-n} \]
for large \( n \); which is a contradiction.

Proof of Theorem 2. Let \( q, \alpha, \) and \( \gamma \) be as in Theorem 2. Since
\[ \sum_{m=1}^{\infty} \frac{\gamma^m}{1-\alpha q^m} = \gamma^{-l} \left( \sum_{m=1}^{\infty} \frac{\gamma^m}{1-\alpha q^m} - \sum_{m=1}^{l} \frac{\gamma^m}{1-\alpha q^m} \right) \quad (l \geq 1), \]
we can assume that \( \alpha \) is a generalized Pisot number, by replacing \( \alpha \) by \( q^l \alpha \) with suitable \( l \). We modify Borwein's integral in [4] as follows:
\[ G_n(q, \alpha, \gamma) = \frac{1}{2\pi i} \int_{|t|=1} \prod_{k=1}^{n} \left( \frac{1-\alpha q^k/t}{1-q^k t} \right) \frac{-1/t}{1-q^n t} \sum_{m=1}^{\infty} \frac{\gamma^m}{1-\alpha q^m} \ dt. \]
Theorem 2 can be proved by replacing \( F_n(q) \) by \( G_n(q, \alpha, \gamma) \) in Lemmas.

References


[7] D. Duverney, A propos de la série \( \sum_{n=1}^{+\infty} \frac{x^n}{q^n-1} \), J. Théor. Nombres Bordeaux 8 (1996), 173–181.


