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PROPERTIES OF DOUBLE GEL'FAND POLYNOMIALS AND THEIR APPLICATIONS TO MULTIPLICITY-FREE PROBLEMS

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ABSTRACT

Properties of double Gel'fand polynomials are reviewed with some new derivation processes, by making the most of the integration method in the Bargmann space. The vector-coupling expressions of double Gel'fand polynomials in two-fold ways are shown not only to be of basic importance for the application to nuclear many-body problems, but also to have many fruitful contents when they are used in combination with other properties. Applications are discussed with respect to the multiplicity-free problems of the SU_2 and SU_3 algebras, coherent states and Talmi-Moshinsky-Smirnov coefficients.

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§ 1. Introduction and summary

In nuclear many-body problems, products of harmonic oscillator (abbreviated to h.o.) wave functions $\Psi=\prod_{i=1}^{n}\chi_{N_il_im_i}(\boldsymbol{\xi}_i)$ (N_i denotes a number of h.o. quanta) are often adopted as orthonormal basis wave functions, where $\boldsymbol{\xi}_i$ $(i=1\sim n)$ are the usual coordinates of constituent particles or the Jacobi coordinates among these particles. When the h.o. wave functions are represented in terms of the creation operators $\boldsymbol{a}^\dagger$ of the h.o. quanta as $\chi_{Nlm}(\boldsymbol{\xi})=U_{Nlm}(\boldsymbol{a}^\dagger)|0\rangle$, the product wave function Ψ is represented as $\Psi=\prod_{i=1}^{n}U_{N_il_im_i}(\boldsymbol{a}_i^\dagger)|0\rangle_n$ by a polynomial of $\boldsymbol{a}_i^\dagger$ $(i=1\sim n)$ operated to the n-particle vacuum $|0\rangle_n$. Since $U_{Nlm}(\boldsymbol{a}^\dagger)$ belongs to $(N,0)$ representation of the Elliott SU_3 group, we can adopt the following coupling scheme for the basis wave functions,

$$[\cdots[[U_{(N_10)}(\boldsymbol{a}_1^\dagger)U_{(N_20)}(\boldsymbol{a}_2^\dagger)]_{(\sigma_2\tau_2)}U_{(N_30)}(\boldsymbol{a}_3^\dagger)]_{(\sigma_3\tau_3)}$$
$$\cdots U_{(N_n0)}(\boldsymbol{a}_n^\dagger)]_{(\lambda\mu)KJM}|0\rangle_n\,, \qquad (1\text{-}1)$$

where $[[\cdots]_{(\sigma\tau)}U_{(N0)}(\boldsymbol{a}^\dagger)]_{(\sigma'\tau')}$ means an SU_3 coupling $(\sigma\tau)\times(N0)\to(\sigma'\tau')$. Polynomials of $\boldsymbol{a}_i^\dagger$ $(i=1\sim n)$ coupled as in Eq. (1-1) were noted by Moshinsky and Chacón[1)] to be just equivalent to the so-called doubly-indexed Gel'fand polynomials (of the type $U_3\times U_n$), which we hereafter simply call double Gel'fand polynomials or DG polynomials. Thus the investigation of the properties of the DG polynomials is very important for nuclear many-body problems.

DG polynomials have been extensively investigated by Moshinsky and his coworkers[2),3)] and also by Biedenharn, Louck and their coworkers[4),5),49)]. From these investigations, many remarkable properties of DG polynomials have been clarified. Thus we can now utilize these fruitful results on DG polynomials for the sake of nuclear many-body problems. However, to the present authors at least, it seems that the importance of the above-mentioned equivalence between the DG polynomials and the vector-coupled many-body wave functions of the type of Eq. (1-1) has not been realized seriously till now except in Ref. 1).

The main purpose of this paper is to show the importance and usefulness of the vector-coupling expressions of the DG polynomials. Besides the basic importance for the applications to nuclear many-body problems, we can show that the vector-coupling expressions of the DG polynomials give us in a straightforward way fruitful applications in many other problems when they are used in combination with other remarkable properties of the DG polynomials. All our discussions in this paper are done in the so-called Bargmann space,[6)] since we can then acquire wider flexibility in treating the DG polynomials than the "boson-calculus technique"[2),5)]. In this Bargmann space, we can embed entirely the "boson-calculus technique", and furthermore utilize many nice properties of the integration with the Bargmann measure.

In the first part of this paper, before discussing the applications, we review several basically important properties of DG polynomials, including the full proof of the vector-coupling expressions in two-fold ways. Here we have tried as far as possible to give simple and straightforward derivations newly for those kinds of properties already shown to hold in the above-cited works of Refs. 2) ∼5). For example, our derivation of the transformation property of the DG polynomials under a general linear transformation of argument matrix variables is done in a simple and novel way with the Bargmann integration concept.

With this preparation, we then discuss how the DG polynomials are powerfully applied to many kinds of problems. We will see there that the use of the vector-coupling expressions certainly provides us a straightforward method in handling the DG polynomials. For some kinds of problems which have been proved already, our emphasis lies in new and straightforward processes of our derivations. In this paper, our applications are restricted to the multiplicity-free problems, which yet involves a wide field of problems.

The composition of this paper is as follows. In the next section (§ 2) we discuss several basically important properties of the DG polynomials. They include completeness of the DG polynomials as orthonormal basis states, vector-coupling expressions in two-fold ways and transformation property under the general linear transformation of argument matrix variables. We further briefly discuss on Clebsch-Gordan (C-G) series and complex conjugate representation. For the sake of later applications we give explicit expressions (including vector-coupling ones) of the DG polynomials of $U_n \times U_m$ with $n, m \leqq 3$. In § 3 applications to SU_2 coefficients are discussed focussing on the Regge symmetries of 3-j and 6-j coefficients. Section 4 treats applications to the coherent states of SU_2, SU_3 and symmetric top wave functions. Here we also discuss briefly invariant polynomials of SU_2 and SU_3. In this section, we will see that the use of the Bargmann integration technique is essential. In § 5 applications to multiplicity-free SU_3 co-efficients are treated. We show that some special 3-$(\lambda\mu)$, 6-$(\lambda\mu)$ and 9-$(\lambda\mu)$ coefficients are directly expressed by SU_2 co-efficients. Final section (§ 6) discusses the Talmi-Moshinsky-Smirnov coefficients of an n-particle system, which are shown there to be given by the DG polynomials of $U_{n-1}\times U_{n-1}$ type if the basis wave functions are SU_3-coupled ones. Appendices give elementary and straightforward derivations of some important formulas or relations. For example, Appendix A gives straightforward procedure to calculate the normalization coefficients of the DG polynomials with the highest weights, and Appendix B gives a proof of the completeness of the DG polynomials by directly calculating the number of independent basis states.

§ 2. Properties of double Gel'fand polynomials

2-1 Double Gel'fand polynomials

Let us consider a Hilbert space $\mathcal{H}(n, m)$, which is sometimes called Bargmann space[6] composed of polynomial functions $f(R)$, $g(R)$,$\cdots$ of nm complex variables $R_{\alpha i}(\alpha=1\sim n,\ i=1\sim m)$, where we use the shorthand notation R as an $n\times m$ matrix

$$R\equiv(R_{\alpha i})=\begin{pmatrix}R_{11}\cdots R_{1m}\\ \vdots \quad \vdots\\ R_{n1}\cdots R_{nm}\end{pmatrix}=(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_m)=\begin{pmatrix}\boldsymbol{R}_1^r\\ \vdots\\ \boldsymbol{R}_n^r\end{pmatrix} \tag{2-1-1}$$

The unitary inner product of two elements $f(R)$ and $g(R)$ of $\mathcal{H}(n, m)$ was introduced by Bargmann[6] as

$$\langle f(R)|g(R)\rangle=\int d\mu(R)f(R)^*g(R), \tag{2-1-2a}$$

where the measure $d\mu(R)$ is

$$d\mu(R)\equiv\prod_{\alpha=1}^{n}\prod_{i=1}^{m}d\mu(R_{\alpha i}), \tag{2-1-2b}$$

$$d\mu(z)\equiv\pi^{-1}e^{-z^*z}d(\mathrm{Re}\,z)\,d(\mathrm{Im}\,z), \tag{2-1-2c}$$

and the integration is extended over the whole nm-dimensional complex Euclidean space. Integration measure $d\mu(z)$ for a single complex variable z gives us the following relation

$$\left\langle\frac{z^p}{\sqrt{p!}}\middle|\frac{z^q}{\sqrt{q!}}\right\rangle=\delta_{p,q}\,, \tag{2-1-3a}$$

with the inner product $\langle u(z)|v(z)\rangle = \int d\mu(z) u^*(z) v(z)$. From this we obtain

$$\langle u(z)|z|v(z)\rangle = \left\langle \frac{d}{dz} u(z) \middle| v(z) \right\rangle,$$

$$\left\langle u(z) \middle| \frac{d}{dz} \middle| v(z) \right\rangle = \langle zu(z)|v(z)\rangle. \qquad (2\text{-}1\text{-}3b)$$

Equation (2-1-3b), together with $[(d/dz), z] = 1$, assures that (d/dz) and z can be regarded as the usual boson annihilation and creation operators, respectively. We define U_n and U_m generators as

$$A_{\alpha\beta} \equiv \sum_{i=1}^{m} R_{\alpha i} \frac{\partial}{\partial R_{\beta i}} \quad \text{for} \quad \alpha, \beta = 1 \sim n, \qquad (2\text{-}1\text{-}4a)$$

$$T_{ij} \equiv \sum_{\alpha=1}^{n} R_{\alpha i} \frac{\partial}{\partial R_{\alpha j}} \quad \text{for} \quad i, j \equiv 1 \sim m, \qquad (2\text{-}1\text{-}4b)$$

which fulfill the following commutation relations;

$$[A_{\alpha\beta}, A_{\gamma\delta}] = \delta_{\beta\gamma} A_{\alpha\delta} - \delta_{\alpha\delta} A_{\gamma\beta}, \qquad (2\text{-}1\text{-}5a)$$

$$[T_{ij}, T_{kl}] = \delta_{jk} T_{il} - \delta_{il} T_{kj}, \qquad (2\text{-}1\text{-}5b)$$

$$[A_{\alpha\beta}, T_{ij}] = 0 \quad \text{for any } \alpha, \beta, i \text{ and } j. \qquad (2\text{-}1\text{-}5c)$$

The generators T_{ij} are the so-called Moshinsky operators[7] and the property of Eq. (2-1-5c) is of basic importance.

The simplest invariant composed of the generator algebras $A_{\alpha\beta}$ and T_{ij} is the Casimir operator of the first rank

$$\hat{N} \equiv \sum_{\alpha=1}^{n} A_{\alpha\alpha} = \sum_{i=1}^{m} T_{ii}$$
$$= \sum_{\alpha=1}^{n} \sum_{i=1}^{m} R_{\alpha i} \frac{\partial}{\partial R_{\alpha i}}, \qquad (2\text{-}1\text{-}6)$$

by which the Hilbert space $\mathscr{H}(n, m)$ is divided into the direct sum of the subspaces $\mathscr{B}_N(n, m)$ composed of all the homogeneous polynomial functions of degree N with respect to $R_{\alpha i}$;

$$\mathscr{H}(n, m) = \sum_{N=0}^{\infty} \mathscr{B}_N(n, m). \qquad (2\text{-}1\text{-}7)$$

Further decomposition of the subspace $\mathscr{B}_N(n, m)$ is accomplished by constructing the basis of the irreducible representation (abbreviated to BIR) with respect to the U_n and U_m groups governed by mutually commutable generator algebras $A_{\alpha\beta}$ and T_{ij}, respectively. In the following we assume $n \leq m$ without loss of generality. According to Moshinsky[2] let us consider the following homogeneous polynomial function with degree $N = f_1 + f_2 + \cdots + f_n$;

$$\varphi_{H,H}^{(n,n)[f]}(R) = N_H[f] (\Delta_1^1)^{f_1-f_2} (\Delta_{12}^{12})^{f_2-f_3} \cdots (\Delta_{12\cdots n}^{12\cdots n})^{f_n}, \qquad (2\text{-}1\text{-}8a)$$

where

$$\Delta^{1\cdots r}_{1\cdots r} = \begin{vmatrix} R_{11} & \cdots & R_{1r} \\ \vdots & & \vdots \\ R_{r1} & \cdots & R_{rr} \end{vmatrix} \quad \text{for} \quad r=1\sim n\,. \tag{2-1-8b}$$

and we have used the shorthand notation $[f]\equiv[f_1,f_2,\cdots,f_n]$ with $f_1\geqq f_2\geqq\cdots\geqq f_n\geqq 0$. $N_H[f]\equiv N_H[f_1,\cdots,f_n]$ is a normalization constant determined from $\langle\varphi^{(nn)[f]}_{H,H}(R)|\varphi^{(nn)[f]}_{H,H}(R)\rangle=1$ and is calculated to be*

$$N_H[f_1\cdots f_n]=\left[\frac{\prod\limits_{\nu>\mu=1}^{n}(f_\mu-f_\nu+\nu-\mu)}{\prod\limits_{\mu=1}^{n}(f_\mu+n-\mu)!}\right]^{1/2} \tag{2-1-9}$$

A new straightforward method to verify Eq. (2-1-9) is given in Appendix A. The function $\varphi^{(nn)[f]}_{H,H}(R)$ satisfies the following set of differential equations;

$$\begin{aligned}
&\hat{N}\ \varphi^{(nn)[f]}_{H,H}(R)=N\ \ \varphi^{(nn)[f]}_{H,H}(R),\\
&A_{\alpha\alpha}\varphi^{(nn)[f]}_{H,H}(R)=f_\alpha\ \ \varphi^{(nn)[f]}_{H,H}(R)\quad \text{for}\quad \alpha=1\sim n\,,\\
&T_{ii}\ \varphi^{(nn)[f]}_{H,H}(R)=f_i\ \ \varphi^{(nn)[f]}_{H,H}(R)\quad \text{for}\quad i=1\sim m\,,\\
&A_{\alpha\beta}\varphi^{(nn)[f]}_{H,H}(R)=T_{ij}\,\varphi^{(nn)[f]}_{H,H}(R)=0\quad\text{for}\\
&\qquad \text{all}\quad 1\leqq\alpha<\beta\leqq n\quad\text{and}\quad 1\leqq i<j\leqq m\,.
\end{aligned} \tag{2-1-10}$$

This means that $\varphi^{(nn)[f]}_{H,H}(R)$ is a BIR with the highest weights belonging to the partition $[f_1\cdots f_n]$ and $[f_1\cdots f_n\ 0\cdots 0]$ with respect to the U_n and U_m groups associated with the generators $A_{\alpha\beta}$ and T_{ij}, respectively. In the following, we sometimes use the parametrization of the partition $[f]$ as

$$\begin{aligned}
&[f]=N(\lambda)\equiv N(\lambda_1\lambda_2\cdots\lambda_{n-1}),\\
&N=\sum_{\alpha=1}^{n} f_\alpha\,,\\
&\lambda_\alpha=f_\alpha-f_{\alpha+1}\quad\text{for}\quad \alpha=1\sim n-1
\end{aligned} \tag{2-1-11}$$

when we show explicitly the labeling of the irreducible representation (IR) of the SU_n group.

A subspace $\mathscr{D}^{[f_1\cdots f_n]}(n,m)\equiv\mathscr{D}^{[f]}(n,m)$ of $\mathscr{B}_N(n,m)$ $(N=f_1+\cdots+f_n)$ is defined as the vector space obtained by operating the generators $A_{\alpha\beta}$ and T_{ij} on $\varphi^{(nn)[f]}_{H,H}(R)$. A double Gel'fand polynomial of the type $U_n\times U_m$, which we abbreviate to an $n\times m$ DG polynomial, is defined to be a BIR $\varphi^{(nm)[f]}_{a,b}(R)$ belonging to $\mathscr{D}^{[f]}(n,m)$. In order to specify the inner quantum numbers a and b of the IR of the U_n and U_m groups, respectively, we adopt the canonical chain

* Here we adopt the phase convention $N_H[f_1\cdots\cdots f_n]\geqq 0$. Furthermore, $N_H[f_1\cdots\cdots f_{n-1}\,0]=N_H[f_1\cdots\cdots f_{n-1}]$.

$$U_n \supset U_{n-1} \supset \cdots \supset U_1,$$
$$U_m \supset U_{m-1} \supset \cdots \supset U_1, \tag{2-1-12}$$

according to the works by Gel'fand and Zetlin,[8] Baird and Biedenharn,[4] Moshinsky,[2] Nagel and Moshinsky[9] and many other authors. Namely, by specifying

$$a \equiv \begin{bmatrix} f_{1,n-1} \cdots\cdots f_{n-1,n-1} \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ f_{12} \quad f_{22} \\ f_{11} \end{bmatrix}, \quad b \equiv \begin{bmatrix} g_{1,m-1} \cdots\cdots g_{m-1,m-1} \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ g_{12} \quad g_{22} \\ g_{11} \end{bmatrix}, \tag{2-1-13}$$

in the Gel'fand patterns, we write

$$\varphi_{a,b}^{(nn)[f]}(R) \equiv \left| \begin{matrix} f_{1n} f_{2n} \cdots\cdots\cdots\cdots f_{nn} \\ f_{1,n-1} \cdots\cdots f_{n-1,n-1} \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ f_{11} \end{matrix} \quad \begin{matrix} g_{1m} g_{2m} \cdots\cdots\cdots g_{mm} \\ g_{1,m-1} \cdots\cdots g_{m-1,m-1} \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ g_{11} \end{matrix} \right\rangle, \tag{2-1-14}$$

where $[f_{1n} f_{2n} \cdots f_{nn}] = [f_1 f_2 \cdots f_n]$ and $[g_{1m} g_{2m} \cdots g_{nm} g_{n+1,m} \cdots g_{mm}] = [f_1 f_2 \cdots f_n\ 0 \cdots 0]$. The quantum numbers $[f_{\alpha\beta}]$ and $[g_{ij}]$ denote that $\varphi_{a,b}^{(nm)[f]}(R)$ is a BIR with a partition $[f_{1\gamma} \cdots f_{\gamma\gamma}]$ of the subgroup U_γ governed by the generator algebra $\{A_{\alpha\beta};\ \alpha, \beta = 1 \sim \gamma\}$ for each $\gamma = 1 \sim n$, and is also a BIR with a partition $[g_{1k} \cdots g_{kk}]$ of the subgroup U_k governed by $\{T_{ij};\ i, j = 1 \sim k\}$ for each $k = 1 \sim m$. $[f_{\alpha\beta}]$ and $[g_{ij}]$ should satisfy the following so-called betweenness conditions;

$$f_{\alpha,\beta+1} \geqq f_{\alpha\beta} \geqq f_{\alpha+1,\beta+1} \quad \text{for} \quad 1 \leqq \alpha \leqq \beta \leqq n-1,$$
$$g_{i,j+1} \geqq g_{ij} \geqq g_{i+1,j+1} \quad \text{for} \quad 1 \leqq i \leqq j \leqq m-1. \tag{2-1-15}$$

In this notation, we can write the highest weights state Eq. (2-1-8) as

$$\varphi_{H,H}^{(nm)[f]}(R) \equiv \varphi_{H,H}^{(nn)[f]}(R) = \left| \begin{matrix} \\ \\ f_1 f_2 \cdots\cdots\cdots f_n \\ f_1 f_2 \cdots\cdots f_{n-1} \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ f_1 \quad f_2 \\ f_1 \end{matrix} \quad \begin{matrix} f_1 f_2 \cdots\cdots\cdots f_n\ 0 \cdots 0 \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ f_1 f_2 \cdots\cdots\cdots f_n\ 0 \\ f_1 f_2 \cdots\cdots\cdots f_n \\ f_1 f_2 \cdots\cdots f_{n-1} \\ \ddots \quad \cdot^{\cdot^{\cdot}} \\ f_1 \quad f_2 \\ f_1 \end{matrix} \right\rangle, \tag{2-1-16}$$

where we are assuming $n \leqq m$. Explicit expressions of lowering operators to get a BIR of the U_n and U_m groups in the chain Eq. (2-1-12) were obtained by Nagel and Moshinsky,[9] together with the normalization of the lowering operators $N\begin{bmatrix} f_1\cdots f_{n-1} \\ f_1\cdots f_n \\ q_1\cdots q_{n-1}\end{bmatrix}^{-1}$,[10] as follows;

$$L_k^l(C) = \sum_{p=0}^{k-l-1}\{\sum_{k>\mu_p>\cdots>\mu_1>l} C_{\mu_1}^l C_{\mu_2}^{\mu_1}\cdots C_k^{\mu_p}\cdot \prod_{\substack{\nu=l+1 \\ \nu\neq\mu_1\cdots\mu_p}}^{k-1} \mathcal{E}_{l\nu}\} \tag{2-1-17a}$$

$$= \sum_{p=0}^{k-l-1}\{\sum_{k>\mu_p>\cdots>\mu_1>l} (\prod_{\substack{\nu=l+1 \\ \nu\neq\mu_1\cdots\mu_p}}^{k-1} \mathcal{E}_{l\nu}) C_k^{\mu_p}\cdots C_{\mu_2}^{\mu_1} C_{\mu_1}^l\}, \tag{2-1-17b}$$

where $k>l$, $\mathcal{E}_{l\nu}=C_l^l-C_\nu^\nu+\nu-l$, $\prod_{\nu=k}^{k-1}\mathcal{E}_{k-1,\nu}=1$ and $C\equiv\{C_\nu^\mu;\ \mu,\nu=1\sim k\}$ are the U_k generators of $A\equiv\{A_{\alpha\beta};\ \alpha,\beta=1\sim k\}$ or $T\equiv\{T_{ij};\ i,j=1\sim k\}$. By using these lowering operators, $n\times m$ DG polynomial Eq. (2-1-14) is implicitly written as[3]

$$\begin{aligned}\varphi_{a,b}^{(nm)[f]}(R) &= |[f_{\alpha\beta}],[g_{ij}]\rangle \\ &= \mathcal{N}[f_{\alpha\beta}]\cdot\prod_{\beta=2}^{n}\prod_{\alpha=1}^{\beta-1}\{L_\beta^\alpha(A)\}^{f_{\alpha\beta}-f_{\alpha,\beta-1}} \\ &\cdot\mathcal{N}[g_{ij}]\cdot\prod_{j=2}^{m}\prod_{i=1}^{j-1}\{L_j^i(T)\}^{g_{ij}-g_{i,j-1}} \\ &\cdot\varphi_{H,H}^{(nm)[f]}(R),\end{aligned} \tag{2-1-18}*$$

where the normalization $\mathcal{N}[f_{\alpha\beta}]$ and $\mathcal{N}[g_{ij}]$ are, for instance,

$$\begin{aligned}\mathcal{N}[f_{\alpha\beta}] &= \prod_{\gamma=2}^{n} N\begin{bmatrix} f_{1\gamma}\cdots\cdots f_{\gamma-1,\gamma} \\ f_{1\gamma}\cdots\cdots f_{\gamma\gamma} \\ f_{1,\gamma-1}\cdots\cdots f_{\gamma-1,\gamma-1}\end{bmatrix}^{-1} \\ &= \Bigg[\prod_{\gamma=2}^{n}\Bigg\{\prod_{\beta\geqq\alpha=1}^{\gamma-1}\frac{(f_{\alpha,\gamma-1}-f_{\beta,\gamma-1}+\beta-\alpha)!}{(f_{\alpha\gamma}-f_{\beta,\gamma-1}+\beta-\alpha)!} \\ &\cdot\prod_{\beta>\alpha=1}^{\gamma}\frac{(f_{\alpha,\gamma-1}-f_{\beta\gamma}+\beta-\alpha-1)!}{(f_{\alpha,\gamma}-f_{\beta\gamma}+\beta-\alpha-1)!}\Bigg\}\Bigg]^{1/2}.\end{aligned} \tag{2-1-19}**$$

A general explicit expression after the lowering operation in Eq. (2-1-18) is obtained when $a=H$, $b=\begin{bmatrix} q_1\cdots q_{n-1} \\ H\end{bmatrix}$ or $a=\begin{bmatrix} q_1\cdots q_{n-1} \\ H\end{bmatrix}$, $b=H$ (the semi-maximum-weight state).[4] For instance, we can show

$$\varphi_{H,\,{}^{q_1\cdots q_{n-1}}_{H}}^{(nn)[f_1\cdots f_n]}(R) = \left|\begin{matrix} f_1\cdots\cdots f_n & f_1\cdots\cdots\cdots f_n \\ H & q_1\cdots\cdots q_{n-1} \\ & H\end{matrix}\right\rangle$$

* The order in the product of the lowering operators should be strictly kept with respect to β and j; namely, the lowering operators with larger β and j values should be operated prior to those with smaller β and j values, respectively.

** Here we adopt the phase convention that the normalization of the lowering operators is non-negative, according to Nagel and Moshinsky,[9)10)] which is equivalent to that the matrix elements of U_m generators with respect to Gel'fand bases are always non-negative.[10]

$$=N\begin{bmatrix} f_1\cdots\cdots f_{n-1} \\ f_1\cdots\cdots f_n \\ q_1\cdots\cdots q_{n-1}\end{bmatrix}^{-1} \prod_{m=1}^{n-1}\{L_n^m(T)\}^{f_m-q_m}\varphi_{H,H}^{(nn)[f_1\cdots f_n]}(R)$$

$$=N_H[f_1\cdots f_n]N\begin{bmatrix} f_1\cdots\cdots f_{n-1} \\ f_1\cdots\cdots f_n \\ q_1\cdots\cdots q_{n-1}\end{bmatrix}^{-1}\left\{\prod_{\nu>m=1}^{n}\frac{(f_m-f_\nu+\nu-m-1)!}{(q_m-f_\nu+\nu-m-1)!}\right\}$$

$$\cdot\prod_{m=1}^{n-1}\{(\Delta_{1\cdots m}^{1\cdots m})^{q_m-f_{m+1}}(\Delta_{1\cdots n}^{1\cdots m})^{f_m-q_m}\}\cdot(\Delta_{1\cdots n}^{1\cdots n})^{f_n}\,, \tag{2-1-20}$$

from the relation

$$L_n^m|\lambda_m\lambda_{m+1}\cdots\lambda_{n-1}\rangle=\{\prod_{\mu=m+1}^{n}(\sum_{\nu=m}^{\mu-1}\lambda_\nu+\mu-m-1)\}$$
$$\cdot\Delta_{1\cdots n}^{1\cdots m}|\lambda_m-1,\lambda_{m+1}\cdots\lambda_{n-1}\rangle\,,$$
$$|\lambda_m\lambda_{m+1}\cdots\lambda_{n-1}\rangle$$
$$\equiv(\Delta_{1\cdots m}^{1\cdots m})^{\lambda_m}(\Delta_{1\cdots m+1}^{1\cdots m+1})^{\lambda_{m+1}}\cdots(\Delta_{1\cdots n-1}^{1\cdots n-1})^{\lambda_{n-1}}\,. \tag{2-1-21}$$

The proof of Eq. (2-1-21) is due to mathematical induction with respect to n, where we should use the recursion formula[9]

$$L_n^m=\sum_{\mu=m}^{n-1}(\prod_{\nu=\mu+1}^{n-1}\mathcal{E}_{m\nu})C_n^\mu L_\mu^m$$

$(L_m^m=1)$ and the relation

$$\sum_{\mu=m}^{n}\Delta_{1\cdots\mu}^{1\cdots m}C_{n+1}^\mu|\lambda_m\cdots\lambda_n\rangle=(\sum_{\nu=m}^{n}\lambda_\nu)\,\Delta_{1\cdots n+1}^{1\cdots m}|\lambda_m\cdots\lambda_n\rangle\,,$$

which can be proved by cofactor expansions like Eq. (A-13) of Appendix A.

The $n\times m$ DG polynomials thus constructed satisfy the following orthonormality with respect to the inner product defined as Eq. (2-1-2);

$$\langle\varphi_{a,b}^{(nm)[f]}(R)|\varphi_{a',b'}^{(nm)[f']}(R)\rangle=\delta_{[f],[f']}\delta_{a,a'}\delta_{b,b'} \tag{2-1-22}$$

Let us consider the direct sum of the subspace $\mathcal{D}^{[f]}(n,m)$ in $\mathcal{B}_N(n,m)$, namely, $\sum_{f_1+\cdots+f_n=N}\mathcal{D}^{[f]}(n,m)$. The most important fact is that this subspace precisely coincides with the subspace $\mathcal{B}_N(n,m)$[2,3,5];

$$\mathcal{B}_N(n,m)=\sum_{f_1+\cdots+f_n=N}\mathcal{D}^{[f]}(n,m)\,. \tag{2-1-23}$$

This fact, the completeness of the $n\times m$ DG polynomials, can be verified by directly calculating the number of independent basis states of the $n\times m$ DG polynomials with the degree $N=f_1+\cdots+f_n$. This elementary method is demonstrated in Appendix B. In this way, we have obtained the full decomposition of the Hilbert space $\mathcal{H}(n,m)$;

$$\mathcal{H}(n, m) = \sum_{N=0}^{\infty} \mathcal{B}_N(n, m) = \sum_{N=0}^{\infty} \sum_{f_1+\cdots+f_n=N} \mathcal{D}^{[f]}(n, m)$$
$$= \sum_{N=0}^{\infty} \sum_{(\lambda)} \mathcal{D}^{N(\lambda)}(n, m), \qquad (2\text{-}1\text{-}24)$$

where we have used the notation $[f] = N(\lambda)$.

Finally, we briefly discuss transposition relation of the $n \times m$ DG polynomials and two kinds of general linear transformations. The effect of a transposition operator $\mathcal{T}$ for an element $f(R)$ of $\mathcal{H}(n, m)$ is defined as

$$(\mathcal{T}f)(R) \equiv f({}^tR), \qquad (2\text{-}1\text{-}25)$$

where $f({}^tR)$ is a function obtained by replacing $R_{\alpha i}$ by $R_{i\alpha}$ in $f(R)$ and is an element of $\mathcal{H}(m, n)$. Especially, $\mathcal{T}\boldsymbol{R}_i = \boldsymbol{R}_i^r$, $\mathcal{T}\boldsymbol{R}_\alpha^r = \boldsymbol{R}_\alpha$ or in matrix notation $\mathcal{T}R\mathcal{T}^{-1} = {}^tR$ and $\mathcal{T}^2 = 1$. By this transposition, the generator algebras $\{A_{\alpha\beta}\}$ and $\{T_{ij}\}$ in Eq. (2-1-4) transform to $\{T_{ij}\}$ and $\{A_{\alpha\beta}\}$, respectively, defined in the Hilbert space $\mathcal{H}(m, n)$. This fact and $\varphi_{H,H}^{(nn)[f]}({}^tR) = \varphi_{H,H}^{(nn)[f]}(R)$ from Eq. (2-1-8) give the following transposition relation of the DG polynomials from the definition Eq. (2-1-18);

$$\mathcal{T}\varphi_{a,b}^{(nm)[f]}(R) \equiv \varphi_{a,b}^{(nm)[f]}({}^tR)$$
$$= \varphi_{b,a}^{(mn)[f]}(R). \qquad (2\text{-}1\text{-}26)$$

For an element $f(R)$ of $\mathcal{H}(n, m)$, we can consider two kinds of general linear transformations associated with the $GL(n, C)$ and $GL(m, C)$ groups. The effects of a left transformation T_G^L and of a right transformation T_G^R for a element $f(R)$ of $\mathcal{H}(n, m)$ are defined to be

$$(T_G^L f)(R) \equiv f({}^tGR), \qquad (2\text{-}1\text{-}27\text{a})$$

$$(T_G^R f)(R) \equiv f(RG), \qquad (2\text{-}1\text{-}27\text{b})$$

respectively, where G is an $n \times n$ matrix in T_G^L and $m \times m$ one in T_G^R. When we parametrize the transformation matrix G as $G = \exp\{i \sum_{k,l=1} g_{kl} e_{kl}\}$ (e_{kl} are the so-called $n \times n$ or $m \times m$ matrix units), we can write these operators explicitly as follows;

$$T_G^L = \exp\{i \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta} A_{\alpha\beta}\}, \qquad (2\text{-}1\text{-}28\text{a})$$

$$T_G^R = \exp\{i \sum_{i,j=1}^{m} g_{ij} T_{ij}\}. \qquad (2\text{-}1\text{-}28\text{b})$$

2-2. Vector-coupling expressions of the DG polynomials in two-fold ways

In this subsection, we show vector-coupling expressions of the DG polynomials in two-fold ways, which are of basic importance in this article. The proof is based on the decomposition relationship Eq. (2-1-23) of the subspace $\mathscr{B}_N(n, m)$ into $\mathscr{D}^{[f]}(n, m)$ with $N=f_1+\cdots+f_n$. In the following we assume $n\geqq m$ without loss of generality. The relationship of Eq. (2-1-23) gives us the following theorem; namely, *a polynomial function of* $R_{\alpha i}$ ($\alpha=1\sim n$, $i=1\sim m$) *which belongs to an IR of the* U_m *group governed by the generator algebra* $\{T_{ij};\ i,\ j=1\sim m\}$ *and is characterized by the partion* $[f_1\cdots f_m]$, *also belongs to a definite IR of the* U_n *group governed by the generator algebra* $\{A_{\alpha\beta};\ \alpha,\ \beta=1\sim n\}$ *and is characterized by the same partion* $[f_1\cdots f_m\ 0\cdots 0]\equiv[f_1\cdots f_m]$[2]. For instance, any monomial of degree Λ composed of single n-dimensional vector $\boldsymbol{R}$ belongs to the IR of the partition $[\Lambda 0\cdots 0]=[\Lambda]$. We write this monomial as

$$U_{[\Lambda]\boldsymbol{\Lambda}}(\boldsymbol{R})\equiv\frac{R_1^{\Lambda_1}\cdots R_n^{\Lambda_n}}{\sqrt{\Lambda_1!\cdots\Lambda_n!}},\quad \boldsymbol{\Lambda}\equiv(\Lambda_1\cdots\Lambda_n),\quad \Lambda=|\boldsymbol{\Lambda}|\equiv\Lambda_1+\cdots+\Lambda_n\,, \tag{2-2-1}$$

by which the $n\times 1$ DG polynomials are written as

$$\left|\begin{matrix} \Lambda & 0\cdots\cdots\cdots\cdots 0 \\ \Lambda_1+\cdots+\Lambda_{n-1} & 0\cdots\cdots\cdots 0 \\ \ddots & \ddots \quad \cdot^{\cdot^{\cdot}} \\ \Lambda_1+\Lambda_2 & 0 \\ \Lambda_1 & \end{matrix}\quad \Lambda\right\rangle=U_{[\Lambda]\boldsymbol{\Lambda}}(\boldsymbol{R}_1)\,. \tag{2-2-2}$$

It should be noted that the Gel'fand pattern $[g_{ij}]$ in Eq. (2-1-14) shows that the degree (the weight) of a vector $\boldsymbol{R}_j$ ($j=1\sim m$) is $N_j\equiv\sum_{i=1}^{j}g_{ij}-\sum_{i=1}^{j-1}g_{i,j-1}$ ($N_1\equiv g_{11}$), which means that $\boldsymbol{R}_j$ is contained in the form of $U_{[N_j]\boldsymbol{\Lambda}}(\boldsymbol{R}_j)$ ($|\boldsymbol{\Lambda}|=N_j$). Next, let us consider the transformation properties of $\varphi_{a,b}^{(nm)[f]}(R)$ with respect to the U_n and U_k groups governed by generator algebras

$$A_{\alpha\beta}^{(k)}=\sum_{i=1}^{k}R_{\alpha i}\frac{\partial}{\partial R_{\beta i}}\quad \text{for}\quad \alpha,\beta=1\sim n\,, \tag{2-2-3}$$

and $\{T_{ij};\ i,\ j=1\sim \mathrm{k}\}$ in Eq. (2-1-4b). By this we can determine the dependence of $\varphi_{a,b}^{(nm)[f]}(R)$ on $\boldsymbol{R}_1,\ \boldsymbol{R}_2,\cdots,\boldsymbol{R}_k$. According to the above theorem, the fact that $\varphi_{a,b}^{(nm)[f]}(R)$ is a BIR with a partition $[g_{1k},\cdots,g_{kk}]$ of the U_k group governed by the generator algebra $\{T_{ij};\ i,\ j=1\sim k\}$ means that the dependence of the vectors $\boldsymbol{R}_1,\cdots,\boldsymbol{R}_k$ is kept in the form of the BIR of the U_n group governed by the generator algebra Eq. (2-2-3) and the IR is characterized by the partition $[g_{1k}\cdots g_{kk}\ 0\cdots 0]=[g_{1k}\cdots g_{kk}]$. Thus we obtain the following vector-coupling expression of $\varphi_{a,b}^{(nm)[f]}(R)$ in Eq. (2-1-14) ($n\geqq m$);

$$\varphi_{a,b}^{(nm)[f]}(R)\equiv\left|\begin{matrix} f_{1n}\ f_{2n}\cdots\cdots\cdots f_{nn} & g_{1m}\ g_{2m}\cdots\cdots\cdots g_{mm} \\ f_{1,n-1}\cdots\cdots f_{n-1,n-1} & g_{1,m-1}\cdots\cdots g_{m-1,m-1} \\ \ddots \quad \cdot^{\cdot^{\cdot}} & \ddots \quad \cdot^{\cdot^{\cdot}} \\ f_{11} & g_{11} \end{matrix}\right\rangle$$

$$= (-)^{P\binom{[f]}{a,b}} [\cdots[[U_{[N_1]}(\boldsymbol{R}_1) U_{[N_2]}(\boldsymbol{R})]_{[g_2]} U_{[N_3]}(\boldsymbol{R}_3)]_{[g_3]}\cdots U_{[N_m]}(\boldsymbol{R}_m)]_{[f]a}, \tag{2-2-4a}$$

where

$$N_1 \equiv g_{11},\ N_j \equiv \sum_{i=1}^{j} g_{ij} - \sum_{i=1}^{j-1} g_{i,j-1} \quad \text{for} \quad j=2\sim m,$$
$$[g_j] \equiv [g_{1j}\cdots g_{jj}] \quad \text{for} \quad j=2\sim m-1,$$
$$[f_{1n}\cdots f_{nn}] = [f_1\cdots f_m 0\cdots 0] = [f],$$
$$[g_{1m}\cdots g_{mm}] = [f_1\cdots f_m] = [f], \tag{2-2-4b}$$

and $(-)^{P\binom{[f]}{a,b}}$ is a phase factor to be determined.

Let us determine the phase factor $(-)^{P\binom{[f]}{a,b}}$. First, let us take $a=H$ (the highest weight) in Eq. (2-2-4) and operate the lowering operators associated with the generator algebra $\{A_{\alpha\beta}\}$ down to a. By this procedure, we find

$$(-)^{P\binom{[f]}{a,b}} = (-)^{P\binom{[f]}{H,b}} \tag{2-2-5}$$

for an arbitrary a. Thus it is enough for us to determine $(-)^{P\binom{[f]}{H,b}}$, which we write $(-)^{P\binom{[f]}{b}}$. This phase factor can be taken to be unity as will be shown below, if we adopt the following phase convention for an arbitrary positive integer l;

i) $$N_H[f_1\cdots f_l] \geqq 0, \tag{2-2-6a}$$

ii) $$N\begin{bmatrix} f_1\cdots\cdots f_{l-1} \\ f_1\cdots\cdots f_l \\ q_1\cdots\cdots q_{l-1}\end{bmatrix}^{-1} \geqq 0, \tag{2-2-6b}$$

iii) $$\langle [q_1\cdots q_{l-1}]H[\Lambda]\boldsymbol{\Lambda} | [f_1\cdots f_l]H\rangle_l \geqq 0,$$
$$\Lambda = \sum_{\mu=1}^{l} f_\mu - \sum_{\mu=1}^{l-1} q_\mu, \quad \Lambda_\mu = f_\mu - q_\mu \ (\mu=1\sim l-1),$$
$$\Lambda_l = f_l. \tag{2-2-6c}$$

The conditions i) and ii) mean that the normalizations of the highest weight states and of the lowering operators, respectively, are non-negative as we already adopted. The condition iii) is a natural extension of the phase convention in the angular momentum theory given by Condon and Shortley;[11] namely, $\langle j_1 j_1 j_2 j_3 - j_1 | j_3 j_3\rangle \geqq 0$.

Let us take $a=H$ and $b=\begin{bmatrix} q_1\cdots q_{m-1} \\ c\end{bmatrix}$ in Eq. (2-2-4) and integrate it by $U_{[\Lambda]\boldsymbol{\Lambda}}(\boldsymbol{R}_m)$ with $\Lambda = \sum_{\mu=1}^{m} f_\mu - \sum_{\mu=1}^{m-1} q_\mu$, $\Lambda_\mu = f_\mu - q_\mu$ $(\mu=1\sim m-1)$ and $\Lambda_m = f_m$.

By this adoption of quantum numbers, the summation over the internal quantum numbers d of

$$[\cdots[U_{[N_1]}(\boldsymbol{R}_1)U_{[N_2]}(\boldsymbol{R}_2)]_{[g_2]}\cdots U_{[N_{m-1}]}(\boldsymbol{R}_{m-1})]_{[q_1\cdots q_{m-1}]d} \tag{2-2-7}$$

in the right hand side of Eq. (2-2-4) gives non-zero contribution only when $d=H$ from the consideration of the weights and the betweenness conditions Eq. (2-1-15). Since Eq. (2-2-7) is written as

$$(-)^{-P\binom{q_1\cdots q_{m-1}}{c}}\varphi_{d,c}^{(n,m-1)[q_1\cdots q_{m-1}]}(R), \tag{2-2-8}$$

from Eqs. (2-2-4) and (2-2-5), we obtain

$$\begin{aligned}&\langle U_{[A]A}(\boldsymbol{R}_m)|\varphi_{H,\substack{q_1\cdots q_{m-1}\\c}}^{(nm)[f_1\cdots f_m]}(R)\rangle\\ &=(-)^{P\left(\substack{f_1\cdots f_m\\q_1\cdots q_{m-1}\\c}\right)-P\binom{q_1\cdots q_{m-1}}{c}}\\ &\cdot\langle[q_1\cdots q_{m-1}]H[A]\boldsymbol{A}|[f_1\cdots f_m]H\rangle_m\\ &\cdot\varphi_{H,c}^{(n,m-1)[q_1\cdots q_{m-1}]}(R).\end{aligned} \tag{2-2-9}$$

On the other hand, if $c=H$ (the highest weight), we can easily perform the integration in the left hand side of Eq. (2-2-9) by the use of the explicit expression of the semi-maximum weight state Eq. (2-1-20); namely by using $\varphi_{H,b}^{(nm)[f_1\cdots f_m]}(R)=\varphi_{H,b}^{(mm)[f_1\cdots f_m]}(R)$, we obtain

$$\begin{aligned}&\langle U_{[A]A}(\boldsymbol{R}_m)|\varphi_{H,\substack{q_1\cdots q_{m-1}\\H}}^{(nm)[f_1\cdots f_m]}(R)\rangle=\langle U_{[A]A}(\boldsymbol{R}_m)|\varphi_{H,\substack{q_1\cdots q_{m-1}\\H}}^{(mm)[f_1\cdots f_m]}(R)\rangle\\ &=\frac{N_H[f_1\cdots f_m]}{N_H[q_1\cdots q_{m-1}]}N\begin{bmatrix}f_1\cdots\cdots f_{m-1}\\ f_1\cdots\cdots f_m\\ q_1\cdots\cdots q_{m-1}\end{bmatrix}^{-1}\left\{\prod_{\nu>\mu=1}^{m}\frac{(f_\mu-f_\nu+\nu-\mu-1)!}{(q_\mu-f_\nu+\nu-\mu-1)!}\right\}\\ &\cdot\sqrt{\prod_{\mu=1}^{m}(A_\mu!)}\,\varphi_{H,H}^{(m-1,m-1)[q_1\cdots q_{m-1}]}(R).\end{aligned} \tag{2-2-10}$$

From Eqs. (2-2-10) and (2-2-9) with $c=H$, we obtain

$$\begin{aligned}&\langle[q_1\cdots q_{m-1}]H[A]\boldsymbol{A}|[f_1\cdots f_m]H\rangle_m=(-)^{P\binom{q_1\cdots q_{m-1}}{H}-P\left(\substack{f_1\cdots f_m\\q_1\cdots q_{m-1}\\H}\right)}\\ &\cdot\frac{N_H[f_1\cdots f_m]}{N_H[q_1\cdots q_{m-1}]}N\begin{bmatrix}f_1\cdots\cdots f_{m-1}\\ f_1\cdots\cdots f_m\\ q_1\cdots\cdots q_{m-1}\end{bmatrix}^{-1}\left\{\prod_{\nu>\mu=1}^{m}\frac{(f_\mu-f_\nu+\nu-\mu-1)!}{(q_\mu-f_\nu+\nu-\mu-1)!}\right\}\\ &\cdot\sqrt{\prod_{\mu=1}^{m}(A_\mu!)}\end{aligned} \tag{2-2-11}$$

Thus the phase convension i), ii) and iii) leads to

$$(-)^{P\left(\substack{f_1\cdots f_m\\q_1\cdots q_{m-1}\\H}\right)}=(-)^{P\binom{q_1\cdots q_{m-1}}{H}}. \tag{2-2-12}$$

Next, starting from Eq. (2-2-9) with $c=H$, we operate on it the lowering operators associated with $T_{ij}(i,\ j\leqq m-1)$ down to c. Since these lowering operators have no dependence on $\boldsymbol{R}_m$, we find that Eq. (2-2-9) is also equal to

$$\langle[q_1\cdots q_{m-1}]H[\Lambda]\boldsymbol{\Lambda}|[f_1\cdots f_m]H\rangle_m\cdot\varphi_{H,c}^{(n,m-1)[q_1\cdots q_{m-1}]}(R). \qquad (2\text{-}2\text{-}13)$$

Considering that the Clebsch-Gordan (C-G) coefficient Eq. (2-2-11) is non-zero if the coupling $[q_1\cdots q_{m-1}]\times[\Lambda]\rightarrow[f_1\cdots f_m]$ is allowed, we finally obtain the recursion relation of the phase factor;

$$(-)^{P\begin{pmatrix} f_1\cdots f_m \\ q_1\cdots q_{m-1} \\ c \end{pmatrix}}=(-)^{P\begin{pmatrix} q_1\cdots q_{m-1} \\ c \end{pmatrix}}. \qquad (2\text{-}2\text{-}14)$$

Since the phase factor of the $m\times 1$ DG polynomial Eq. (2-2-2) is unity, we can conclude that the phase factor in Eq. (2-2-4) is unity under the phase convention i), ii) and iii).

Let us take the transposition operation on Eq. (2-2-4). Exchanging a and b (at the same time, n and m) after the use of Eq. (2-1-26) and $\mathcal{T}\boldsymbol{R}_i=\boldsymbol{R}_i^T$, we obtain another kind of vector-coupling expression of the $n\times m$ DG polynomial Eq. (2-2-4);

$$\varphi_{a,b}^{(nm)[f]}(R)=[\cdots[[U_{[P_1]}(\boldsymbol{R}_1^T)\,U_{[P_2]}(\boldsymbol{R}_2^T)]_{[f_2]}$$
$$\cdot U_{[P_3]}(\boldsymbol{R}_3^T)]_{[f_3]}\cdots U_{[P_n]}(\boldsymbol{R}_n^T)]_{[f]b}, \qquad (2\text{-}2\text{-}15\text{a})$$

where

$$P_1\equiv f_{11}\quad P_\beta\equiv\sum_{\alpha=1}^{\beta}f_{\alpha\beta}-\sum_{\alpha=1}^{\beta-1}f_{\alpha,\beta-1}\quad\text{for}\quad\beta=2\sim n\,,$$
$$[f_\beta]\equiv[f_{1\beta}\cdots f_{\beta\beta}]\quad\text{for}\quad\beta=2\sim n-1\,. \qquad (2\text{-}2\text{-}15\text{b})$$

We have proved Eq. (2-2-4) with a unity phase factor and Eq. (2-2-15) in the case $n\geqq m$ and $n\leqq m$, respectively. In order to show the validity of these expressions for arbitrary n and m, we should repeat the above discussions. This procedure, however, is not necessary because we can obtain any $n\times m$ DG polynomial from the $n\times n$ ($n>m$ case) or $m\times m$ ($n<m$ case) DG polynomial by taking special quantum numbers. This reduction formula is easily obtained from the definition of the DG polynomial Eqs. (2-1-14), (2-1-18) and (2-1-19); namely, if we assume $n<m$,

$$\varphi_{a',b}^{(mm)[f_1\cdots f_n0\cdots0]}(R)=\varphi_{a,b}^{(nm)[f_1\cdots f_n]}(R),$$

$$a'=\begin{pmatrix} f_1\cdots\cdots\cdots f_n\,0\cdots 0 \\ \ddots \qquad \ddots\ \ \ddots \\ f_1\cdots\cdots\cdots f_n\,0 \\ f_1\cdots\cdots\cdots f_n \\ a \end{pmatrix}, \qquad (2\text{-}2\text{-}16\text{a})$$

$$\varphi_{a,b'}^{(mm)[f_1\cdots f_n 0\cdots 0]}(R)=\varphi_{a,b}^{(mn)[f_1\cdots f_n]}(R),$$

$$b'=\begin{pmatrix} f_1\cdots\cdots f_n\ 0\cdots 0 \\ \ddots \qquad \ddots \quad \ddots \\ f_1\cdots\cdots f_n\ 0 \\ f_1\cdots\cdots f_n \\ b \end{pmatrix}, \tag{2-2-16b}$$

where the argument matrix $R=(R_{\alpha i})$ is of an appropriate type associated with the type of the DG polynomial considered.

Use of the vector-coupling expression gives us another type of the reduction relation as shown below, where we assume $n\leqq m$ without loss of generality;

$$\varphi_{a,b}^{(nm)[f]}(R_{pq})=\begin{cases} \varphi_{\tilde{a},\tilde{b}}^{(pq)[f]}(\tilde{R}_{pq}) & \text{if}\quad [f],\ a,\ b \text{ satisfy the condition } (\#), \\ 0 & \text{otherwise}, \end{cases}$$

$$R_{pq}=\begin{matrix} & \leftarrow q \rightarrow & \\ \updownarrow p & \begin{pmatrix} \tilde{R}_{pq} & 0 \\ 0 & 0 \end{pmatrix} & \updownarrow n \\ & \longleftarrow m \longrightarrow & \end{matrix} \tag{2-2-17a}$$

where the condition (#) is for $p\leqq q$

$$[f]\subset U_p,$$

$$a=\begin{pmatrix} f_1\cdots\cdots f_p\ 0\cdots 0 \\ \ddots \qquad \ddots \quad \ddots \\ f_1\cdots\cdots f_p\ 0 \\ f_1\cdots\cdots f_p \\ \tilde{a} \end{pmatrix}\updownarrow p, \quad b=\begin{pmatrix} f_1\cdots\cdots f_p\ 0\cdots\cdots\cdots 0 \\ \ddots \qquad \ddots \qquad \ddots \\ f_1\cdots\cdots f_p\ 0\cdots 0 \\ \tilde{b} \end{pmatrix}\updownarrow q \tag{2-2-17b}$$

and is for $p\geqq q$

$$[f]\subset U_q$$

$$a=\begin{pmatrix} f_1\cdots\cdots f_q\ 0\cdots\cdots\cdots 0 \\ \ddots \qquad \ddots \qquad \ddots \\ f_1\cdots\cdots f_q\ 0\cdots 0 \\ \tilde{a} \end{pmatrix}\updownarrow p, \quad b=\begin{pmatrix} f_1\cdots\cdots f_q\ 0\cdots 0 \\ \ddots \qquad \ddots \quad \ddots \\ f_1\cdots\cdots f_q\ 0 \\ f_1\cdots\cdots f_q \\ \tilde{b} \end{pmatrix}\updownarrow q \tag{2-2-17c}$$

Finally, we comment that the betweenness condition Eq. (2-1-15) of two Gel'fand patterns in Eq. (2-1-14) is naturally understood from the vector-coupling expressions Eqs. (2-2-4) and (2-2-15) of the DG polynomial in two-fold ways.

Vector-coupling expression of the DG polynomial was noted by Moshinsky and Chacón.[1)] The purpose of this subsection is to determine the phase factor exactly and to remark that there are *two* kinds of vector-coupling expressions for the DG polynomial which is a direct consequence of the transposition property given in Eq. (2-1-26)

2-3. Transformation formula of the DG polynomials

2-3-1. Expansion of $(Tr\ R_1R_2)^N$

The decomposition Eq. (2-1-24) of $\mathscr{H}(n, m)$ and the orthonormality relationship Eq. (2-1-22) means that $\{\varphi_{a,b}^{(nm)N(\lambda)}(R)\}$ $(N(\lambda)=[f])$ with all posible labels N, (λ), *a and b* constitutes a complete orthonormal basis set of the Hilbert space $\mathscr{H}(n, m)$ composed of all entire functions of $R_{\alpha i}$ $(\alpha=1 \sim n,\ i=1\sim m)$. On the other hand, as is well known, the function

$$\exp\{\sum_{\alpha=1}^{n}\sum_{i=1}^{m} R'_{\alpha i}R^*_{\alpha i}\} = \exp\{\mathrm{Tr}\ {}^tR'\cdot R^*\} \tag{2-3-1}$$

behaves like a Dirac delta function in Bargmann space as

$$\int d\mu(R)\exp\{\mathrm{Tr}\,{}^tR'\cdot R^*\}\,f(R) = f(R'), \tag{2-3-2}$$

for an arbitrary element $f(R)$ of $\mathscr{H}(n, m)$. Thus we get the following relation

$$\begin{aligned}\exp\{\mathrm{Tr}\,{}^tR'\cdot R^*\} &= \sum_{N(\lambda)ab}\varphi_{ab}^{(nm)N(\lambda)}(R')\,\varphi_{ab}^{(nm)N(\lambda)}(R^*)\\ &= \sum_{N(\lambda)ab}\varphi_{ba}^{(mn)N(\lambda)}({}^tR')\,\varphi_{ab}^{(nm)N(\lambda)}(R^*),\end{aligned} \tag{2-3-3a}$$

where of course R' and R are both $n\times m$ matrices. The last equation of Eq. (2-3-3a) is due to Eq. (2-1-26). From this equation we get

$$\frac{1}{N!}(\mathrm{Tr}\,{}^tR'R^*)^N = \sum_{(\lambda)ab}\varphi_{ba}^{(mn)N(\lambda)}({}^tR')\,\varphi_{ab}^{(nm)N(\lambda)}(R^*). \tag{2-3-4a}$$

Now we introduce a matrix notation $\varphi^{(nm)N(\lambda)}(R)$ which is a matrix composed of matrix elements $\varphi_{a,b}^{(nm)N(\lambda)}(R)$. We can rewrite Eqs. (2-3-3a) and (2-3-4a) as follows;

$$\exp\{\mathrm{Tr}\,R_1R_2\} = \sum_{N(\lambda)}\mathrm{Tr}\{\varphi^{(mn)N(\lambda)}(R_1)\,\varphi^{(nm)N(\lambda)}(R_2)\}, \tag{2-3-3b}$$

$$\frac{1}{N!}(\mathrm{Tr}\,R_1R_2)^N = \sum_{(\lambda)}\mathrm{Tr}\{\varphi^{(mn)N(\lambda)}(R_1)\,\varphi^{(nm)N(\lambda)}(R_2)\}, \tag{2-3-4b}$$

where R_1 is an $m\times n$ matrix and R_2 an $n\times m$ matrix both with arbitrary complex matrix elements.

2-3-2. Expansion of $(Tr\ R_1R_2R_3)^N$ *and the transformation formula of the DG polynomials*

Let us consider the following quantity

$$\sum_{i=1}^{l}\sum_{j=1}^{m} C_{ji}(\boldsymbol{X}_i\cdot \boldsymbol{Y}_j^*)=\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{\alpha=1}^{n} C_{ji}X_{\alpha i}Y_{\alpha j}^*$$
$$=\mathrm{Tr}({}^tXY^*C). \qquad (2\text{-}3\text{-}5)$$

We see that $Tr({}^tXY^*C)$ is invariant under the simultaneous unitary transformation U_n of vectors $\boldsymbol{X}_i(i=1\sim l)$, $\boldsymbol{Y}_j(j=1\sim m)$. Therefore the expansion of $(Tr{}^tXY^*C)^N$ by the BIR of U_n, $\varphi^{(nl)N(\lambda)}(X)$ and $\varphi^{(nm)N(\lambda)}(Y)$, becomes

$$\frac{1}{N!}(\mathrm{Tr}\,{}^tXY^*C)^N=\sum_{(\lambda)bc} f_{cb}^{N(\lambda)}(C)\sum_{a}\varphi_{ab}^{(nl)N(\lambda)}(X)\varphi_{ac}^{(nm)N(\lambda)}(Y^*). \qquad (2\text{-}3\text{-}6)$$

Notice that $\sum_a \varphi_{ab}^{(nl)N(\lambda)}(X)\varphi_{ac}^{(nm)N(\lambda)}(Y^*)$ is invariant under U_n transformations of $\boldsymbol{X}_i(i=1\sim l)$, $\boldsymbol{Y}_j(j=1\sim m)$ for any suffices b and c. The expansion coefficient $f_{cb}^{N(\lambda)}(C)$ is a polynomial of matrix elements $C_{ij}(i=1\sim l,\ j=1\sim m)$ with total degree N. Next we notice

$$\mathrm{Tr}({}^tXY^*C)=\sum_{\alpha=1}^{n}\sum_{i=1}^{l}X_{\alpha i}\sum_{j=1}^{m}C_{ji}Y_{\alpha j}^*$$
$$=\sum_{\alpha=1}^{n}\sum_{i=1}^{l}X_{\alpha i}(\boldsymbol{C}_i\cdot \boldsymbol{Y}_\alpha^{r*}), \qquad (2\text{-}3\text{-}7)$$

which means that $Tr({}^tXY^*C)$ is invariant under the simultaneous unitary transformation U_m of vectors $\boldsymbol{C}_i$ $(i=1\sim l)$ and $\boldsymbol{Y}_\alpha^r$ $(\alpha=1\sim m)$. Thus in a similar way as in Eq. (2-3-6) we get

$$\frac{1}{N!}(\mathrm{Tr}\,{}^tXY^*C)^N=\sum_{(\lambda)b'c'}\tilde{f}_{b'c'}^{N(\lambda)}({}^tX)\sum_{a'}\varphi_{a'b'}^{(ml)N(\lambda)}(C)\varphi_{a'c'}^{(mn)N(\lambda)}({}^tY^*), \qquad (2\text{-}3\text{-}8)$$

where the expansion coefficient $\tilde{f}_{b'c'}^{N(\lambda)}({}^tX)$ is a polynomial of matrix elements $X_{\alpha i}(\alpha=1\sim n,\ i=1\sim l)$ with total degree N. Thirdly we consider the quantity

$$\mathrm{Tr}({}^tXYC^*)=\sum_{\alpha=1}^{n}\sum_{j=1}^{m}Y_{\alpha j}\sum_{i=1}^{l}C_{ji}^*X_{\alpha i}$$
$$=\sum_{\alpha=1}^{n}\sum_{j=1}^{m}Y_{\alpha j}(\boldsymbol{C}_j^{r*}\cdot \boldsymbol{X}_\alpha^r), \qquad (2\text{-}3\text{-}9)$$

which is invariant under the simultaneous unitary transformation U_l of vectors $\boldsymbol{C}_j^r$ $(j=1\sim m)$, $\boldsymbol{X}_\alpha^r$ $(\alpha=1\sim n)$. We obtain

$$\frac{1}{N!}(\mathrm{Tr}\,{}^tXYC^*)^N=\sum_{(\lambda)b''c''}\hat{f}_{b''c''}^{N(\lambda)}(Y)\sum_{a''}\varphi_{a''b''}^{(ln)N(\lambda)}({}^tX)\varphi_{a''c''}^{(lm)N(\lambda)}({}^tC^*), \qquad (2\text{-}3\text{-}10)$$

where $\hat{f}_{b''c''}^{N(\lambda)}(Y)$ is an expansion coefficient and is a polynomial of total degree N composed of matrix elements $Y_{\alpha j}(\alpha=1\sim n, j=1\sim m)$. From Eqs. (2-3-6), (2-3-8) and (2-3-10), we get

$$\frac{1}{N!}(\mathrm{Tr}\,R_1R_2R_3)^N$$
$$= \sum_{(\lambda)abc} \varphi_{ab}^{(ln)N(\lambda)}(R_1)\varphi_{bc}^{(nm)N(\lambda)}(R_2) f_{ca}^{N(\lambda)}(R_3)$$
$$= \sum_{(\lambda)abc} \tilde{f}_{ab}^{N(\lambda)}(R_1)\varphi_{bc}^{(nm)N(\lambda)}(R_2)\varphi_{ca}^{(ml)N(\lambda)}(R_3)$$
$$= \sum_{(\lambda)abc} \varphi_{ab}^{(ln)N(\lambda)}(R_1)\hat{f}_{bc}^{N(\lambda)}(R_2)\varphi_{ca}^{(ml)N(\lambda)}(R_3), \qquad (2\text{-}3\text{-}11)$$

where R_1 R_2 and R_3 are arbitrary complex matrices of the type $l\times n$, $n\times m$ and $m\times l$, respectively.

From the second equality of Eq. (2-3-11) we have

$$\tilde{f}_{ab}^{N(\lambda)}(R_1) = \sum_{a'} \langle \varphi_{ca}^{(ml)N(\lambda)}(R_3) \,|\, f_{ca'}^{N(\lambda)}(R_3)\rangle \varphi_{a'b}^{(ln)N(\lambda)}(R_1), \qquad (2\text{-}3\text{-}12)$$

where the bracket $\langle P(R_3)|Q(R_3)\rangle$ means an integration of $P(R_3)^*\cdot Q(R_3)$ with $d\mu(R_3)$. The fact that Eq. (2-3-12) is valid for any label c means that $\langle \varphi_{ca}^{(ml)N(\lambda)}(R_3)|f_{ca'}^{N(\lambda)}(R_3)\rangle$ is independent of c, which gives us

$$\tilde{f}_{ab}^{N(\lambda)}(R_1) = \sum_{a'} F_{aa'}^{N(\lambda)}\varphi_{a'b}^{(ln)N(\lambda)}(R_1),$$
$$F_{aa'}^{N(\lambda)} = \langle \varphi_{ca}^{(ml)N(\lambda)}(R_3)\,|\,f_{ca'}^{N(\lambda)}(R_3)\rangle\,. \qquad (2\text{-}3\text{-}13)$$

By using Eq. (2-3-13) for the third equality of Eq. (2-3-11) we obtain

$$\hat{f}_{bc}^{N(\lambda)}(R_2) = F_{aa}^{N(\lambda)}\varphi_{bc}^{(nm)N(\lambda)}(R_2). \qquad (2\text{-}3\text{-}14)$$

Eq. (2-3-14) means that $F_{aa}^{N(\lambda)}$ is independent of the label a, by which we can denote $F_{aa}^{N(\lambda)}$ simply as $F^{N(\lambda)}$. Thus we can write Eq. (2-3-11) in the following form

$$\frac{1}{N!}(\mathrm{Tr}\,R_1R_2R_3)^N$$
$$= \sum_{(\lambda)} F^{N(\lambda)} \sum_{abc} \varphi_{ab}^{(ln)N(\lambda)}(R_1)\varphi_{bc}^{(nm)N(\lambda)}(R_2)\varphi_{ca}^{(ml)N(\lambda)}(R_3). \qquad (2\text{-}3\text{-}15)$$

Here, without loss of generality, we can set $l<\bar{n}\equiv\mathrm{Min}\{m,n\}$. Then the label (λ) appearing in Eq. (2-3-15) is restricted to those $(\lambda)=(\lambda_1,\lambda_2,\cdots,\lambda_{l-1})$ which corresponds to the partition $[f]=[f_1,f_2,\cdots,f_l]$ with $N=\sum_{i=1}^{l} f_i$, $\lambda_i=f_i-f_{i+1}$ $(i=1\sim l-1)$. This kind of (λ) which is compatible with U_l is denoted as $(\lambda)\subset U_l$. Now we compare Eq. (2-3-15) with Eq. (2-3-4b). From this comparison we obtain

$$\varphi_{cb}^{(mn)N(\lambda)}(R_3R_1)$$
$$= \begin{cases} 0 & \text{unless} \quad (\lambda)\subset U_l \\ F^{N(\lambda)} \sum_{a} \varphi_{ca}^{(ml)N(\lambda)}(R_3)\varphi_{ab}^{(ln)N(\lambda)}(R_1) & \text{for} \quad (\lambda)\subset U_l\,, \end{cases} \qquad (2\text{-}3\text{-}16)$$

Adopting the labels b and c to be the maximum weight ones, we insert to this equation the special matrices $\tilde{R}_3$, $\tilde{R}_1$ given by

$$\tilde{R}_3 \equiv m\underbrace{\begin{pmatrix} 1 & & \\ & 1 & 0 \\ & 0 & \ddots \\ & & & 1 \\ & 0 & \end{pmatrix}}_{l}\Big\} l\,, \quad \tilde{R}_1 \equiv l\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & 0 & \\ & 0 & \ddots & 0 \\ & & & 1 \end{pmatrix}}_{n}$$

$$\tilde{R}_3\tilde{R}_1 = l\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & 0 & \\ & 0 & \ddots & 0 \\ & & & 1 \\ & 0 & & 0 \end{pmatrix}}_{n} m\,. \tag{2-3-17}$$

Since we can easily obtained the following relations from Eqs. (2-2-17), (2-1-8) and (2-1-18)

$$\begin{aligned} &\varphi_{H,H}^{(mn)N(\lambda)}(\tilde{R}_3\tilde{R}_1) = \varphi_{H,H}^{(ll)N(\lambda)}(E_l) = N_H[N(\lambda)], \\ &\varphi_{H,a}^{(ml)N(\lambda)}(\tilde{R}_3) = \varphi_{H,a}^{(ll)N(\lambda)}(E_l) = N_H[N(\lambda)]\cdot\delta_{a,H}, \\ &\varphi_{a,H}^{(ln)N(\lambda)}(\tilde{R}_1) = \varphi_{a,H}^{(ll)N(\lambda)}(E_l) = N_H[N(\lambda)]\cdot\delta_{a,H}, \\ &\quad E_l \equiv l\times l \quad \text{unit matrix,} \end{aligned} \tag{2-3-18}$$

we find the value of $F^{N(\lambda)}$ as follows;

$$F^{N(\lambda)} = \frac{1}{N_H[N(\lambda)]}\,. \tag{2-3-19}$$

Thus we can state the transformation formula of the DG polynomials as follows;

$$\varphi_{ab}^{(n_1n_2)N(\lambda)}(R_1R_2) = \begin{cases} 0 & \text{if} \quad (\lambda) \not\subset U_{n_3}, \\ \dfrac{1}{N_H[N(\lambda)]}\sum_c \varphi_{ac}^{(n_1n_3)N(\lambda)}(R_1')\,\varphi_{cb}^{(n_3n_2)N(\lambda)}(R_2) & \text{otherwise}, \end{cases} \tag{2-3-20a}$$

or

$$\varphi^{(n_1n_2)N(\lambda)}(R_1R_2) = \begin{cases} 0 & \text{if} \quad (\lambda) \not\subset U_{n_3}, \\ \dfrac{1}{N_H[N(\lambda)]}\varphi^{(n_1n_3)N(\lambda)}(R_1)\,\varphi^{(n_3n_2)N(\lambda)}(R_2) & \text{otherwise}, \end{cases} \tag{2-3-20b}$$

where R_1 and R_2 are arbitrary complex matrices of the type $n_1 \times n_3$ and $n_3 \times n_2$, respectively, and the order of the magnitudes of n_1, n_2, n_3 is arbitrary. The statement that $\varphi^{(n_1n_2)N(\lambda)}(R_1R_2)=0$ when $(\lambda) \not\subset U_{n_3}$ can be directly shown as follows. $(\lambda) \not\subset U_{n_3}$ happens only when $n_3 < \mathrm{Min}\{n_1, n_2\}$, and (λ) contains at least one-non-vanishing λ_i with $n_3 < i \leqq \mathrm{Min}\{n_1, n_2\}$. For $i > n_3$, we can show that $\Delta^{1,2,\cdots,i}_{1,2,\cdots,i}$ of R_1R_2 defined in Eq. (2-1-8b) vanishes. This is due to the well-known theorem which demands $\det CD=0$ if C and D are matrices of the type $p \times q$ and $q \times p$, respectively, with q smaller than $p (q<p)$.

Finally we should note the general transformation formula of Eq. (2-3-20a, b) can be derived from the special transformation formula

$$\varphi^{(nn)N(\lambda)}_{ab}(R_1R_2)=\frac{1}{N_H[N(\lambda)]}\sum_c \varphi^{(nn)N(\lambda)}_{ac}(R_1)\varphi^{(nn)N(\lambda)}_{cb}(R_2) \quad \text{(2-3-20c)}$$

by using the reduction relation of Eq. (2-2-17).

2-3-3. $\varphi^{(nn)N(\lambda)}(G)$ *as a representation matrix of* $GL(n;C)$

Let us consider a special case $n_1=n_3$ in Eq. (2-3-20). For an arbitrary $n \times n$ matrix G and $n \times \mathrm{m}$ argument matrices R, we have

$$\varphi^{(nm)N(\lambda)}_{a,b}({}^tGR)=\sum_c \frac{1}{N_H[N(\lambda)]}\varphi^{(nn)N(\lambda)}_{c,a}(G)\varphi^{(nm)N(\lambda)}_{c,b}(R). \quad \text{(2-3-21)}$$

By taking the $n \times n$ unit matrix E_n for G, we can see

$$\varphi^{(nn)N(\lambda)}_{c,a}(E_n)=N_H[N(\lambda)]\delta_{c,a}. \quad \text{(2-3-22)}$$

When the matrix G is parametrized as $G=\exp\{i\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}e_{\alpha\beta}\}$ and the transformation operator Eq. (2-1-28a) and its property Eq. (2-1-27a) are used, we can write Eq. (2-3-21) as

$$\begin{aligned} T^L_G\varphi^{(nm)N(\lambda)}_{a,b}(R) &= \varphi^{(nm)N(\lambda)}_{a,b}({}^tGR) \\ &= \sum_c D^{(n)N(\lambda)}_{c,a}(G)\varphi^{(nm)N(\lambda)}_{c,b}(R), \end{aligned} \quad \text{(2-3-23a)}$$

where

$$D^{(n)N(\lambda)}_{c,a}(G)\equiv\frac{1}{N_H[N(\lambda)]}\varphi^{(nn)N(\lambda)}_{c,a} \quad \text{(2-3-24)*}$$

and from Eq. (2-3-22)

$$D^{(n)N(\lambda)}_{c,a}(E_n)=\delta_{c,a}. \quad \text{(2-3-25)}$$

In the same way, we can write the transformation formula for the right transformation T^R_G in Eqs. (2-1-28b) and (2-1-27b);

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* When G is an $n \times n$ unimodular matrix U, Eq. (2-3-24) is independent of N and we may write

$$D^{(n)N(\lambda)}_{c,a}(U)\equiv D^{(n)(\lambda)}_{c,a}(U). \quad \text{(2-3-26)}$$

$$T_G^R \varphi_{a,b}^{(nm)N(\lambda)}(R) = \varphi_{a,b}^{(nm)N(\lambda)}(RG)$$
$$= \sum_c D_{c,b}^{(m)N(\lambda)}(G)\varphi_{a,c}^{(nm)N(\lambda)}(R), \qquad (2\text{-}3\text{-}23b)$$

where G is an arbitrary $m\times m$ matrix. Equation (2-3-23a) means that the set SET $I\equiv\{\varphi_{a,b}^{(nm)N(\lambda)}(R)\}$ with arbitrary but fixed b constitutes a representation basis of $GL(n;C)$, whose representation matrix is $D^{(n)N(\lambda)}(G)\equiv\{D_{a,b}^{(n)N(\lambda)}(G)\}$, and Eq. (2-3-23b) that the set SET II$\equiv\{\varphi_{a,b}^{(nm)N(\lambda)}(R)\}$ with arbitrary but fixed a constitutes a representation basis of $GL(m;C)$, whose representation matrix is given by $D^{(m)N(\lambda)}(G)$.

The fact that $\varphi^{(nn)N(\lambda)}(G)$ is proportional to the representation matrix of $GL(n;C)$ was noted first by Louck[5] for U_n and later extended to $GL(n;C)$ by Brunet and Seligman[12] in a quite different way from ours.

2-4. The Clebsch-Gordan series

Let us consider a product of two representation matrices of $GL(n;C)$ with the common argument matrix R; $D_{a_1,b_1}^{(n)N_1(\lambda_1)}(R)\,D_{a_2,b_2}^{(n)N_2(\lambda_2)}(R)$. Combining the internal quantum numbers a_1 and a_2 by the use of the SU_n Clebsch-Gordan (C-G) coefficients and using the completeness relation of $n\times n$ DG polynomials, we can expand as

$$\sum_{a_1,a_2}\langle(\lambda_1)\,a_1(\lambda_2)\,a_2|\,(\lambda)\,a\,;\rho\rangle_n D_{a_1,b_1}^{(n)N_1(\lambda_1)}(R)\,D_{a_2,b_2}^{(n)N_2(\lambda_2)}(R)$$
$$=\sum_b C_{b_1,b_2,b} D_{ab}^{(n)N_1+N_2(\lambda)}(R), \qquad (2\text{-}4\text{-}1)$$

where ρ is an index to classify the multiplicity. The expansion coefficient $C_{b_1,b_2,b}$ is obtained by setting $R=E_n$ in Eq. (2-4-1). From Eq. (2-3-25) we obtain $C_{b_1,b_2,a}=\langle(\lambda_1)\,b_1(\lambda_2)\,b_2|\,(\lambda)\,a\,;\rho\rangle_n$. As a result, we obtain the following C-G series of the representation matrices of $GL(n;C)$;

$$\sum_{a_1,a_2}\langle(\lambda_1)\,a_1(\lambda_2)\,a_2|\,(\lambda)\,a\,;\rho\rangle_n D_{a_1,b_1}^{(n)N_1(\lambda_1)}(R)\,D_{a_2,b_2}^{(n)N_2(\lambda_2)}(R)$$
$$=\sum_b\langle(\lambda_1)\,b_1(\lambda_2)\,b_2|\,(\lambda)\,b\,;\rho\rangle_n D_{a,b}^{(n)N_1+N_2(\lambda)}(R), \qquad (2\text{-}4\text{-}2)$$

From the orthogonality relation of the C-G coefficients, we have

$$\sum_{a_1,a_2}\sum_{b_1,b_2}\langle(\lambda_1)\,a_1(\lambda_2)\,a_2|\,(\lambda)\,a\,;\rho\rangle_n\langle(\lambda_1)\,b_1(\lambda_2)\,b_2|\,(\lambda)\,b\,;\rho\rangle_n$$
$$\times D_{a_1,b_1}^{(n)N_1(\lambda_1)}(R)\,D_{a_2,b_2}^{(n)N_2(\lambda_2)}(R) = D_{a,b}^{(n)N_1+N_2(\lambda)}(R) \qquad (2\text{-}4\text{-}3)$$

and

$$D_{a_1,b_1}^{(n)N_1(\lambda_1)}(R)\,D_{a_2,b_2}^{(n)N_2(\lambda_2)}(R)$$
$$=\sum_{(\lambda)}\sum_{a,b}\sum_{\rho}\langle(\lambda_1)\,a_1(\lambda_2)\,a_2|\,(\lambda)\,a\,;\rho\rangle_n$$
$$\times\langle(\lambda_1)\,b_1(\lambda_2)\,b_2|\,(\lambda)\,b\,;\rho\rangle_n D_{a,b}^{(n)N_1+N_2(\lambda)}(R). \qquad (2\text{-}4\text{-}4)$$

The multiplicity quantum number ρ can be specified according to the Biedenharn-Louck canonical coupling prescription and can be described in terms of upper Gel'fand patterns.[13]

2-5. The Complex Conjugate Representation of the SU_n Group

In this subsection we consider only the IR of SU_n by the $n\times n$ DG polynomials $\varphi^{N(\lambda)}_{ab}(R)$. The restriction of G to a unimodular unitary $n\times n$ matrix U in Eq. (2-3-23) with $m=n$ gives us such representations. A basis of complex conjugate representation (abbreviated to BCCR) $\varphi^{N(\lambda);c}_{ab}(R)$ of SU_n is defined by the following equations;

$$T^L_U\varphi^{N(\lambda);c}_{ab}(R)=\sum_c D^{(\lambda)}_{ca}(U)^*\varphi^{N(\lambda);c}_{cb}(R), \tag{2-5-1a}$$

$$T^R_U\varphi^{N(\lambda);c}_{ab}(R)=\sum_c D^{(\lambda)}_{cb}(U)^*\varphi^{N(\lambda);c}_{ac}(R), \tag{2-5-1b}$$

where T^L_U and T^R_U should be considered as in Eqs. (2-1-28a) and (2-1-28b) (with $m=n$) by parametrizing U as

$$U=\exp\{i\sum_{\alpha,\beta=1}^{n}u_{\alpha\beta}e_{\alpha\beta}\},$$

$$u^*_{\alpha\beta}=u_{\beta\alpha},\quad \sum_{\alpha=1}^{n}u_{\alpha\alpha}=0\,. \tag{2-5-2}$$

According to Louck,[5),14)] we can find a BCCR in the space of $\{\varphi^{N(\lambda)}_{a,b}(R)\}$, by specifying the label of the IR as follows. First, for the quantum numbers $N(\lambda)a\equiv[f_{\alpha\beta}]$ $(\alpha=1\sim\beta,\ \beta=1\sim n)$, we define the conjugate quantum numbers $(\bar\lambda)$ and $\bar a$ as $-N(\bar\lambda)\bar a\equiv[\bar f_{\alpha\beta}]$ where $\bar f_{\alpha\beta}=-f_{\beta-\alpha+1,\beta}$. If we use the notation $\lambda_{\alpha\beta}\equiv f_{\alpha\beta}-f_{\alpha+1,\beta}$, we can write $\bar\lambda_{\alpha\beta}=\lambda_{\beta-\alpha,\beta}$. Although this transformation of quantum numbers breaks the condition that the partition $[f]$ is non-negative, it should be noted that the condition is recovered by adding some positive number commonly to all partition numbers in the two Gel'fand patterns, which is valid if we consider only unimodular transformations of DG polynomials. For a BIR with these conjugate quantum numbers we can write for the left transformation, for instance,

$$T^L_U\varphi^{N(\bar\lambda)}_{\bar a\bar b}(R)=\sum_c D^{(\bar\lambda)}_{\bar c\bar a}(U)\varphi^{N(\bar\lambda)}_{\bar c\bar b}(R). \tag{2-5-3}$$

On the other hand, we can verify for the representation matrix of SU_n that

$$D^{(\lambda)}_{ca}(U)^*=(-)^{\phi^{(\lambda)}(c)-\phi^{(\lambda)}(a)}D^{(\bar\lambda)}_{\bar c\bar a}(U),$$

$$\phi^{(\lambda)}(c)=\sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{\beta}f_{\alpha\beta}\quad\text{for}\quad(\lambda)c\equiv[f_{\alpha\beta}]\,(\beta=1\sim n-1)\ \text{etc.}, \tag{2-5-4}$$

the proof of which is given in Appendix C based on the relation between matrix elements of the U_n generators by the Gel'fand bases (Eq. (C-1)).

By comparing Eq. (2-5-3) with Eq. (2-5-1a) after the substitution of Eq. (2-5-4) and by the similar procedure for the right transformation, we obtain the result

$$\varphi_{ab}^{N(\lambda);c}(R) = (-)^{\psi^{(\lambda)}(a)+\psi^{(\lambda)}(b)}\varphi_{\bar{a}\bar{b}}^{N(\bar{\lambda})}(R), \tag{2-5-5}$$

where the polynomials of R with degree N are adopted.

Another method to find a BCCR in the space of $\{\varphi_{a,b}^{N(\lambda)}(R)\}$ is due to the explicit $\mathcal{R}$-conjugation transformation;[15)]

$$\mathcal{R}_R; R \rightarrow \delta_R, \tag{2-5-6}$$

where $\delta_R \equiv (\delta_{\alpha i}) = {}^tR^{-1}|R|$ is a $n\times n$ matrix composed of (α, i) cofactors $\delta_{\alpha i}$ of R. Considering that U is a unimodular unitary matrix, we can easily find, from Eq. (2-1-27),

$$T_U^L\delta_R = {}^tU^*\cdot\delta_R, \quad T_U^R\delta_R = \delta_R\cdot U^*. \tag{2-5-7}$$

Thus $\mathcal{R}_R\varphi_{ab}^{N(\lambda)}(R) = \varphi_{ab}^{N(\lambda)}(\delta_R)$ satisfies Eq. (2-5-1) and should be proportional to $\varphi_{ab}^{N'(\lambda);c}(R)$ where $N' = (n-1)N$ since $\delta_{\alpha i}$ is a homogeneous polynomial with degree $n-1$. In the following we always consider only the DG polynomials with the lowest degree; namely,

$$N = \sum_{k=1}^{n-1} k\lambda_k, \quad \overline{N} \equiv \sum_{k=1}^{n-1} k\bar{\lambda}_k = \sum_{k=1}^{n-1} k\lambda_{n-k}, \tag{2-5-8}$$

as long as we treat the BIR of SU_n. Then we can write

$$\begin{aligned}\varphi_{ab}^{\overline{N}(\lambda);c}(R) &= (-)^{\psi^{(\lambda)}(a)+\psi^{(\lambda)}(b)}\varphi_{\bar{a}\bar{b}}^{\overline{N}(\bar{\lambda})}(R) \\ &= C\varphi_{ab}^{N(\lambda)}(\delta_R)\cdot|R|^{-w},\end{aligned} \tag{2-5-9}$$

where

$$w = \frac{1}{n}\{(n-1)N - \overline{N}\} = \sum_{k=1}^{n-1}(k-1)\lambda_k. \tag{2-5-10}$$

The constant C is obtained by setting $R = E_n$ (the $n\times n$ unit matrix) and, by using Eq. (2-3-22), we have

$$C = \frac{N_H[\overline{N}(\bar{\lambda})]}{N_H[N(\lambda)]}, \tag{2-5-11}$$

where N and $\overline{N}$ are defined by Eq. (2-5-8).

2-6. Explicit expressions of DG polinomials

In this subsection we show explicit expressions of the DG polynomials with respect to $SU_2\times SU_2$, $SU_3\times SU_2$ and $SU_3\times SU_3$ cases. For later applications we use shorthand notations of DG polynomials and simple parametrizations of quantum numbers in each case so far as no confusion takes place. Reduction of the DG polynomials to lower ranks, the vector-coupling expressions in two-fold ways and the explicit expressions of the BCCR are also shown.

2-6-1. $SU_2 \times SU_2$

The parametrization we use in this paper is

$$\varphi_{ab}^{(22)N(\lambda)}(R) \equiv \varphi_{r,r'}^{(\lambda)\mu}(R)$$

$$= \left| \begin{matrix} \lambda+\mu & \mu & \lambda+\mu & \mu \\ \lambda+\mu-r & & \lambda+\mu-r' & \end{matrix} \right\rangle$$

$$= \mathcal{N}(\lambda;r)\, A_{yx}^{r} \cdot \mathcal{N}(\lambda;r')\, T_{21}^{r'} \varphi_{HH}^{(\lambda)\mu}(R), \tag{2-6-1}$$

where $N=\lambda+2\mu$ and the ranges of r and r' are from zero to λ. $\mathcal{N}(\lambda;r)$ is the normalization of the SU_2 lowering operator and is

$$\mathcal{N}(\lambda;r) \equiv \left[\frac{(\lambda-r)!}{\lambda!\,r!}\right]^{1/2}. \tag{2-6-2}$$

The polynomial of the highest weights with respect to two kinds of U_2 generator algebras is

$$\varphi_{HH}^{(\lambda)\mu}(R) = N_H(\lambda\mu)\, R_{x1}^{\lambda} |R|^{\mu}, \tag{2-6-3}$$

$$|R| \equiv \begin{vmatrix} R_{x1} R_{x2} \\ R_{y1} R_{y2} \end{vmatrix}, \quad N_H(\lambda\mu) \equiv \left[\frac{\lambda+1}{(\lambda+\mu+1)!\,\mu!}\right]^{1/2} \tag{2-6-4}$$

The result of lowering operation in Eq. (2-6-1) is

$$\varphi_{rr'}^{(\lambda)\mu}(R) = N(\lambda\mu rr') \sum_{a=\mathrm{Max}\{0,r+r'-\lambda\}}^{\mathrm{Min}\{r,r'\}} \frac{1}{(\lambda-r-r'+a)!\,(r-a)!\,(r'-a)!\,a!}$$
$$R_{x1}^{\lambda-r-r'+a} R_{y1}^{r-a} R_{x2}^{r'-a} R_{y2}^{a} |R|^{\mu}, \tag{2-6-5}$$

where

$$N(\lambda\mu rr') = \left[\frac{(\lambda+1)\,r!\,r'!\,(\lambda-r)!\,(\lambda-r')!}{(\lambda+\mu+1)!\,\mu!}\right]^{1/2}. \tag{2-6-6}$$

When the argument matrix R is a unimodular unitary matrix, we can parametrize it as

$$R = \begin{pmatrix} R_{x1} & R_{x2} \\ R_{y1} & R_{y2} \end{pmatrix} = \begin{pmatrix} e^{-i(\varphi/2)} \cos\frac{\theta}{2} e^{-i(\psi/2)} & -e^{-i(\varphi/2)} \sin\frac{\theta}{2} e^{i(\psi/2)} \\ e^{i(\varphi/2)} \sin\frac{\theta}{2} e^{-i(\psi/2)} & e^{i(\varphi/2)} \cos\frac{\theta}{2} e^{i(\psi/2)} \end{pmatrix} \tag{2-6-7}$$

and the relation of the DG polynomial to the rotation matrix of the angular momentum algebra is

$$\frac{1}{N_H(\lambda\mu)}\varphi_{rr'}^{(\lambda)\mu}(R)=e^{-i((\lambda/2)-r)\varphi}d^{\lambda/2}_{\lambda/2-r,\lambda/2-r'}(\theta)\,e^{-i((\lambda/2)-r')\phi}$$
$$=D^{\lambda/2}_{\lambda/2-r,\lambda/2-r'}(\varphi,\theta,\psi)\,. \qquad (2\text{-}6\text{-}8)$$

2-6-2. $SU_3\times SU_2$

The parametrization in this case is

$$\varphi_{ab}^{(32)N(\lambda\mu)}(R)\equiv\varphi^{(\lambda\mu)}_{[\varepsilon\Lambda\nu],r'}(R)\equiv\varphi^{(\lambda\mu)}_{pqr,r'}(R)$$
$$=\left|\begin{matrix}\lambda+\mu & \mu & 0\\ \lambda+\mu-p & \mu-q & \\ \lambda+\mu-p-r & & \end{matrix}\right.\left.\begin{matrix}\lambda+\mu & \mu\\ \lambda+\mu-r' & \end{matrix}\right\rangle$$
$$=\mathscr{N}(\lambda\mu;pqr)\,A^r_{yx}A^q_{zy}O^p_{zx}\cdot\mathscr{N}(\lambda;r')\,T^{r'}_{21}\varphi^{(\lambda\mu)}_{HH}(R), \qquad (2\text{-}6\text{-}9)$$

where $\varphi^{(\lambda\mu)}_{HH}(R)$ is $\varphi^{(\lambda)\mu}_{HH}(R)$ given in Eq. (2-6-3) and

$$O_{zx}\equiv A_{zx}(A_{xx}-A_{yy}+1)+A_{yx}A_{zy}$$
$$=(A_{xx}-A_{yy}+1)A_{zx}+A_{zy}A_{yx}\,. \qquad (2\text{-}6\text{-}10)$$

The ranges of internal quantum numbers are $p=0\sim\lambda$, $q=0\sim\mu$, $r=0\sim\lambda-p+q$ and $r'=0\sim\lambda$, and $N=\lambda+2\mu$ is the total degree of the DG polynomial. The normalization of the SU_3 lowering operators is

$$\mathscr{N}(\lambda\mu;pqr)$$
$$=\left[\frac{(\lambda-p)!(\mu-q)!(\lambda-p+q+1)!(\lambda+\mu-p+1)!}{p!q!(\lambda+q+1)!\lambda!\mu!(\lambda+\mu+1)!}\cdot\frac{(\lambda-p+q-r)!}{r!(\lambda-p+q)!}\right]^{1/2},$$
$$(2\text{-}6\text{-}11)$$

which satisfies $\mathscr{N}(\lambda\mu;oor)=\mathscr{N}(\lambda;r)$. ε, Λ, ν are due to the so-called Elliott's notation and are the eigen-values of the operators Q_0, $\boldsymbol{\Lambda}^2$ and $2\Lambda_0$, respectively, where $Q_0=2A_{zz}-A_{xx}-A_{yy}$, $\Lambda_0=(A_{xx}-A_{yy})/2$, $\Lambda_+=A_{xy}$ and $\Lambda_-=A_{yx}$. Their relation with quantum numbers p, q and r, however, is a little different from the usual one; namely,

$$\varepsilon=3(p+q)-\lambda-2\mu\,,$$
$$\Lambda=\frac{1}{2}(\lambda-p+q),$$
$$\frac{1}{2}\nu=\Lambda-r\,. \qquad (2\text{-}6\text{-}12)$$

The calculated result of lowering operation is

$$\varphi^{(\lambda\mu)}_{pqr,H}(R)=N(\lambda\mu;pqr)\sum_{k=\mathrm{Max}\{0,p+r-\lambda\}}^{\mathrm{Min}\{r,q\}}(-)^{q-k}r!\binom{\lambda-p}{r-k}\binom{q}{k}$$
$$\times R^{\lambda-p-r+k}_{x1}R^{r-k}_{y1}R^{p}_{z1}(\delta_{12})^k_x(\delta_{12})^{q-k}_y(\delta_{12})^{\mu-q}_z \qquad (2\text{-}6\text{-}13)$$

and

$$\varphi^{(\lambda\mu)}_{pqr,r'}(R)=N(\lambda\mu;pqr)\mathscr{N}(\lambda;r')\sum_{k=\mathrm{Max}\{0,p+r-\lambda\}}^{\mathrm{Min}\{r,q\}}(-)^{q-k}r!\binom{\lambda-p}{r-k}\binom{q}{k}$$
$$\times\sum_{a+b+c=r'}r'!\binom{\lambda-p-r+k}{a}\binom{r-k}{b}\binom{p}{c}$$
$$\times R_{x1}^{\lambda-p-r+k-a}R_{y1}^{r-k-b}R_{z1}^{p-c}R_{x2}^{a}R_{y2}^{b}R_{z2}^{c}$$
$$\times(\delta_{12})_x^k(\delta_{12})_y^{q-k}(\delta_{12})_z^{\mu-q}, \qquad (2\text{-}6\text{-}14)$$

where, for instance,

$$(\delta_{12})_x\equiv\begin{vmatrix}R_{y1}&R_{y2}\\R_{z1}&R_{z2}\end{vmatrix} \qquad (2\text{-}6\text{-}15)$$

and the normalization $N(\lambda\mu;pqr)$ is

$$N(\lambda\mu;pqr)\equiv\mathscr{N}(\lambda\mu;pqr)N_H(\lambda\mu)\frac{\lambda!\mu!(\lambda+\mu+1)!}{(\lambda-p)!(\mu-q)!(\lambda+\mu-p+1)!}$$
$$=\left[\frac{(\lambda+1)!(\lambda-p+q+1)!}{p!q!(\lambda-p)!(\mu-q)!(\lambda+\mu-p+1)!(\lambda+q+1)!}\cdot\frac{(\lambda-p+q-r)!}{r!(\lambda-p+q)!}\right]^{1/2} \qquad (2\text{-}6\text{-}16)$$

The expression of Eq. (2-6-13) is essentially the same as that used by Resnikoff[15] for the calculation of C-G coefficients and Racah coefficients of the SU_3 algebra.

2-6-3. $SU_3\times SU_3$

In this case we write

$$\varphi^{(33)N(\lambda\mu)}_{a,b}(R)\equiv\varphi^{N(\lambda\mu)}_{[\varepsilon\Lambda\nu],[eSt]}(R)\equiv\varphi^{(\lambda\mu)\omega}_{pqr,p'q'r'}(R)$$
$$=\left|\begin{matrix}\lambda+\mu+\omega&\mu+\omega&\omega\\\lambda+\mu+\omega-p&\mu+\omega-q&\\\lambda+\mu+\omega-p-r&&\end{matrix}\;\begin{matrix}\lambda+\mu+\omega&\mu+\omega&\omega\\\lambda+\mu+\omega-p'&\mu+\omega-q'&\\\lambda+\mu+\omega-p'-r'&&\end{matrix}\right\rangle$$
$$=\mathscr{N}(\lambda\mu;pqr)A_{yx}^{r}A_{zy}^{q}O_{zx}^{p}$$
$$\times\mathscr{N}(\lambda\mu;p'q'r')T_{21}^{r'}T_{32}^{q'}Q_{31}^{p'}\varphi^{(\lambda\mu)\omega}_{H,H}(R), \qquad (2\text{-}6\text{-}17)$$

where $N=\lambda+2\mu+3\omega$ and

$$Q_{31}=T_{31}(T_{11}-T_{22}+1)T_{21}T_{32}$$
$$=(T_{11}-T_{22}+1)T_{31}+T_{32}T_{21}. \qquad (2\text{-}6\text{-}18)$$

The relation of the notations e, S, t and p', q', r' is given in the similar way as in the case of ε, Λ, ν and p, q, r (Eq. (2-6-12)). The polynomial of the highest weight $\varphi^{(\lambda\mu)\omega}_{HH}(R)$ is given by

$$\varphi_{HH}^{(\lambda\mu)\omega}(R) = N_H(\lambda\mu\omega) R_{x1}^{\lambda}(\delta_{z3})^{\mu}|R|^{\omega}\,, \tag{2-6-19}$$

where δ_{z3} is a $(z, 3)$ cofactor of the 3×3 determinant $|R|$; namely,

$$\delta_{z3} \equiv \begin{vmatrix} R_{x1} & R_{x2} \\ R_{y1} & R_{y2} \end{vmatrix} \quad \text{and} \quad |R| \equiv \begin{vmatrix} R_{x1} & R_{x2} & R_{x3} \\ R_{y1} & R_{y2} & R_{y3} \\ R_{z1} & R_{z2} & R_{z3} \end{vmatrix}. \tag{2-6-20}$$

The normalization $N_H(\lambda\mu\omega)$ is

$$N_H(\lambda\mu\omega) \equiv \left[\frac{(\lambda+1)(\mu+1)(\lambda+\mu+2)}{(\lambda+\mu+\omega+2)!(\mu+\omega+1)!\omega!}\right]^{1/2} \tag{2-6-21}$$

and, especially, $N_H(\lambda\mu 0) = N_H(\lambda\mu)$. The lowering operation in Eq. (2-6-17) is straightforward but very tedious. Although this was done by Holman,[16] we show a method to perform the lowering operations with respect to p and p' in Appendix D, since in Ref. 16) the process is not written explicitly. The result is

$$\begin{aligned} &O_{zx}^{p} Q_{31}^{p'} R_{x1}^{\lambda}(\delta_{z3})^{\mu} \\ &= \frac{\lambda!\mu!(\lambda+\mu+1)!p!p'!}{(\lambda+\mu-p+1)!(\lambda+\mu-p'+1)!} \sum_{a=\mathrm{Max}\{0,p+p'-\lambda\}}^{\mathrm{Min}\{p,p'\}} \sum_{b=\mathrm{Max}\{0,a-\mu\}}^{a} \\ &\quad \times (-)^{a+b} \frac{(\lambda+\mu-a+b+1)!}{(\lambda-p-p'+a)!(p'-a)!(p-a)!b!(\mu-a+b)!(a-b)!} \\ &\quad \times R_{x1}^{\lambda-p-p'+a} R_{x3}^{p'-a} R_{z1}^{p-a} R_{z3}^{b} (\delta_{z3})^{\mu-a+b}|R|^{a-b}\,. \end{aligned} \tag{2-6-22}$$

From this expression we can easily obtain the polynomials with the highest weights in SU_2 parts;

$$\begin{aligned} &\varphi_{[\varepsilon A,2A];[\varepsilon S,2S]}^{N(\lambda\mu)}(R) = \varphi_{pq0,p'q'0}^{(\lambda\mu)\omega}(R) \\ &= N_H(\lambda\mu\omega)\mathcal{N}(\lambda\mu;pq0)\mathcal{N}(\lambda\mu;p'q'0) \\ &\quad \times \frac{\lambda!\mu!(\lambda+\mu+1)!p!p'!q!q'!}{(\lambda+\mu-p+1)!(\lambda+\mu-p'+1)!} \sum_{a=\mathrm{Max}\{0,p+p'-\lambda\}}^{\mathrm{Min}\{p,p'\}} \sum_{b=\mathrm{Max}\{0,a-\mu\}}^{a} \\ &\quad \times \sum_{c=\mathrm{Max}\{0,q+q'-\mu+a-b\}}^{\mathrm{Min}\{q,q'\}} (-)^{a+b+q+q'}(\lambda+\mu-a+b+1)! \\ &\quad \times [(\lambda-p-p'+a)!(p'-a)!(p-a)!b!(a-b)! \\ &\quad \times c!(q-c)!(q'-c)!(\mu-q-q'-a+b+c)!]^{-1} \\ &\quad \times R_{x1}^{\lambda-p-p'+a} R_{x3}^{p'-a} R_{z1}^{p-a} R_{z3}^{b} (\delta_{y2})^{c}(\delta_{y3})^{q-c} \\ &\quad \times (\delta_{z2})^{q'-c}(\delta_{z3})^{\mu-q-q'-a+b+c}|R|^{\omega+a-b}\,. \end{aligned} \tag{2-6-23}$$

In order to perform the lowering operation of r and r', it is convenient to expand the cofactors $(\delta_{y2})^{c}$ in Eq. (2-6-23). Final explicit expression of the 3×3 DG polynomial is

$$
\begin{aligned}
&\varphi^{(\lambda\mu)\omega}_{pqr,p'q'r'}(R)\\
&=N_H(\lambda\mu\omega)\mathcal{N}(\lambda\mu;pqr)\mathcal{N}(\lambda\mu;p'q'r')\\
&\quad\times\frac{\lambda!\mu!(\lambda+\mu+1)!p!p'!q!q'!r!r'!}{(\lambda+\mu-p+1)!(\lambda+\mu-p'+1)!}\\
&\quad\times\sum_{a=\mathrm{Max}\{0,p+p'-\lambda\}}^{\mathrm{Min}\{p,p'\}}\ \sum_{b=\mathrm{Max}\{0,a-\mu\}}^{a}\ \sum_{c=\mathrm{Max}\{0,q+q'-\mu+a-b\}}^{\mathrm{Min}\{q,q'\}}\ \sum_{d=0}^{c}\\
&\quad\times(-)^{q+q'+a+b+d}\frac{(\lambda+\mu-a+b+1)!(\lambda-p-p'+a+c-d)!}{(\lambda-p-p'+a)!(p'-a)!(p-a)!b!(a-b)!}\\
&\quad\times\frac{(p'-a+d)!(p-a+d)!}{(\mu-q-q'-a+b+c)!d!(c-d)!}\\
&\quad\times\sum_{n_1=0}^{r}\ \sum_{n_1'=0}^{r'}\ \sum_{e=\mathrm{Max}\{0,r+r'+p+p'-\lambda-n_1-n_1'-a-c+d\}}^{\mathrm{Min}\{r-n_1,r'-n_1'\}}\\
&\quad\times\sum_{n_2=\mathrm{Max}\{0,a-d+n_1-p'\}}^{\mathrm{Min}\{n_1,q-c\}}\ \sum_{n_2'=\mathrm{Max}\{0,a-d+n_1'-p\}}^{\mathrm{Min}\{n_1',q'-c\}}(-)^{n_2+n_2'}\\
&\quad\times[(\lambda-p-p'+a+c-d-r-r'+n_1+n_1'+e)!(r'-n_1'-e)!\\
&\quad\times(p'-a+d-n_1+n_2)!(r-n_1-e)!e!(n_1-n_2)!(p-a+d-n_1'+n_2')!\\
&\quad\times(n_1'-n_2')!n_2!(q-c-n_2)!n_2'!(q'-c-n_2')!]^{-1}\\
&\quad\times R_{x1}^{\lambda-p-p'+a+c-d-r-r'+n_1+n_1'+e}R_{x2}^{r'-n_1'-e}\\
&\quad\times R_{x3}^{p'-a+d-n_1+n_2}R_{y1}^{r-n_1-e}R_{y2}^{e}R_{y3}^{n_1-n_2}R_{z1}^{p-a+d-n_1'+n_2'}\\
&\quad\times R_{z2}^{n_1'-n_2'}R_{z3}^{b+c-d}(\delta_{x3})^{n_2}(\delta_{y3})^{q-c-n_2}(\delta_{z1})^{n_2'}\\
&\quad\times(\delta_{z2})^{q'-c-n_2'}(\delta_{z3})^{\mu-q-q'-a+b+c}|R|^{\omega+a-b}. \qquad (2\text{-}6\text{-}24)
\end{aligned}
$$

2-6-4. *Reduction of the DG polynomials to lower ranks and the vector-coupling expressions in two-fold ways*

Equations (2-6-24), (2-6-14) and (2-6-5) satisfy the following reduction formulas;

$$\varphi^{(33)(\lambda\mu)0}_{pqr,00r'}(R)=\varphi^{(32)(\lambda\mu)}_{pqr,r'}(R), \qquad (2\text{-}6\text{-}25)$$

$$\varphi^{(32)(\lambda\mu)}_{00r,r'}(R)=\varphi^{(22)(\lambda)\mu}_{rr'}(R), \qquad (2\text{-}6\text{-}26)$$

where the argument matrix R is of an appropriate type associated with that of the respective DG polynomials. By setting $\mu=q=0$ in Eq. (2-6-13) we obtain the reduction from the $SU_3\times SU_2$ DG polynomial to a $SU_3\times SU_1$ one, which we write

$$U_{(\lambda0)pr}(\boldsymbol{R}_1)\equiv\frac{R_{x1}^{\lambda-p-r}R_{y1}^{r}R_{z1}^{p}}{\sqrt{(\lambda-p-r)!r!p!}}=\varphi^{(32)(\lambda0)}_{p0r,0}(R). \qquad (2\text{-}6\text{-}27)$$

In shorthand notation we sometimes write $U_{(\lambda0)}(\boldsymbol{R})$ as $U_\lambda(\boldsymbol{R})$. Furthermore, by setting $p=0$ in Eq. (2-6-27) we obtain a $SU_2\times SU_1$ DG polynomial, which we write

$$v_{\lambda/2,\lambda/2-r}(\boldsymbol{R}) = \frac{R_x^{\lambda-r}R_y^r}{\sqrt{(\lambda-r)!r!}} = U_{(\lambda 0)0r}(\boldsymbol{R}). \tag{2-6-28}$$

Here, we use the notation of an angular momentum state according to Bargmann;[6] namely,

$$v_{jm}(\boldsymbol{R}) \equiv \frac{R_x^{j+m}R_y^{j-m}}{\sqrt{(j+m)!(j-m)!}}. \tag{2-6-29}$$

The vector coupling expression of the $SU_2\times SU_2$ DG polynomial is

$$\varphi_{rr'}^{(22)(\lambda)\mu}(R) = [v_{(\lambda+\mu-r')/2}(\boldsymbol{R}_1)\,v_{(\mu+r')/2}(\boldsymbol{R}_2)]_{\lambda/2,\lambda/2-r} \tag{2-6-30a}$$

$$= [v_{(\lambda+\mu-r)/2}(\boldsymbol{R}_x^r)\,v_{(\mu+r)/2}(\boldsymbol{R}_y^r)]_{\lambda/2,\lambda/2-r'}, \tag{2-6-30b}$$

where the coupling is the angular momentum one. In the case of $SU_3\times SU_2$, we have

$$\varphi_{[\varepsilon\Lambda\nu],r'}^{(32)(\lambda\mu)}(R) = \varphi_{pqr,r'}^{(32)(\lambda\mu)}(R)$$

$$= [U_{\lambda+\mu-r'}(\boldsymbol{R}_1)\,U_{\mu+r'}(\boldsymbol{R}_2)]_{(\lambda\mu)\varepsilon\Lambda\nu} \tag{2-6-31a}$$

$$= [[v_{(\lambda+\mu-p-r)/2}(\boldsymbol{R}_x^r)\,v_{(\mu-q+r)/2}(\boldsymbol{R}_y^r)]_{\Lambda=(\lambda-p+q)/2}v_{(p+q)/2}(\boldsymbol{R}_z^r)]_{\lambda/2,\lambda/2-r'} \tag{2-6-31b}$$

where the coupling in Eq. (2-6-31a) is the SU_3 coupling $(\lambda+\mu-r',0)\times(\mu+r',0)\to(\lambda\mu)$, while in Eq. (2-6-31b) the angular momentum one.* All the couplings in the $SU_3\times SU_3$ DG polynomial are the SU_3 couplings and

$$\varphi_{[\varepsilon\Lambda\nu],[eSt]}^{(33)N(\lambda\mu)}(R) = \varphi_{pqr,p'q'r'}^{(33)(\lambda\mu)\omega}(R)$$

$$= [[U_{N_1}(\boldsymbol{R}_1)\,U_{N_2}(\boldsymbol{R})]_{(\sigma\tau)}U_{N_3}(\boldsymbol{R}_3)]_{(\lambda\mu)\varepsilon\Lambda\nu} \tag{2-6-32a}$$

$$= [[U_{P_x}(\boldsymbol{R}_x^r)\,U_{P_y}(\boldsymbol{R}_y^r)]_{(lm)}U_{P_z}(\boldsymbol{R}_z^r)]_{(\lambda\mu)eSt}, \tag{2-6-32b}$$

where

$$N_1 = \lambda+\mu+\omega-p'-r' = \sigma+\tau-r',$$

$$N_2 = \mu+\omega-q'+r' = \tau+r',$$

$$N_3 = \omega+p'+q' = N-\sigma-2\tau,$$

$$(\sigma,\tau) = (\lambda-p'+q',\ \mu+\omega-q')$$

---

* In order to obtain Eq. (2-6-31b) from Eq. (2-2-15), we should use the reduction formula Eq. (5-2-3) from the U_2 C-G coefficients to the SU_2 ones; namely,

$$\langle[\lambda+\mu-p,\ \mu-q]\lambda+\mu-p-r_1[p+q]p+q-r_2|[\lambda+\mu,\ \mu]\lambda+\mu-r'\rangle_2$$

$$=\langle[\lambda-p+q,0]\lambda-p+q-r_1[p+q]p+q-r_2|[\lambda+q,\ q]\lambda+q-r'\rangle_2$$

$$=\left\langle\frac{\lambda-p+q}{2}\ \frac{\lambda-p+q}{2}-r_1\ \frac{p+q}{2}\ \frac{p+q}{2}-r_2\middle|\frac{\lambda}{2}\ \frac{\lambda}{2}-r'\right\rangle.$$

$$=\left(2S,\frac{1}{2}(N_1+N_2-\sigma)\right), \tag{2-6-33a}$$

$$\begin{aligned}
&P_x=\lambda+\mu+\omega-p-r=l+m-r\,,\\
&P_y=\mu+\omega-q+r=m+r\\
&P_z=\omega+p+q=N-l-2m\,,\\
&(l,m)=(\lambda-p+q,\mu+\omega-q)\\
&\qquad=\left(2\Lambda,\frac{1}{2}(P_x-P_y-l)\right).
\end{aligned} \tag{2-6-33b}$$

2-6-5. *The basis of complex conjugate representation (BCCR) of the 2×2 and 3×3 DG polynomials*

The quantum number of the complex conjugate representation of SU_2 is determined, as discussed in § 2-5, from the following transformation;

$$\left|\begin{matrix}\lambda & 0\\ \lambda-r & \end{matrix}\right\rangle \rightarrow \left|\begin{matrix}0 & -\lambda\\ -\lambda+r & \end{matrix}\right\rangle \rightarrow \left|\begin{matrix}\lambda & 0\\ r & \end{matrix}\right\rangle \rightarrow \left|\begin{matrix}\lambda & 0\\ \lambda-(\lambda-r) & \end{matrix}\right\rangle. \tag{2-6-34}$$

Thus $\overline{(\lambda)0;r}=(\lambda)0;\lambda-r$ and

$$\varphi_{rr'}^{(22)(\lambda)0;c}(R)=(-)^{r+r'}\varphi_{\lambda-r,\lambda-r'}^{(22)(\lambda)0}(R)=\varphi_{rr'}^{(22)(\lambda)0}(\delta_R), \tag{2-6-35}$$

where

$$R=\begin{pmatrix}R_{x1} & R_{x2}\\ R_{y1} & R_{y2}\end{pmatrix},\quad \delta_R=\begin{pmatrix}R_{y2} & -R_{y1}\\ -R_{x2} & R_{x1}\end{pmatrix}. \tag{2-6-36}$$

In the case of the 3×3 DG polynomial, the transformation

$$\begin{aligned}
&\left|\begin{matrix}\lambda+\mu & \mu & 0\\ \lambda+\mu-p & \mu-q & \\ \lambda+\mu-p-r & & \end{matrix}\right\rangle \rightarrow \left|\begin{matrix}0 & -\mu & -\lambda-\mu\\ -\mu+q & -\lambda-\mu+p & \\ -\lambda-\mu+p+r & & \end{matrix}\right\rangle \rightarrow \left|\begin{matrix}\lambda+\mu & \lambda & 0\\ \lambda+q & p & \\ p+r & & \end{matrix}\right\rangle\\
&\quad=\left|\begin{matrix}\mu+\lambda & \lambda & 0\\ (\mu+\lambda)-(\mu-q) & \lambda-(\lambda-p) & \\ (\lambda+q)-(\lambda-p+q-r) & & \end{matrix}\right\rangle
\end{aligned} \tag{2-6-37}$$

gives

$$\overline{(\lambda\mu)0;pqr}=(\mu\lambda)0;\mu-q,\lambda-p,\lambda-p+q-r\,, \tag{2-6-38}$$

and

$$\begin{aligned}
&\varphi_{pqr,p'q'r'}^{(33)(\lambda\mu)0;c}(R)\\
&\quad=(-)^{q+r+q'+r'}\varphi^{(33)(\mu\lambda)0}_{\mu-q,\lambda-p,\lambda-p+q-r;\mu-q',\lambda-p',\lambda-p'+q'-r'}(R)\\
&\quad=\left(\frac{(\mu+1)!}{(\lambda+1)!}\right)^{1/2}\varphi_{pqr,p'q'r'}^{(\lambda\mu)0}(\delta_R)\,|R|^{-\mu}
\end{aligned} \tag{2-6-39}$$

from Eqs. (2-5-9) $\sim$ (2-5-11) and (2-6-4), where the $\mathscr{R}$-conjugation transformation $R\rightarrow\delta_R={}^tR^{-1}|R|$ leads $\delta_R\rightarrow R|R|$. Thus we find that $\varphi^{(\lambda\mu)0}_{pq0,p'q'0}(\delta_R)\cdot|R|^{-\mu}$ is obtained by the exchange $R\leftrightarrow\delta_R$ in the functional form $f(R,\delta_R,|R|)=\varphi^{(\lambda\mu)0}_{pq0,p'q'0}(R)$ of Eq. (2-6-23) with $\omega=0$.[15]

§ 3. Application to SU_2 coefficients

By using the vector-coupling expression of the DG polynomials, we can derive various C-G coefficients, Racah coefficients and their properties of the SU_n algebra by considering appropriate DG polynomials. In this section we show the application to the SU_2 coefficients with respect to the derivation of their explicit expressions and to their Regge symmetry briefly, since the SU_2 algebra has been studied very widely by many authors.

3-1. Expression of the C-G coefficient and its Regge symmetry

In order to obtain the explicit expression of the C-G coefficient, we start from Eq. (2-6-30a);

$$\varphi^{(22)(\lambda)\mu}_{rr'}(R)$$
$$=\sum_{r_1r_2}\left\langle\frac{\lambda+\mu-r'}{2}\;\frac{\lambda+\mu-r'}{2}-r_1\;\frac{\mu+r'}{2}\;\frac{\mu+r'}{2}-r_2\middle|\frac{\lambda}{2}\;\frac{\lambda}{2}-r\right\rangle$$
$$v_{(\lambda+\mu-r')/2,(\lambda+\mu-r')/2-r_1}(\boldsymbol{R}_1)\,v_{(\mu+r')/2,(\mu+r')/2-r_2}(\boldsymbol{R}_2)\,. \qquad (3\text{-}1\text{-}1)$$

By substituting Eqs. (2-6-5) and (2-6-29) to Eq. (3-1-1) and comparing the coefficient of the monomials of $R_{\alpha i}$ ($\alpha=x,y$ and $i=1,2$) with the same degrees, we obtain

$$\left\langle\frac{\lambda+\mu-r'}{2}\;\frac{\lambda+\mu-r'}{2}-r_1\;\frac{\mu+r'}{2}\;\frac{\mu+r'}{2}-r_2\middle|\frac{\lambda}{2}\;\frac{\lambda}{2}-r\right\rangle$$
$$=N(\lambda\mu rr')\sqrt{(\lambda+\mu-r'-r_1)!\,r_1!\,(\mu+r'-r_2)!\,r_2!}$$
$$\times\sum_a(-)^{r_1-r-a}\binom{\mu}{r_1-r+a}\frac{1}{(\lambda-r-r'+a)!\,(r-a)!\,(r'-a)!\,a!}\,, \qquad (3\text{-}1\text{-}2)$$

where $r_1+r_2=r+\mu$ and the index a runs over all integers such that none of the factorial arguments are negative. By the substitution

$$j_1\equiv\frac{\lambda+\mu-r'}{2},\quad m_1\equiv\frac{\lambda+\mu-r'}{2}-r_1\,,$$
$$j_2\equiv\frac{\mu+r'}{2},\quad m_2\equiv\frac{\mu+r'}{2}-r_2\,,$$
$$j_3\equiv\frac{\lambda}{2},\qquad m_3\equiv\frac{\lambda}{2}-r\,, \qquad (3\text{-}1\text{-}3)$$

we obtain the Racah's expression of the C-G coefficient.[17]

In the same way, the equality of the vector-coupling expressions in two-fold ways Eqs. (2-6-30a) and (2-6-30b) gives us the following relation of two C-G coefficients;

$$\left\langle \frac{\lambda+\mu-r'}{2}\ \frac{\lambda+\mu-r'}{2}-r_1\ \frac{\mu+r'}{2}\ \frac{\mu+r'}{2}-r_2 \middle| \frac{\lambda}{2}\ \frac{\lambda}{2}-r \right\rangle$$
$$= \left\langle \frac{\lambda+\mu-r}{2}\ \frac{\lambda+\mu-r}{2}-r_1'\ \frac{\mu+r}{2}\ \frac{\mu+r}{2}-r_2 \middle| \frac{\lambda}{2}\ \frac{\lambda}{2}-r' \right\rangle, \qquad (3\text{-}1\text{-}4)$$

where $\mu=r_1+r_2-r=r_1'+r_2-r'$. By the transformation Eq. (3-1-3) of notation, we obtain the Regge symmetry[18] of the C-G coefficient;

$$\begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{bmatrix}$$
$$= \begin{bmatrix} j_3+m_3 & j_3-m_3 & j_1+j_2-j_3 \\ j_2+m_2 & j_2-m_2 & j_1-j_2+j_3 \\ j_1+m_1 & j_1-m_1 & -j_1+j_2+j_3 \end{bmatrix}, \qquad (3\text{-}1\text{-}5a)$$

where

$$\begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{bmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$
$$= (-)^{j_1-j_2-m_3} \frac{1}{\sqrt{2j_3+1}} \langle j_1m_1j_2m_2|j_3-m_3\rangle . \qquad (3\text{-}1\text{-}5b)$$

The procedure to obtain Eq. (3-1-4) is also applicable to the U_n case by using the two kinds of the vector-coupling expressions Eqs. (2-2-4) and (2-2-15); namely, we can prove

$$\langle [q_1'\cdots q_{n-1}']r_1\cdots r_{n-1}[N_n]N_{n-1}\| [f_1\cdots f_n]q_1\cdots q_{n-1}\rangle_n$$
$$=\langle [q_1\cdots q_{n-1}]r_1\cdots r_{n-1}[P_n]P_{n-1}\| [f_1\cdots f_n]q_1'\cdots q_{n-1}'\rangle_n ,$$
$$N_n = \sum_{i=1}^{n} f_i - \sum_{i=1}^{n-1} q_i', \quad N_{n-1} = \sum_{i=1}^{n-1} (q_i-r_i),$$
$$P_n = \sum_{i=1}^{n} f_i - \sum_{i=1}^{n-1} q_i, \quad P_{n-1} = \sum_{i=1}^{n-1} (q_i'-r_i), \qquad (3\text{-}1\text{-}6)$$

as the Regge symmetry of the multiplicity-free U_n reduced C-G coefficient.[16] (A proof of Eq. (3-1-6) is given in Appendix E.) The fact that the essence of the Regge symmetry of the multiplicity-free U_n C-G coefficients is the transposition property Eq. (2-1-26) (of the $n\times n$ DG polynomial) was already noted by Bincer,[19] Jucys[20] and Holman.[16] We consider that the use of the two-fold vector-coupling expressions of the DG polynomial is one of the simple way to realize the relation between the Regge symmetry and the transposition symmetry.

3-2. Expression of the Racah coefficients and its Regge symmetry

In the case of the Racah coefficient, we start from the transformation formula of the DG polynomials Eq. (2-3-20) with $n_1=n_3=3$ and $n_2=2$;

$$\varphi^{(32)(\lambda\mu)}_{pqr,s}(AR)=\frac{1}{N_H(\lambda\mu)}\sum_{p'q'r'}\varphi^{(33)(\lambda\mu)0}_{pqr,p'q'r'}(A)\,\varphi^{(32)(\lambda\mu)}_{p'q'r',s}(R), \qquad (3\text{-}2\text{-}1)$$

where we take

$$A\equiv\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad AR=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}\boldsymbol{R}^r_x\\\boldsymbol{R}^r_y\\\boldsymbol{R}^r_z\end{bmatrix}=\begin{bmatrix}\boldsymbol{R}^r_x\\\boldsymbol{R}^r_z\\\boldsymbol{R}^r_y\end{bmatrix}. \qquad (3\text{-}2\text{-}2)$$

By using the vector-coupling expression Eq. (2-6-31b) and the definition of the Racah coefficient, we obtain

$$\begin{aligned}
&\varphi^{(32)(\lambda\mu)}_{pqr,s}(AR)\\
&=[[v_{(\lambda+\mu-p-r)/2}(\boldsymbol{R}^r_x)\,v_{(\mu-q+r)/2}(\boldsymbol{R}^r_z)]_{(\lambda-p+q)/2}v_{(p+q)/2}(\boldsymbol{R}^r_y)]_{\lambda/2,\lambda/2-s}\\
&=\sum_{\Lambda'}(-)^{(\lambda+\mu+q+r)/2+\Lambda'}[(\lambda-p+q+1)(2\Lambda'+1)]^{1/2}\\
&\quad\times\begin{Bmatrix}\frac{\mu-q+r}{2}&\frac{\lambda+\mu-p-r}{2}&\frac{\lambda-p+q}{2}\\\frac{p+q}{2}&\frac{\lambda}{2}&\Lambda'\end{Bmatrix}\\
&\quad\times[[v_{(\lambda+\mu-p-r)/2}(\boldsymbol{R}^r_x)\,v_{(p+q)/2}(\boldsymbol{R}^r_y)]_{\Lambda'}v_{(\mu-q+r)/2}(\boldsymbol{R}^r_z)]_{\lambda/2,\lambda/2-s}\\
&=\sum_{p'q'r'}(-)^{\lambda+q+q'}[(\lambda-p+q+1)(\lambda-p'+q'+1)]^{1/2}\\
&\quad\begin{Bmatrix}\frac{\mu-q+r}{2}&\frac{\lambda+\mu-p-r}{2}&\frac{\lambda-p+q}{2}\\\frac{p+q}{2}&\frac{\lambda}{2}&\frac{\lambda-p'+q'}{2}\end{Bmatrix}\cdot\varphi^{(32)(\lambda\mu)}_{p'q'r',s}(R),
\end{aligned}\qquad (3\text{-}2\text{-}3)$$

where the sum over p', q', r' is restricted as $p'+q'=\mu-q+r$ and $p'+r'=p+r$. Comparing Eqs. (3-2-1) and (3-2-3) we find that the Racah coefficient is given by the 3×3 DG polynomial with the specific argument matrix $-\mathrm{A}$;

$$\begin{Bmatrix}\frac{\mu-q+r}{2}&\frac{\lambda+\mu-p-r}{2}&\frac{\lambda-p+q}{2}\\\frac{p+q}{2}&\frac{\lambda}{2}&\frac{\lambda-p'+q'}{2}\end{Bmatrix}=\begin{Bmatrix}\frac{p'+q'}{2}&\frac{\lambda+\mu-p'-r'}{2}&\frac{\lambda-p+q}{2}\\\frac{\mu-q'-r'}{2}&\frac{\lambda}{2}&\frac{\lambda-p'+q'}{2}\end{Bmatrix}$$

$$= (-)^{q+q'}[(\lambda-p+q+1)(\lambda-p'+q'+1)]^{-1/2}\frac{1}{N_H(\lambda\mu)}\varphi^{(33)(\lambda\mu)0}_{pqr,p'q'r'}(-A), \tag{3-2-4}$$

where $p'+q'=\mu-q+r$, $p'+r'=p+r$ and $\varphi^{(32)(\lambda\mu)0}_{pqr,p'q'r'}(-A) = (-)^{\lambda}\varphi^{(32)(\lambda\mu)0}_{pqr,p'q'r'}(A)$. In the notation of the angular momentum, we have

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} = (-)^{J_{12}+j_3-J_{23}-j_1}(\hat{J}_{12}\hat{J}_{23})^{-1}\frac{1}{N_H(\lambda\mu)}\varphi^{(33)(\lambda\mu)0}_{pqr,p'q'r'}(-A), \tag{3-2-5}$$

where $\hat{a}=\sqrt{2a+1}$ and the correspondence of the quantum numbers is

$$\begin{aligned} j_1 &= \frac{\mu-q+r}{2}, \quad j_2=\frac{\lambda+\mu-p-r}{2}, \quad j_3=\frac{p+q}{2}, \\ J_{12} &= \frac{\lambda-p+q}{2}, \quad J_{23}=\frac{\lambda-p'+q'}{2}, \quad J=\frac{\lambda}{2}, \end{aligned} \tag{3-2-6a}$$

or

$$\begin{aligned} &\lambda=2J, \quad \mu=j_1+j_2+j_3-J \\ &p=J-J_{12}+j_3, \quad q=-J+J_{12}+j_3, \quad r=j_1-j_2+J_{12}, \\ &p'=J-J_{23}+j_1, \quad q'=-J+J_{23}+j_1, \quad r'=j_3-j_2+J_{23}. \end{aligned} \tag{3-2-6b}$$

By using the expression from Eq. (2-6-24)

$$\begin{aligned} &\frac{1}{N_H(\lambda\mu)}\varphi^{(33)(\lambda\mu)0}_{pqr,p'q'r'}(-A) \\ &= \mathscr{N}(\lambda\mu;pqr)\mathscr{N}(\lambda\mu;p'q'r')\ \frac{\lambda!\mu!(\lambda+\mu+1)!p!p'!q!q'!r!r'!}{(\lambda+\mu-p+1)!(\lambda+\mu-p'+1)!} \\ &\quad\times(-)^{q+q'}\sum_a(-)^{\lambda+\mu-a}(\lambda+\mu-a+1)! \\ &\quad\times[(\lambda-p-p'+a)!(p'-a)!(p-a)! \\ &\quad\times a!(\mu-q'-a)!(\mu-q-a)!(q+q'-\mu+a)!]^{-1}, \end{aligned} \tag{3-2-7}$$

we obtain from Eq. (3-2-4) or (3-2-5) the well-known formula of the Racah coefficient.[17)]*

The Regge symmetry of the Racah coefficients[21)] is a direct result of the complex conjugate property of the SU_3 representation matrix Eq. (2-5-4). Since $-A$ is a unimodular unitary matrix, we can write Eq. (2-5-4), from Eq. (2-6-38), as

$$\begin{aligned} D^{(33)(\lambda\mu)}_{pqr,p'q'r'}(-A)^* &= (-)^{q+r+q'+r'}D^{(33)(\overline{\lambda\mu})}_{\bar{p}\bar{q}\bar{r},\bar{p}'\bar{q}'\bar{r}'}(-A) \\ &= (-)^{q+r+q'+r'}D^{(33)(\mu\lambda)}_{\mu-q,\lambda-p,\lambda-p+q-r;\mu-q',\lambda-p',\lambda-p'+q'-r'}(-A), \end{aligned} \tag{3-2-8}$$

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* Moshinsky anc Chacón[1)] obtained the expression of the Racah coefficient by direct integration in the matrix element of a Weyl operator with respect to the Gel'fand bases.

where

$$D^{(33)(\lambda\mu)}_{pqr,p'q'r'}(U) \equiv \frac{1}{N_H(\lambda\mu)} \varphi^{(33)(\lambda\mu)\omega}_{pqqr,p'q'r'}(U),$$
$$U = \text{unimodular } 3\times 3 \text{ matrix}. \tag{3-2-9}$$

Recalling Eq. (3-2-8) is a real number and using Eqs. (3-2-5), (3-2-6a) and (3-2-6b), we obtain

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} = \begin{Bmatrix} \frac{-j_1+j_2+j_3+J}{2} & \frac{j_1-j_2+j_3+J}{2} & J_{12} \\ \frac{j_1+j_2-j_3+J}{2} & \frac{j_1+j_2+j_3-J}{2} & J_{23} \end{Bmatrix}. \tag{3-2-10}$$

3-3. The 9-j symbol expressed by a 4×4 DG polynomial

The discussion in the preceding subsection can be also applied to the 9-j symbol. In this case we must introduce the 4×2 DG polynomial and its vector-coupling expression;

$$\varphi^{(42)(\lambda\mu)}_{uvpqr,s}(R)$$
$$= \left| \begin{matrix} \lambda+\mu & \mu & 0 & 0 \\ \lambda+\mu-u & \mu-v & 0 & \\ \lambda+\mu-u-p & \mu-v-q & & \\ \lambda+\mu-u-p-r & & & \end{matrix} \quad \begin{matrix} \lambda+\mu & \mu \\ \lambda+\mu-s & \end{matrix} \right\rangle$$
$$= [[[v_{(\lambda+\mu-u-p-r)/2}(\boldsymbol{R}^r_1)\, v_{(\mu-v-q+r)/2}(\boldsymbol{R}^r_2)]_{(\lambda-u+v-p+q)/2}$$
$$\times v_{(p+q)/2}(\boldsymbol{R}^r_3)]_{(\lambda-u+v)/2} v_{(u+v)/2}(\boldsymbol{R}^r_4)]_{\lambda/2,\lambda/2-s}. \tag{3-3-1}$$

The transformation of the argument matrix

$$(14)R \equiv (14)\begin{pmatrix} \boldsymbol{R}^r_1 \\ \boldsymbol{R}^r_2 \\ \boldsymbol{R}^r_3 \\ \boldsymbol{R}^r_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}^r_4 \\ \boldsymbol{R}^r_2 \\ \boldsymbol{R}^r_3 \\ \boldsymbol{R}^r_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{R}^r_1 \\ \boldsymbol{R}^r_2 \\ \boldsymbol{R}^r_3 \\ \boldsymbol{R}^r_4 \end{pmatrix} \equiv \Omega R \tag{3-3-2}$$

makes us to treat the recoupling coefficient

$$\langle [[(j_1j_2)J_{12}, j_3]J_{123}, j_4]_J | [[(j_4j_2)J_{42}, j_3]J_{423}, j_1]_J \rangle$$
$$= (-)^{J_{423}+j_1-J_{123}-j_4} \hat{J}_{12}\hat{J}_{123}\hat{J}_{42}\hat{J}_{423} \begin{Bmatrix} j_2 & J_{12} & j_1 \\ J_{42} & j_3 & J_{423} \\ j_4 & J_{123} & J \end{Bmatrix}, \tag{3-3-3}$$

the equation of which can be proved by using the symmetry properties of the C-G coefficients. Thus the 9-j symbol is expressed by a 4×4 DG polynomial with the argument matrix Ω;

$$\begin{Bmatrix} j_2 & J_{12} & j_1 \\ J_{42} & j_3 & J_{423} \\ j_4 & J_{123} & J \end{Bmatrix} = (-)^{J_{123}+j_4-J_{423}-j_1} (\hat{J}_{12}\hat{J}_{123}\hat{J}_{42}\hat{J}_{423})^{-1}$$

$$\times \frac{1}{N_H(\lambda\mu)} \varphi^{(44)(\lambda\mu 0)0}_{uvpqr,u'v'p'q'r'}(\Omega), \tag{3-3-4}$$

where the correspondence of the quantum numbers is

$$\begin{aligned} &\lambda = 2J, \quad \mu = j_1 + j_2 + j_3 + j_4 - J, \\ &u = J - J_{123} + j_4, \quad v = -J + J_{123} + j_4, \\ &p = J_{123} - J_{12} + j_3, \quad q = -J_{123} + J_{12} + j_3, \quad r = J_{12} - j_1 + j_2, \\ &u' = J - J_{423} + j_1, \quad v' = -J + J_{423} + j_1, \\ &p' = J_{423} - J_{42} + j_3, \quad q' = -J_{423} + J_{42} + j_3, \quad r' = J_{42} - j_4 + j_2. \end{aligned} \tag{3-3-5}$$

The Regge symmetry of the 9-j symbol, however, can never be obtained from Eq. (3-3-4) by the complex conjugate argument. The reason is as follows. By performing the similar transformation as Eq. (2-6-37) for the SU_4-partition $(\lambda\mu 0)$, we find that its complex conjugate representation has the partition $(0\mu\lambda)$. Nevertheless, the 4×4 DG polynomial with the partition $(0\mu\lambda)$ cannot be connected with a 9-j symbol by Eq. (3-3-4) except in the case of $\lambda=0$, when $J=0$ from Eq. (3-3-5) and the Regge symmetry of the Racah coefficients is obtained.

§ 4. Application to coherent states and to invariant polynomials

4-1. Spin coherent states

Let $\Phi_{(\lambda)a}$ be states classified by SU_n state labels $(\lambda)a$. We define $\Phi^R_{N(\lambda)b}$ by

$$\Phi^R_{N(\lambda)b} \equiv \sum_a \varphi^{(nn)N(\lambda)}_{ab}(R)\Phi_{(\lambda)a}, \tag{4-1-1}$$

where R is $n\times n$ matrix of complex numbers. $\Phi^R_{N(\lambda)b}$ satisfy the following completeness relation

$$\int d\mu(R)\,|\Phi^R_{N(\lambda)b}\rangle\langle\Phi^R_{N'(\lambda')b'}| = \delta_{N,N'}\delta_{(\lambda),(\lambda')}\delta_{b,b'} \times \sum_a |\Phi_{(\lambda)a}\rangle\langle\Phi_{(\lambda)a}|, \tag{4-1-2}$$

and can be regarded as SU_n coherent states.

In the case of SU_2, the above prescription gives us SU_2 coherent states

$$\Phi^R_{N(\lambda)r} = \sum_{r'=0}^{\lambda} \varphi^{(\lambda)\mu}_{r',r}(R)\Phi_{(\lambda)r'}, \tag{4-1-3}$$

where $N=\lambda+2\mu$ and $\varphi_{r',r}^{(\lambda)\mu}(R)$ are defined by Eq. (2-6-1). In the usual notation of angular momentum algebra $\Phi_{(\lambda)r}=\Phi_{jm}$ with $\lambda=2j$, $r=j-m$ $(m=-j, -j+1, \cdots, j)$. The above coherent states $\Phi_{N(\lambda)r}^{R}$ are functions of four complex parameter variables expressed by a 2×2 matrix R. We can reduce the number of parameter variables from four to one as in the following way. First we choose $N=\lambda$ and $r=0$ (the highest weight) in $\Phi_{N(\lambda)r}^{R}$ which we denote as $\Phi_{\lambda}^{(R_{11}R_{21})}$:

$$\begin{aligned}\Phi_{\lambda}^{(R_{11}R_{21})}\equiv\Phi_{\lambda(\lambda)0}^{R}&=\sum_{r'=0}^{\lambda}\varphi_{r',0}^{(\lambda)0}(R)\,\Phi_{(\lambda)r'}\\&=\sum_{r'=0}^{\lambda}\frac{R_{11}^{\lambda-r'}R_{21}^{r'}}{\sqrt{(\lambda-r')!\,r'!}}\Phi_{(\lambda)r'}\\&=R_{11}^{\lambda}\sum_{r'=0}^{\lambda}\frac{(R_{21}/R_{11})^{r'}}{\sqrt{(\lambda-r')!\,r'!}}\Phi_{(\lambda)r'}\,.\end{aligned}\tag{4-1-4}$$

By transforming the integral variables from (R_{11}, R_{21}) to (u, z) defined by

$$z\equiv R_{21}/R_{11},\quad u\equiv\sqrt{1+z\cdot z^{*}}\cdot R_{11}\,,\tag{4-1-5}$$

we can rewrite the completeness relation of $\Phi_{\lambda}^{(R_{11}R_{21})}$ as

$$\begin{aligned}\sum_{m=-j}^{j}|\Phi_{jm}\rangle\langle\Phi_{jm}|&=\sum_{r=0}^{\lambda}|\Phi_{(\lambda)r}\rangle\langle\Phi_{(\lambda)r}|\\&=\int d\mu(R_{11})\,d\mu(R_{21})\,|\Phi_{\lambda}^{(R_{11}R_{21})}\rangle\langle\Phi_{\lambda}^{(R_{11}R_{21})}|\\&=\frac{1}{\pi^{2}}\int d^{2}R_{11}d^{2}R_{21}\exp\{-R_{11}R_{11}^{*}-R_{21}R_{21}^{*}\}\,(R_{11}R_{11}^{*})^{\lambda}\\&\quad\times\sum_{r,r'=0}^{\lambda}\frac{(R_{21}/R_{11})^{r}}{\sqrt{(\lambda-r)!\,r!}}\frac{(R_{21}^{*}/R_{11}^{*})^{r'}}{\sqrt{(\lambda-r')!\,r'!}}|\Phi_{(\lambda)r}\rangle\langle\Phi_{(\lambda)r'}|\\&=\frac{1}{\pi^{2}}\int d^{2}u\,d^{2}z\exp\{-uu^{*}\}\frac{(uu^{*})^{\lambda+1}}{(1+zz^{*})^{\lambda+2}}\\&\quad\times\sum_{r,r'=0}^{\lambda}\frac{z^{r}}{\sqrt{(\lambda-r)!\,r!}}\frac{(z^{*})^{r'}}{\sqrt{(\lambda-r')!\,r'!}}|\Phi_{(\lambda)r}\rangle\langle\Phi_{(\lambda)r'}|\\&=\frac{\lambda+1}{\pi}\int\frac{d^{2}z}{(1+zz^{*})^{\lambda+2}}\sum_{r,r'=0}^{\lambda}\sqrt{\binom{\lambda}{r}\binom{\lambda}{r'}}\,z^{r}(z^{*})^{r'}\\&\quad|\Phi_{(\lambda)r}\rangle\langle\Phi_{(\lambda)r'}|\,,\end{aligned}\tag{4-1-6}$$

where $d^{2}\omega\equiv d(\mathrm{Re}\,\omega)\,d(\mathrm{Im}\,\omega)$ for any complex variable ω. In treating the transformation of integral variables, use of the technique of the so-called Grassmann algebra makes the calculation quite easy; for example

$$d^{2}z=d(\mathrm{Re}\,z)_{\wedge}d(\mathrm{Im}\,z)=\frac{i}{2}dz_{\wedge}dz^{*}\,,$$

$$d^2R_{11}\wedge d^2R_{21} = -\frac{1}{4}dR_{11}\wedge dR_{11}^*\wedge(R_{11}dz+zdR_{11})\wedge(R_{11}^*dz^*+z^*dR_{11}^*)$$

$$= -\frac{1}{4}dR_{11}\wedge dR_{11}^*\wedge R_{11}dz\wedge R_{11}^*dz^*$$

$$= R_{11}R_{11}^*d^2R_{11}\wedge d^2z\ ,$$

$$d^2R_{11}\wedge d^2z = \frac{1}{1+zz^*}d^2u\wedge d^2z\ . \tag{4-1-7}$$

From Eq. (4-1-6), we know that the states $\Phi_j(z)$ defined by

$$\Phi_j(z) \equiv \sum_{r=0}^{2j}\sqrt{\binom{2j}{r}}z^r\Phi_{j,j-r} \tag{4-1-8}$$

satisfy the following completeness relation

$$\int dM_j(z)\,|\Phi_j(z)\rangle\langle\Phi_j(z)| = \sum_{m=-j}^{j}|\Phi_{jm}\rangle\langle\Phi_{jm}|,$$

$$dM_j(z) \equiv \frac{2j+1}{\pi}\frac{d^2z}{(1+zz^*)^{2j+2}}\ . \tag{4-1-9}$$

$\Phi_j(z)$ is just the spin coherent state introduced by Radcriffe.[22)]

If we express the SU_2 states Φ_{jm} by the Schwinger bosons $a_{11}^\dagger$ and $a_{12}^\dagger$ as

$$\Phi_{jm} = \frac{(a_{11}^\dagger)^{j+m}(a_{12}^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle = \frac{(a_{11}^\dagger)^{\lambda-r}(a_{12}^\dagger)^r}{\sqrt{(\lambda-r)!r!}}|0\rangle$$

$$= \varphi_{0,r}^{(\lambda)0}(a^\dagger)\,|0\rangle\ ,$$

$$a^\dagger \equiv \begin{pmatrix} a_{11}^\dagger & a_{12}^\dagger \\ a_{21}^\dagger & a_{22}^\dagger \end{pmatrix}, \tag{4-1-10}$$

we obtain from Eq. (4-1-4)

$$\Phi_\lambda^{(R_{11}R_{21})} = \frac{1}{\lambda!}(R_{11}a_{11}^\dagger + R_{21}a_{12}^\dagger)^\lambda|0\rangle\ . \tag{4-1-11}$$

We know thus that the h.o. coherent state $\Phi^{(R_{11}R_{21})}$ defined by

$$\Phi^{(R_{11}R_{21})} \equiv \exp\{(R_{11}a_{11}^\dagger + R_{21}a_{12}^\dagger)\}\,|0\rangle = \sum_{\lambda=0}^{\infty}\Phi_\lambda^{(R_{11}R_{21})} \tag{4-1-12}$$

can be regarded as the general SU_2 coherent state since $\Phi^{(R_{11}R_{21})}$ satisfy the completeness relation

$$\int d\mu(R_{11})\,d\mu(R_{21})\,|\Phi^{(R_{11}R_{21})}\rangle\langle\Phi^{(R_{11}R_{21})}|$$

$$= \sum_{j=0,1/2,1,\cdots}\ \sum_{m=-j}^{j}|\Phi_{jm}\rangle\langle\Phi_{jm}|\ . \tag{4-1-13}$$

Spin coherent state minimizes the product of uncertainties of the angular momentum components. Namely, the left-hand-side quantity in the following general inequality can be made minimum if we choose $\Psi=$ (spin coherent state);

$$\frac{(\Delta J_i^2)(\Delta J_j^2)}{\langle J_k\rangle^2}\geqq\frac{1}{4},$$
$$\Delta J_i^2\equiv\langle\Psi|(J_i-\langle J_i\rangle)^2|\Psi\rangle,$$
$$\langle J_i\rangle\equiv\langle\Psi|J_i|\Psi\rangle, \tag{4-1-14}$$

where (i, j, k) is any permutation of (x, y, z). This property of spin coherent state can be proved by parametrizing R_{11} and R_{21} as in Eq. (2-6-7),

$$R_{11}=r\exp\left\{-i\frac{\varphi}{2}\right\}\cos\frac{\theta}{2}\exp\left\{-i\frac{\psi}{2}\right\},$$
$$R_{21}=r\exp\left\{i\frac{\varphi}{2}\right\}\sin\frac{\theta}{2}\exp\left\{-i\frac{\psi}{2}\right\}, \tag{4-1-15}$$

and by noting the following properties of the normalized spin coherent states $\hat{\Phi}_\lambda^{(R_{11}R_{21})}\equiv\Phi_\lambda^{(R_{11}R_{21})}/\|\Phi_\lambda^{(R_{11}R_{21})}\|$ and $\hat{\Phi}^{(R_{11}R_{21})}\equiv\Phi^{(R_{11}R_{21})}/\|\Phi^{(R_{11}R_{21})}\|$;

$$\frac{(\Delta J_i^2)(\Delta J_j^2)}{\langle J_k\rangle^2}=\begin{cases}\frac{1}{4}\frac{(1-n_i^2)(1-n_j^2)}{n_k^2} & \text{for } \Psi=\hat{\Phi}_\lambda^{(R_{11}R_{21})},\\ \frac{1}{4}\frac{1}{n_k^2} & \text{for } \Psi=\hat{\Phi}^{(R_{11}R_{21})},\end{cases}$$
$$\boldsymbol{n}\equiv(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) \tag{4-1-16}$$

Eq. (4-1-16) shows that $(\Delta J_i)^2(\Delta J_j)^2/\langle J_k\rangle^2$ takes its minimum value 1/4 when $\boldsymbol{n}$ is taken to be parallel to the k-direction. We should note here that the magnitude r in Eq. (4-1-15) is irrelevant to this property of the spin coherent states $\hat{\Phi}_\lambda^{(R_{11}R_{21})}$ and $\hat{\Phi}^{(R_{11}R_{21})}$.

Eq. (4-1-11) can be generalized to SU_n with $n\geqq3$ by expressing the SU_n basis states $\Phi_{(\lambda)a}$ by boson operators $a_{ij}^\dagger$ as

$$\Psi_{(\lambda)a}=\varphi_{H,a}^{(nn)N(\lambda)}(a^\dagger)|0\rangle,$$
$$a^\dagger\equiv\begin{pmatrix}a_{11}^\dagger\cdots a_{1n}^\dagger\\ \vdots \quad \vdots\\ a_{n1}^\dagger\cdots a_{nn}^\dagger\end{pmatrix},\quad N=\sum_{k=1}^{n-1}k\lambda_k. \tag{4-1-17}$$

From Eqs. (4-1-1) and (4-1-17), we have

$$\Phi_{N(\lambda)H}^R=N_H[N(\lambda)]\varphi_{H,H}^{(nn)N(\lambda)}(a^\dagger R)|0\rangle,$$
$$N=\sum_{k=1}^{n-1}k\lambda_k. \tag{4-1-18}$$

For $n\geqq3$, however, we have no more simple relation as Eq. (4-1-12) between the h.o. coherent state and the SU_n coherent states.

4-2. Coherent states of rotator

Wave functions $\psi^j_{mm'}$ of an axially symmetric top are basis states of two kinds of angular momentum operators $\boldsymbol{L}$ and $\boldsymbol{K}$; $L^2\psi^j_{mm'}=K^2\psi^j_{mm'}=j(j+1)\psi^j_{mm'}$, $L_z\psi^j_{mm'}=m\psi^j_{mm'}$, $K_z\psi^j_{mm'}=m'\psi^j_{mm'}$. Wave functions of an asymmetric top is obtained by linearly combining $\psi^j_{mm'}$ over m'. Coherent states for $\psi^j_{mm'}$ are desirable to be constructed so that they behave like the spin coherent states for both $\boldsymbol{L}$ and $\boldsymbol{K}$, simultaneously but separately.

For fixed j, the following states

$$\Phi_\lambda\begin{pmatrix}X_{11}X_{21}\\Y_{11}Y_{12}\end{pmatrix}\equiv\sum_{r,r'=0}^{\lambda}\varphi^{(\lambda)}_{r,0}(X)\,\varphi^{(\lambda)0}_{0,r'}(Y)\,\psi^{\lambda/2}_{\lambda/2-r,\,\lambda/2-r'}$$

$$=(X_{11}Y_{11})^\lambda\sum_{r,r'=0}^{\lambda}\frac{(X_{21}/X_{11})^r\,(Y_{12}/Y_{11})^{r'}}{\sqrt{(\lambda-r)!r!}\sqrt{(\lambda-r')!r'!}}\psi^{\lambda/2}_{\lambda/2-r,\,\lambda/2-r'} \qquad (4\text{-}2\text{-}1)$$

have desirable properties for the coherent states including the completeness relation

$$\int d\mu(X_{11})\,d\mu(X_{21})\,d\mu(Y_{11})\,d\mu(Y_{12})\left|\Phi_\lambda\begin{pmatrix}X_{11}X_{21}\\Y_{11}Y_{12}\end{pmatrix}\right\rangle\left\langle\Phi_\lambda\begin{pmatrix}X_{11}X_{21}\\Y_{11}Y_{12}\end{pmatrix}\right|$$

$$=\sum_{m,m'=-j}^{j}|\psi^j_{mm'}\rangle\langle\psi^j_{mm'}|. \qquad (4\text{-}2\text{-}2)$$

Let $\mathcal{O}(\boldsymbol{L})$ be any operator composed of $\boldsymbol{L}$. We have

$$\left\langle\Phi_\lambda\begin{pmatrix}X_{11}X_{21}\\Y_{11}Y_{12}\end{pmatrix}\right|\mathcal{O}(\boldsymbol{L})\left|\Phi_\lambda\begin{pmatrix}X_{11}X_{21}\\Y_{11}Y_{12}\end{pmatrix}\right\rangle$$

$$=\frac{1}{\lambda!}(Y_{11}Y^*_{11}+Y_{12}Y^*_{12})^\lambda\langle\Phi^{(X_{11}X_{21})}_\lambda|\mathcal{O}(\boldsymbol{L})|\Phi^{(X_{11}X_{21})}_\lambda\rangle$$

$$=\frac{1}{\lambda!}\langle\Phi^{(xX_{11},xX_{21})}_\lambda|\mathcal{O}(\boldsymbol{L})|\Phi^{(xX_{11},xX_{21})}_\lambda\rangle,$$

$$x\equiv\sqrt{Y_{11}Y^*_{11}+Y_{12}Y^*_{12}},$$

$$\Phi^{(X_{11}X_{21})}_\lambda\equiv\sum_{r=0}^{\lambda}\varphi^{(\lambda)0}_{r,0}(X)\,\psi^{\lambda/2}_{\lambda/2-r,\,\lambda/2-r_0}\quad(r_0=\text{arbitrary}). \qquad (4\text{-}2\text{-}3)$$

From Eq. (4-2-3) we see $\Phi_\lambda\binom{X_{11}X_{21}}{Y_{11}Y_{12}}$ have the same property as the spin coherent states $\Phi^{(X_{11}X_{21})}_\lambda$ in minimizing the uncertainties of the components of $\boldsymbol{L}$. It is clear that the same conclusion holds also for $\boldsymbol{K}$.

In order to obtain coherent states for non-fixed j, we consider the following states

$$\Phi\begin{pmatrix}X_{11}X_{21}\\Y_{11}Y_{12}\end{pmatrix}=\sum_{\lambda=0}^{\infty}\sqrt{\lambda!}\,\Phi_\lambda\begin{pmatrix}X_{11}X_{21}\\Y_{11}Y_{12}\end{pmatrix}, \qquad (4\text{-}2\text{-}4)$$

which give us by Eq. (4-2-3)

$$\left\langle \Phi\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix} \middle| \mathcal{O}(\boldsymbol{L}) \middle| \Phi\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix} \right\rangle$$

$$= \sum_{\lambda=0}^{\infty} \langle \Phi_\lambda^{(xX_{11}, xX_{21})} | \mathcal{O}(\boldsymbol{L}) | \Phi_\lambda^{(xX_{11}, xX_{21})} \rangle$$

$$= \langle \Phi^{(xX_{11}, xX_{21})} | \mathcal{O}(\boldsymbol{L}) | \Phi^{(xX_{11}, xX_{21})} \rangle ,$$

$$\Phi^{(X_{11}X_{21})} \equiv \sum_{\lambda=0}^{\infty} \Phi_\lambda^{(X_{11}X_{21})} = \sum_{\lambda=0}^{\infty} \sum_{r=0}^{\lambda} \varphi_{r,0}^{(\lambda)0}(X) \psi^{\lambda/2}_{\lambda/2-r, \lambda/2-r_0} . \tag{4-2-5}$$

Eq. (4-2-5) shows that $\Phi(^{X_{11}X_{21}}_{Y_{11}Y_{12}})$ have the same property as the spin coherent states $\Phi^{(X_{11}X_{21})}$ in minimizing the uncertainties of the components of $\boldsymbol{L}$. The same result clearly holds also for $\boldsymbol{K}$. Thus $\Phi(^{X_{11}X_{21}}_{Y_{11}Y_{12}})$ can be considered to be the desired coherent states of the top. In order to obtain the completeness relation

$$\int d\tilde{\mu}\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix} \left| \Phi\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix} \right\rangle \left\langle \Phi\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix} \right|$$

$$= \sum_{j=0, 1/2, \cdots} \sum_{m, m'=-j}^{j} |\psi^j_{mm'}\rangle \langle \psi^j_{mm'}| , \tag{4-2-6}$$

we, however, can not choose as the measure $d\tilde{\mu}(^{X_{11}X_{21}}_{Y_{11}Y_{12}})$ simply $d\mu(X_{11})\, d\mu(X_{21})\, d\mu(Y_{11})\, d\mu(Y_{12})$. The correct measure is derived in the following way. We first notice

$$\varphi^{(\lambda)0}_{r,r'}(R) = \begin{cases} \varphi^{(\lambda)0}_{0,r'}(R)\, \delta_{r,0} & \text{for} \quad R = \begin{pmatrix} R_{11} R_{12} \\ 0 \;\; 0 \end{pmatrix} \\ \varphi^{(\lambda)0}_{r,0}(R)\, \delta_{r',0} & \text{for} \quad R = \begin{pmatrix} R_{11} & 0 \\ R_{21} & 0 \end{pmatrix}, \end{cases} \tag{4-2-7}$$

which gives us

$$\Phi_\lambda\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix} = \sum_{r,r'=0}^{\lambda} \left(\sum_{r''=0}^{\lambda} \varphi^{(\lambda)0}_{r,r''}(\hat{X}) \varphi^{(\lambda)0}_{r'',r'}(\hat{Y}) \right) \psi^{\lambda/2}_{\lambda/2-r, \lambda/2-r'}$$

$$= \frac{1}{\sqrt{\lambda!}} \sum_{r,r'=0}^{\lambda} \varphi^{(\lambda)0}_{r,r'}(C) \psi^{\lambda/2}_{\lambda/2-r, \lambda/2-r'} ,$$

$$\Phi\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix} = \sum_{\lambda=0}^{\infty} \sum_{r,r'=0}^{\infty} \varphi^{(\lambda)0}_{r,r'}(C) \psi^{\lambda/2}_{\lambda/2-r, \lambda/2-r'} ,$$

$$\hat{X} \equiv \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix}, \quad \hat{Y} \equiv \begin{pmatrix} Y_{11} Y_{12} \\ 0 \;\; 0 \end{pmatrix}, \quad C \equiv \hat{X}\hat{Y} = \begin{pmatrix} X_{11}Y_{11} & X_{11}Y_{12} \\ X_{21}Y_{11} & X_{21}Y_{12} \end{pmatrix}. \tag{4-2-8}$$

Since $\det C = 0$, we know that the number of the independent parameters in $\Phi(^{X_{11}X_{21}}_{Y_{11}Y_{12}})$ is three (not four). We therefore adopt the following parametrization

$$C=\begin{pmatrix} y & yx \\ yz & yxz \end{pmatrix}, \quad y\equiv X_{11}Y_{11}, \quad x\equiv\frac{Y_{12}}{Y_{11}}, \quad z\equiv\frac{X_{21}}{X_{11}}, \tag{4-2-9}$$

by which we can write

$$\Phi\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix}=\sum_{j=0,1/2,\cdots}\sum_{m,m'=-j}^{j}\sqrt{\frac{(2j)!}{(j+m)!(j-m)!(j+m')!(j-m')!}} \times z^{j-m}y^{2j}x^{j-m'}\psi^{j}_{m,m'}\,. \tag{4-2-10}$$

From Eq. (4-2-8) we easily see that in order to have the completeness relation of Eq. (4-2-6) we should have

$$\int d\tilde{\mu}\begin{pmatrix} X_{11}X_{21} \\ Y_{11}Y_{12} \end{pmatrix}\varphi^{(\lambda_1)0}_{r_1,r_2}(C)^*\varphi^{(\lambda_2)0}_{r_3,r_4}(C)=\delta_{\lambda_1,\lambda_2}\delta_{r_1,r_3}\delta_{r_2,r_4}\,. \tag{4-2-11}$$

Let us consider an integration measure

$$d\hat{\mu}(D)\equiv\frac{\delta(|\det D|)}{|\det D|}(\mathrm{Tr}\,DD^{\dagger}-1)\,d\mu(D), \tag{4-2-12}$$

which is invariant under the U_2 transformation of the 2×2 matrix D. Due to the presense of $\delta(|\det D|)$, this integration measure reduces the general 2×2 matrix into the special one with the property $\det D=0$. From the U_2 invariance of the measure $d\hat{\mu}(D)$ we clearly have

$$\int d\hat{\mu}(D)\,\varphi^{(\lambda_1)0}_{r_1,r_2}(D)^*\varphi^{(\lambda_2)0}_{r_3,r_4}(D)=a_{\lambda_1}\delta_{\lambda_1,\lambda_2}\delta_{r_1,r_3}\delta_{r_2,r_4}\,. \tag{4-2-13}$$

From the explicit evaluation of

$$\int d\hat{\mu}(D)\,\varphi^{(\lambda)}_{0\,0}(D)^*\varphi^{(\lambda)0}_{0\,0}(D)=\int d\hat{\mu}(D)\frac{(D^*_{11}D_{11})^{\lambda}}{\lambda!} \tag{4-2-14}$$

which is just unity, we obtain

$$a_\lambda=1\,. \tag{4-2-15}$$

On the other hand, by the transformation of the integration variables

$$\begin{pmatrix} D_{11}D_{12} \\ D_{21}D_{22} \end{pmatrix}=\begin{pmatrix} y(\omega+1) & yx \\ yz & yxz \end{pmatrix} \tag{4-2-16}$$

we get

$$\int d\hat{\mu}(D)f(D)=\int d\hat{\mu}(xyz)\,f\begin{pmatrix} y & yx \\ yz & yxz \end{pmatrix},$$

$$d\hat{\mu}(xyz) \equiv \frac{1}{\pi^3} yy^* \{yy^*(1+xx^*)(1+zz^*)-1\}$$
$$\times e^{-yy^*(1+xx^*)(1+zz^*)} d^2y\, d^2x\, d^2z\,. \qquad (4\text{–}2\text{–}17)$$

In deriving Eqs. (4–2–15) and (4–2–17), the technique of the Grassmann algebra explained in Eq. (4–1–7) is useful. Now from Eqs. (4–2–17), (4–2–13) and (4–2–15), we conclude

$$d\tilde{\mu}\begin{pmatrix} X_{11} X_{21} \\ Y_{11} Y_{12} \end{pmatrix} = d\hat{\mu}(xyz)\,. \qquad (4\text{–}2\text{–}18)$$

The rotator coherent state of Eq. (4–2–10) with the integration measure $d\hat{\mu}(xyz)$ of Eq. (4–2–17) was first proposed by Janssen[23] in a different way.

The rotator wave functions $\psi^j_{mm'}$ are nothing but the 2×2 DG polynomials, and if we adopt the usual Euler angle parametrization for the argument 2×2 matrix, they are represented by the rotation matrix $D^j_{mm'}(\varphi\theta\psi)$. In the boson representation of DG polynomials we have

$$\psi^j_{mm'} = \varphi^{(\lambda)0}_{r,r'}(a^\dagger)\,|0\rangle\,,$$
$$a^\dagger = \begin{pmatrix} a^\dagger_{11} & a^\dagger_{12} \\ a^\dagger_{21} & a^\dagger_{22} \end{pmatrix}. \qquad (4\text{–}2\text{–}19)$$

In this representation the rotator coherent states are written as

$$\Phi\begin{pmatrix} X_{11} X_{21} \\ Y_{11} Y_{12} \end{pmatrix} = \sum_{\lambda=0}^{\infty} \sum_{r,r'=0}^{\lambda} \varphi^{(\lambda)0}_{r,r'}(C)\,\varphi^{(\lambda)0}_{r,r'}(a^\dagger)\,|0\rangle$$
$$= \exp\{\mathrm{Tr}({}^tC\cdot a^\dagger)\}\,|0\rangle$$
$$= \exp\{X_{11}(Y_{11}a^\dagger_{11} + Y_{12}a^\dagger_{12}) + X_{21}(Y_{11}a^\dagger_{21} + Y_{12}a^\dagger_{22})\}\,|0\rangle$$
$$= \exp\{y(a^\dagger_{11} + xa^\dagger_{12} + za^\dagger_{21} + xza^\dagger_{22})\}\,|0\rangle\,, \qquad (4\text{–}2\text{–}20)$$

which gives us the relation between the rotator coherent states and the h.o. coherent states.

4–3. SU_3 coherent states

In this subsection, we briefly discuss the application to SU_3 coherent states. Let us introduce boson creation and annihilation operators $a^\dagger_{\alpha i}$ and $a_{\alpha i}$ ($\alpha = x, y, z; i = 1 \sim N$) which satisfy the commutation relation $[a_{\alpha i}, a^\dagger_{\beta j}] = \delta_{\alpha\beta}\cdot\delta_{ij}$, and the U_3 algebra $\hat{A}_{\alpha\beta} = \sum_{i=1}^{N} a^\dagger_{\alpha i} a_{\beta i}$. By using a polynomial function $P_{(\lambda\mu)a}$ of $a^\dagger_{\alpha i}$ or $a_{\alpha i}$, a boson state of a system C, which is classified by the SU_3 label $(\lambda\mu)$ and has its internal quantum number a, is written as

$$\Phi_{(\lambda\mu)a}(C) = P_{(\lambda\mu)a}(a^\dagger_{\alpha i} \text{ or } a_{\alpha i})\,\phi_0\,, \qquad (4\text{–}3\text{–}1)$$

where ϕ_0 is a SU_3-scalar core characterized by the equation

$$\hat{A}_{\alpha\beta}\phi_0 = n\delta_{\alpha\beta}\phi_0 . \tag{4-3-2}$$

Corresponding to a 3×3 unimodular unitary matrix expressed as

$$U = \exp\{i \sum_{\alpha\beta} u_{\alpha\beta} e_{\alpha\beta}\}, \quad (u^*_{\alpha\beta} = u_{\beta\alpha}, \ \sum_{\alpha} u_{\alpha\alpha} = 0), \tag{4-3-3}$$

we can construct a unimodular-unitary-transformation operator $\hat{T}_U \equiv \exp\{i \sum_{\alpha\beta} u_{\alpha\beta}\hat{A}_{\alpha\beta}\}$, which satisfies

$$\hat{T}_U a^\dagger_{\alpha i} \hat{T}_U^{-1} = \sum_{\beta} U_{\beta\alpha} a^\dagger_{\beta i} , \tag{4-3-4a}$$

$$\hat{T}_U a_{\alpha i} \hat{T}_U^{-1} = \sum_{\beta} U^*_{\beta\alpha} a_{\beta i} . \tag{4-3-4b}$$

Here, we use the notations

$$\boldsymbol{a}^\dagger_i \equiv \begin{pmatrix} a^\dagger_{xi} \\ a^\dagger_{yi} \\ a^\dagger_{zi} \end{pmatrix}, \quad \boldsymbol{a}_i \equiv \begin{pmatrix} a_{xi} \\ a_{yi} \\ a_{zi} \end{pmatrix}, \quad \boldsymbol{U}_\alpha \equiv \begin{pmatrix} U_{x\alpha} \\ U_{y\alpha} \\ U_{z\alpha} \end{pmatrix}$$

and write Eq. (4-3-4) as follows;

$$\hat{T}_U a^\dagger_{\alpha i} \hat{T}_U^{-1} = \boldsymbol{U}_\alpha \cdot \boldsymbol{a}^\dagger_i , \tag{4-3-5a}$$

$$\hat{T}_U a_{\alpha i} \hat{T}_U^{-1} = \boldsymbol{U}^*_\alpha \cdot \boldsymbol{a}_i \tag{4-3-5b}$$

Since $\hat{T}_U \phi_0 = \phi_0$ from Eq. (4-3-2), we obtain

$$\begin{aligned} \hat{T}_U \Phi_{(\lambda\mu)a}(C) &= \hat{T}_U P_{(\lambda\mu)a}(a^\dagger_{\alpha i} \text{ or } a_{\alpha i}) \hat{T}_U \cdot \hat{T}_U^{-1} \phi_0 \\ &= P_{(\lambda\mu)a}(\boldsymbol{U}_\alpha \cdot \boldsymbol{a}^\dagger_i \text{ or } \boldsymbol{U}^*_\alpha \cdot \boldsymbol{a}_i)\phi_0 \\ &= \sum_b \Phi_{(\lambda\mu)b}(C) \langle \Phi_{(\lambda\mu)b}(C) | \hat{T}_U | \Phi_{(\lambda\mu)a}(C) \rangle . \end{aligned} \tag{4-3-6}$$

The matrix element $\langle \Phi_{(\lambda\mu)b}(C) | \hat{T}_U | \Phi_{(\lambda\mu)a}(C) \rangle$ is a group theoretical coefficient and does not depend on the specific structure of the system C. Therefore, for the evaluation of the matrix element, we can use the 3×2 DG polynomials and the generator algebra Eq. (2-1-4a);

$$\begin{aligned} &\langle \Phi_{(\lambda\mu)b}(C) | \hat{T}_U | \Phi_{(\lambda\mu)a}(C) \rangle \\ &= \langle \varphi^{(32)(\lambda\mu)}_{b,H}(R) | \exp\{i \sum_{\alpha\beta} u_{\alpha\beta} A_{\alpha\beta}\} | \varphi^{(32)(\lambda\mu)}_{a,H}(R) \rangle \\ &= \langle \varphi^{(32)(\lambda\mu)}_{b,H}(R) | \varphi^{(32)(\lambda\mu)}_{a,H}({}^tUR) \rangle \\ &= \frac{1}{N_H(\lambda\mu)} \varphi^{(33)(\lambda\mu)0}_{a,b}({}^tU) \\ &= \frac{1}{N_H(\lambda\mu)} \varphi^{(33)(\lambda\mu)0}_{b,a}(U) . \end{aligned} \tag{4-3-7}$$

Thus we obtain

$$\widehat{T}_U \Phi_{(\lambda\mu)a}(C) = P_{(\lambda\mu)a}(\boldsymbol{U}_\alpha \cdot \boldsymbol{a}_i^\dagger \text{ or } \boldsymbol{U}_\alpha^* \cdot \boldsymbol{a}_i) \cdot \phi_0$$

$$= \frac{1}{N_H(\lambda\mu)} \sum_b \varphi_{b,a}^{(33)(\lambda\mu)0}(U)\, \Phi_{(\lambda\mu)b}(C) \tag{4-3-8a}$$

$$= \frac{1}{N_H(\mu\lambda)} \sum_b (-)^{\psi(\lambda\mu)(b)-\psi(\lambda\mu)(a)} \varphi_{\bar{b},\bar{a}}^{(33)(\mu\lambda)0}(U)^* \times \Phi_{(\lambda\mu)b}(C), \tag{4-3-8b}$$

where the last equation comes from Eq. (2-5-4).

An SU_3 coherent state is defined as $P_{(\lambda\mu)a}(\boldsymbol{R}_\alpha \cdot \boldsymbol{a}_i^\dagger)\phi_0$ or $P_{(\lambda\mu)a}(\boldsymbol{R}_\alpha^* \cdot \boldsymbol{a}_i)\phi_0$, where $R_{\alpha\beta}$ is a general 3×3 matrix; namely, by replacing $a_{\alpha i}^\dagger$ or $a_{\alpha i}$ in Eq. (4-3-1) with $\boldsymbol{R}_\alpha \cdot \boldsymbol{a}_i^\dagger$ or $\boldsymbol{R}_\alpha^* \cdot \boldsymbol{a}_i$. It is the simplest to take $a=H$ (the highest weight) in $P_{(\lambda\mu)a}(\boldsymbol{R}_\alpha \cdot \boldsymbol{a}_i^\dagger)\phi_0$ and $a=L$ (the lowest weight) in $P_{(\lambda\mu)a}(\boldsymbol{R}_\alpha^* \cdot \boldsymbol{a}_i)\phi_0$. In these cases, the analytical continuations from the unimodular unitary matrix U to a general matrix R in Eqs. (4-3-8a) and (4-3-8b), respectively, give

$$\Phi_{(\lambda\mu)}^R(C) = P_{(\lambda\mu)H}(\boldsymbol{R}_\alpha \cdot \boldsymbol{a}_i^\dagger)\phi_0$$

$$= \frac{1}{N_H(\lambda\mu)} \sum_b \varphi_{b,H}^{(33)(\lambda\mu)0}(R)\, \Phi_{(\lambda\mu)b}(C)$$

$$= \frac{1}{N_H(\lambda\mu)} \sum_{pqr} [U_{\lambda+\mu}(\boldsymbol{R}_x) U_\mu(\boldsymbol{R}_y)]_{(\lambda\mu)pqr} \times \Phi_{(\lambda\mu)pqr}(C), \tag{4-3-9a}$$

$$\widetilde{\Phi}_{(\lambda\mu)}^{R^*}(C) = P_{(\lambda\mu)L}(\boldsymbol{R}_\alpha^* \cdot \boldsymbol{a}_i)\phi_0$$

$$= \frac{1}{N_H(\mu\lambda)} \sum_b (-)^{\psi(\lambda\mu)(b)-\psi(\lambda\mu)(L)} \varphi_{\bar{b},H}^{(33)(\mu\lambda)0}(R)^* \Phi_{(\lambda\mu)b}(C)$$

$$= \frac{1}{N_H(\mu\lambda)} \sum_{pqr} (-)^{q+r} [U_{\mu+\lambda}(\boldsymbol{R}_x) U_\lambda(\boldsymbol{R}_y)]^*_{(\mu\lambda)\mu-q,\lambda-p,\lambda-p+q-r} \times \Phi_{(\lambda\mu)pqr}(C), \tag{4-3-9b}$$

where the vector-coupling expression Eqs. (2-6-32a) and (2-6-33a) is used. Eqs. (4-3-9a) and (4-3-9b) satisfy the following relations;

$$\int d\mu(R)\, \{\Phi_{(\lambda\mu)}^R(C)\}^* \Phi_{(\lambda'\mu')}^R(C) = \delta_{(\lambda\mu),(\lambda'\mu')} \frac{1}{N_H(\lambda\mu)^2} \sum_b \{\Phi_{(\lambda\mu)b}(C)\}^* \Phi_{(\lambda\mu)b}(C), \tag{4-3-10a}$$

$$\int d\mu(R)\, \{\widetilde{\Phi}_{(\lambda\mu)}^{R^*}(C)\}^* \widetilde{\Phi}_{(\lambda'\mu')}^{R^*}(C) = \delta_{(\lambda\mu),(\lambda'\mu')} \frac{1}{N_H(\mu\lambda)^2} \sum_b \{\Phi_{(\lambda\mu)b}(C)\}^* \Phi_{(\lambda\mu)b}(C), \tag{4-3-10b}$$

$$\langle \Phi^{R}_{(\lambda\mu)} | \Phi^{R'}_{(\lambda'\mu')} \rangle = \delta_{(\lambda\mu),(\lambda'\mu')} \frac{1}{N_H(\lambda\mu)} \varphi^{(33)(\lambda\mu)0}_{H,H}({}^tR^*R'), \tag{4-3-11a}$$

$$\langle \tilde{\Phi}^{R^*}_{(\lambda\mu)} | \tilde{\Phi}^{(R')^*}_{(\lambda'\mu')} \rangle = \delta_{(\lambda\mu),(\lambda'\mu')} \frac{1}{N_H(\mu\lambda)} \varphi^{(33)(\mu\lambda)0}_{HH}({}^tR\cdot(R')^*), \tag{4-3-11b}$$

where the inner product in Eq. (4-3-11) is of the boson states. If we take $R=R'=U$ (the unimodular unitary matrix) in Eqs. (4-3-11a) and (4-3-11b), they become $\delta_{(\lambda\mu),(\lambda'\mu')}$.

4-4. Invariant polynomials

Invariant polynomials are used by many authors[6),24),15)] for the calculation of the C-G coefficients. However, the calculation of their normalization constants is very tough in the case of SU_3[24),15)]. From our approach, the invariant polynomials are easily obtained by using the transformation formula Eq. (2-3-20) and the complex conjugate representation by the $\mathfrak{R}$-conjugation transformation.

First let us consider the invariant polynomial of SU_2. The transformation formula of the 2×2 DG polynomial

$$\varphi^{(22)(\lambda)\mu}_{s,\lambda}({}^tAB) = \frac{1}{N_H(\lambda\mu)} \sum_r \varphi^{(22)(\lambda)\mu}_{r,s}(A)\,\varphi^{(22(\lambda)\mu}_{r,\lambda}(B)$$

is rewritten to by

$$\varphi^{(22)(\lambda)0}_{s,\lambda}({}^tAB)\,|A|^{\mu} = \frac{1}{N_H(\lambda\mu)} \sum_r \varphi^{(22)(\lambda)\mu}_{r,s}(A)\,\varphi^{(22)(\lambda)0}_{r,\lambda}(B), \tag{4-4-1}$$

using the relation

$$\varphi^{(\lambda)\mu}_{rr'}(R) = \frac{N_H(\lambda\mu)}{N_H(\lambda 0)} \varphi^{(\lambda)0}_{rr'}(R)\,|R|^{\mu}. \tag{4-4-2}$$

By performing the $\mathfrak{R}$-conjugation transformation (Eq. (2-5-6)) with respect to B in Eq. (4-4-1), we obtain

$$\varphi^{(\lambda)0}_{s,\lambda}(C)\,|A|^{\mu} = \frac{1}{N_H(\lambda\mu)} \sum_r \varphi^{(\lambda)\mu}_{rs}(A)\,\varphi^{(\lambda)0}_{r\lambda}(\delta_B), \tag{4-4-3}$$

where

$$C = \mathfrak{R}_B\,{}^tAB = {}^tA\delta_B$$
$$= \begin{pmatrix} \begin{vmatrix} A_{x1} B_{x2} \\ A_{y1} B_{y2} \end{vmatrix} & -\begin{vmatrix} A_{x1} B_{x1} \\ A_{y1} B_{y1} \end{vmatrix} \\ \begin{vmatrix} A_{x2} B_{x2} \\ A_{y2} B_{y2} \end{vmatrix} & -\begin{vmatrix} A_{x2} B_{x1} \\ A_{y2} B_{y1} \end{vmatrix} \end{pmatrix} \equiv \begin{pmatrix} C_{11} C_{12} \\ C_{21} C_{22} \end{pmatrix} \tag{4-4-4}$$

from Eq. (2-6-36). From Eqs. (2-6-35) and (2-6-30a), we have

$$\varphi_{r\lambda}^{(\lambda)0}(\delta_B)=\varphi_{r\lambda}^{(\lambda)0;\,c}(B)=(-)^{r+\lambda}\varphi_{\lambda-r,0}^{(\lambda)0}(B)$$
$$=(-)^{\lambda+r}v_{\lambda/2,-\lambda/2+r}(\boldsymbol{B}_1)=\omega_{\lambda/2,\lambda/2-r}(\boldsymbol{B}_1). \qquad (4\text{-}4\text{-}5)$$

Here, $\omega_{jm}(\boldsymbol{R})\equiv(-)^{j+m}v_{j,-m}(\boldsymbol{R})$ is a basis of the complex conjugate representation of $v_{jm}(\boldsymbol{R})$.[6] From the explicit expression of $\varphi_{s\lambda}^{(22)(\lambda)0}(C)$, Eq. (4-4-3) becomes

$$\frac{1}{\sqrt{(\lambda-s)!s!}}C_{12}^{\lambda-s}C_{22}^{s}|A|^{\mu}=\frac{1}{N_H(\lambda\mu)}\sum_r\varphi_{rs}^{(\lambda)\mu}(A)\,\omega_{\lambda/2,\lambda/2-r}(\boldsymbol{B}_1). \qquad (4\text{-}4\text{-}6)$$

Substitution of Eq. (4-4-4) and the vector-coupling expression Eq. (2-6-30a) of $\varphi_{rs}^{(\lambda)\mu}(A)$ give the following invariant polynomial of SU_2;*

$$I_s^{(\lambda)\mu}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{B}_1)$$
$$\equiv\sum_{r=0}^{\lambda}\frac{1}{\sqrt{\lambda+1}}[v_{(\lambda+\mu-s)/2}(\boldsymbol{A}_1)\,v_{(\mu+s)/2}(\boldsymbol{A}_2)]_{\lambda/2,\lambda/2-r}\,\omega_{\lambda/2,\lambda/2-r}(\boldsymbol{B}_1)$$
$$=(-)^s\left[\frac{s!\,(\lambda-s)!\,\mu!}{(\lambda+\mu+1)!}\right]^{1/2}\frac{1}{s!\,(\lambda-s)!\,\mu!}\begin{vmatrix}A_{x2}B_{x1}\\A_{y2}B_{y1}\end{vmatrix}^{s}\begin{vmatrix}B_{x1}A_{x1}\\B_{y1}A_{y1}\end{vmatrix}^{\lambda-s}\begin{vmatrix}A_{x1}A_{x2}\\A_{y1}A_{y2}\end{vmatrix}^{\mu}, \qquad (4\text{-}4\text{-}7)$$

which are orthonormalized as

$$\langle I_s^{(\lambda)\mu}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{B}_1)\,|\,I_{s'}^{(\lambda')\mu'}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{B}_1)\rangle=\delta_{\lambda,\lambda'}\delta_{\mu,\mu'}\delta_{s,s'}\,.$$

Substitution of quantum numbers similar to Eq. (3-1-3) gives the expression obtained by Bargmann.[25),18),6)]

The same method is available to obtain the SU_3 invariant polynomial. By considering the 3×3 DG polynomial, the expression which corresponds to Eq. (4-4-1) is

$$\varphi_{p'q'0,L}^{(33)(\lambda\mu)0}({}^tAB)\,|A|^{\omega}=\frac{1}{N_H(\lambda\mu\omega)}\sum_{pqr}\varphi_{pqr,p'q'0}^{(33)(\lambda\mu)\omega}(A)\,\varphi_{pqr,L}^{(33)(\lambda\mu)0}(B), \qquad (4\text{-}4\text{-}8)$$

where $L\equiv(p''=\lambda,\ q''=r''=\mu)$ denotes the lowest weight. By performing the $\mathfrak{R}$-conjugation transformation $\mathfrak{R}_B$ in Eq. (4-4-8) and by using Eq. (2-6-39) and the vector-coupling expression Eq. (2-6-32a), we obtain

$$I_{p'q'}^{(\lambda\mu)\omega}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{B}_1,\boldsymbol{B}_2)$$
$$\equiv\sum_{pqr}\frac{1}{\sqrt{\dim(\lambda\mu)}}[[U_{\sigma+\tau}(\boldsymbol{A}_1)\,U_{\tau}(\boldsymbol{A}_2)]_{(\sigma\tau)}U_{\Lambda}(\boldsymbol{A}_3)]_{(\lambda\mu)pqr}$$
$$\times(-)^{q+r}[U_{\mu+\lambda}(\boldsymbol{B}_1)\,U_{\lambda}(\boldsymbol{B}_2)]_{(\mu\lambda)\mu-q,\lambda-p,\lambda-p+q-r}$$
$$=\frac{N_H(\lambda\mu\omega)}{\sqrt{\dim\,(\lambda\mu)}}\left\{\frac{(\mu+1)!}{(\lambda+1)!}\right\}^{1/2}\varphi_{p'q'0,L}^{(\lambda\mu)0}(C)\,|A|^{\omega}|B|^{-\mu}, \qquad (4\text{-}4\text{-}9)$$

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* We can also obtain the same SU_2 invariant polynomial by comparing the vector coupling expression Eq. (2-6-31b) and the explicit expression of the 3×2 DG polynomial $\varphi_{0qr,0}^{(32)(0\mu)}(R)$.

where $\dim(\lambda\mu)=1/2\cdot(\lambda+1)(\mu+1)(\lambda+\mu+2)$, $(\sigma,\tau)=(\lambda-p'+q',\mu+\omega-q')$, $\Lambda=\omega+p'+q'$ and $C=\mathfrak{R}_B{}^tAB={}^tA\delta_B$. Using the explicit expression*

$$\varphi^{(\lambda\mu)0}_{p'q'0,L}(C)=(-)^{q'}N(\lambda\mu;p'q'0)C_{13}^{\lambda-p'}C_{33}^{p'}(\delta_C)_{21}^{q'}(\delta_C)_{31}^{\mu-q'} \qquad (4\text{-}4\text{-}10)$$

obtained from Eq. (2-6-24) and the relation $\delta_C={}^t\delta_A B|B|$ easily proved, we can obtain the following final expression of the orthonormalized SU_3 invariant polynomial of the type $(\sigma\tau)\times(\Lambda 0)\to(\lambda\mu)$;[15)]**

$$\begin{aligned}
&I^{(\lambda\mu)\omega}_{p'q'}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{B}_1,\boldsymbol{B}_2)\\
&=(-)^{q'}N(\lambda\mu;p'q'0)\frac{N_H(\lambda\mu\omega)}{\sqrt{\dim(\lambda\mu)}}\left\{\frac{(\mu+1)!}{(\lambda+1)!}\right\}^{1/2}\\
&\times\{\boldsymbol{A}_1\cdot(\boldsymbol{\delta}_B)_3\}^{\lambda-p'}\{\boldsymbol{A}_3\cdot(\boldsymbol{\delta}_B)_3\}^{p'}\{\boldsymbol{B}_1\cdot(\boldsymbol{\delta}_A)_2\}^{q'}\{\boldsymbol{B}_1\cdot(\boldsymbol{\delta}_A)_3\}^{\mu-q'}\\
&\times\{\boldsymbol{A}_1\cdot(\boldsymbol{\delta}_A)_1\}^{\omega}, \qquad (4\text{-}4\text{-}11)
\end{aligned}$$

where we have used the 3-dimensional vector expression $\delta_R=((\boldsymbol{\delta}_R)_1,(\boldsymbol{\delta}_R)_2,(\boldsymbol{\delta}_R)_3)$ and the relations $({}^tA\delta_B)_{ij}=\boldsymbol{A}_i\cdot(\boldsymbol{\delta}_B)_j$ and $({}^t\delta_A B)_{ij}=\boldsymbol{B}_j\cdot(\boldsymbol{\delta}_A)_i$.

§ 5. Application to SU_3 coefficients and their relation to SU_2 coefficients

5-1. Expression of the C-G coefficient of $(\sigma\tau)\times(\Lambda 0)\to(\lambda\mu)$ and a relation of $\langle(\sigma\tau)p_1q_1(\Lambda 0)L\|(\lambda\mu)pq\rangle$ to a special 6-$(\lambda\mu)$ coefficient

A special SU_3 reduced C-G coefficient $\langle(\sigma\tau)p_1q_1(\Lambda 0)p_2\|(\lambda\mu)pq\rangle$ associated with the multiplicity-free Kronecker product $(\sigma\tau)\times(\Lambda 0)\to(\lambda\mu)$ was calculated by many authors,[26),27),15),28),29),30)] because of its importance for physical applications. The algebraic expressions of $\langle(\sigma\tau)p_1q_1(\Lambda 0)p_2\|(\lambda\mu)pq\rangle$ obtained by these authors are ranging from two to six summations. In our formalism the vector coupling expression of the 3×3 DG polynomial gives a very simple method to obtain this type of C-G coefficient. By dividing the full C-G coefficient into the reduced SU_3 C-G coefficient and the SU_2 C-G coefficient, we write the vector-coupling expression Eq. (2-6-32a) as

$$\begin{aligned}
&\varphi^{(33)(\lambda\mu)\omega}_{pqr,p'q'0}(R)=[[U_{\sigma+\tau}(\boldsymbol{R}_1)U_\tau(\boldsymbol{R}_2)]_{(\sigma\tau)}U_\Lambda(\boldsymbol{R}_3)]_{(\lambda\mu)pqr}\\
&=\sum_{p_1q_1p_2}\langle(\sigma\tau)p_1q_1(\Lambda 0)p_2\|(\lambda\mu)pq\rangle\\
&\times\sum_{r_1r_2}\left\langle\frac{\sigma-p_1+q_1}{2}\,\frac{\sigma-p_1+q_1}{2}-r_1\,\frac{\Lambda-p_2}{2}\,\frac{\Lambda-p_2}{2}-r_2\middle|\frac{\lambda-p+q}{2}\,\frac{\lambda-p+q}{2}-r\right\rangle\\
&\times[U_{\sigma+\tau}(\boldsymbol{R}_1)U_\tau(\boldsymbol{R}_2)]_{(\sigma\tau)p_1q_1r_1}U_{(\Lambda 0)p_2r_2}(\boldsymbol{R}_3), \qquad (5\text{-}1\text{-}1)
\end{aligned}$$

* This expression is the most simply obtained from the expression of the 3×2 DG polynomial Eq. (2-6-13) by the use of $\mathfrak{R}$-conjugation transformation of the 3×3 DG polynomial (Eq. (2-6-39)).

** There is a misprint in Eq. (3-23) of Ref. 15); $(k_0+k_1+k_6+1)$ should be $(k_0+k_1+k_6+1)!$. Moreover, the right hand side of Eq. (3-27) of Ref. 15) should be multiplied by $\{(\lambda_3+1)!/(\mu_3+1)!\}^{1/2}$, in order to give the correct agreement with our result Eq. (5-1-20).

where from Eq. (2-6-33a)

$$\Lambda \equiv \omega + p' + q' = \lambda + 2\mu + 3\omega - \sigma - 2\tau\,,$$
$$(\sigma, \tau) \equiv (\lambda - p' + q', \mu + \omega - q'), \tag{5-1-2a}$$

or

$$\omega = \frac{1}{3}(\sigma + 2\tau + \Lambda - \lambda - 2\mu),$$
$$p' = \lambda + \mu + \omega - \sigma - \tau = \frac{1}{3}(2\lambda + \mu + \Lambda - 2\sigma - \tau),$$
$$q' = \mu + \omega - \tau = \frac{1}{3}(-\lambda + \mu + \Lambda + \sigma - \tau). \tag{5-1-2b}$$

It should be noted that ω and the SU_3 internal quantum numbers p', q' give the parametrization of the SU_3 coupling $(\sigma\tau) \times (\Lambda 0) \rightarrow (\lambda\mu)$; namely, if $(\sigma\tau)$, $(\Lambda 0)$ and $(\lambda\mu)$ are given, the parameters ω, p' and q' are defined by Eq. (5-1-2b). The relations of the internal quantum numbers p_1, q_1, r_1 and p_2, r_2 with p, q, r originated from the conservation of quanta (polynomial degrees) are, from Eq. (2-6-12),

$$\varepsilon_1 + \varepsilon_2 = \varepsilon \quad \longrightarrow \quad p_1 + q_1 + p_2 = \omega + p + q\,,$$
$$\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 = \frac{1}{2}\nu \quad \longrightarrow \quad r_1 + r_2 = q' + q_1 - q + r\,. \tag{5-1-2c}$$

The straightforward integration of $\varphi^{(33)(\lambda\mu)\omega}_{pq0,p'q'0}(R)$ by the product $\varphi^{(32)(\sigma\tau)}_{p_1q_1r_1,0}(R)^* \cdot U_{(\Lambda 0)p_2 0}(\boldsymbol{R}_3)^*$ gives an expression with five summations. However, we do not show this expression here because some idea proposed by Chacón, Ciftan and Biedenharn[30] works well also in our formalism, which gives the expression with two summations.

First we calculate a special C-G coefficient $\langle(\sigma\tau)p_1q_1(\Lambda 0)L\|(\lambda\mu)pq\rangle$ (L denotes the lowest weight $p_2 = \Lambda$), which we also use in the later application at the end of this subsection. This C-G coefficient is easily obtained by performing the direct integration in the relation

$$\langle U_{(\Lambda 0)L}(\boldsymbol{R}_3)\,|\,\varphi^{(33)(\lambda\mu)\omega}_{pq0,p'q'0}(R)\rangle$$
$$= \langle(\sigma\tau)p_1q_1(\Lambda 0)L\|(\lambda\mu)pq\rangle[U_{\sigma+\tau}(\boldsymbol{R}_1)U_{\tau}(\boldsymbol{R}_2)]_{(\sigma\tau)p_1q_10}\,, \tag{5-1-3}$$

where the values p_1, q_1 are determined as

$$p_1 = p - p',\quad q_1 = q - q' \tag{5-1-4}$$

from the conservation of the angular momentum $(\sigma - p_1 + q_1)/2 = (\lambda - p + q)/2$ and from Eqs. (5-1-2c) and (5-1-2a). The final result by the use of the explicit expression Eq. (2-6-23) of the 3×3 DG polynomial is

$$\langle(\sigma\tau)p_1q_1(\Lambda 0)L\|(\lambda\mu)pq\rangle$$

$$= \frac{N_H(\lambda\mu\omega)\mathcal{N}(\lambda\mu;pq0)\mathcal{N}(\lambda\mu;p'q'0)}{N(\sigma\tau;p_1q_10)} \times \frac{\sqrt{\Lambda!}\,\lambda!\,\mu!(\lambda+\mu+1)!p!q!(\lambda+q+1)!}{(\lambda-p)!(\mu-q)!p_1!q_1!(\lambda+\mu-p+1)!(\lambda-p'+q+1)!}. \tag{5-1-5}$$

In order to calculate a general C-G coefficient $\langle(\sigma\tau)p_1q_1\ (\Lambda0)p_2\|(\lambda\mu)pq\rangle$, we start from the relation

$$\begin{aligned}&\langle\varphi^{(32)(\sigma\tau)}_{p_1q_10,0}(R)\,U_{(\Lambda0)p_2(r_2)_L}(\boldsymbol{R}_3)\,|\varphi^{(33)(\lambda\mu)\omega}_{pqr,p'q'0}(R)\rangle\\ &=\langle\sigma\tau)p_1q_1(\Lambda0)p_2\|(\lambda\mu)pq\rangle\\ &\times\left\langle\frac{\sigma-p_1+q_1}{2}\ \frac{\sigma-p_1+q_1}{2}\ \frac{\Lambda-p_2}{2}\ -\frac{\Lambda-p_2}{2}\right|\left.\frac{\lambda-p+q}{2}\ \frac{\lambda-p+q}{2}-r\right\rangle\end{aligned} \tag{5-1-6}$$

where $(r_2)_L=\Lambda-p_2$ and $r=p'+p_1-p$. The idea in Ref. 30) is as follows; by using the relation

$$\begin{aligned}U_{(\Lambda0)p_2(r_2)_L}(\boldsymbol{R}_3)&=\frac{R_{y3}^{\Lambda-p_2}R_{z3}^{p_2}}{\sqrt{(\Lambda-p_2)!\,p_2!}}\\ &=\sqrt{\frac{p_2!}{\Lambda!(\Lambda-p_2)!}}A_{yz}^{\Lambda-p_2}U_{(\Lambda0)L}(\boldsymbol{R}_3)\end{aligned} \tag{5-1-7}$$

obtained by Eq. (2-6-27), the left hand side of Eq. (5-1-6) is converted as

$$\langle B_1A_{yz}^{\Lambda-p_2}B_2|B_3\rangle=\sum_{a=0}^{\Lambda-p_2}(-)^a\binom{\Lambda-p_2}{a}\langle(A_{yz}^aB_1)\cdot B_2|A_{zy}^{\Lambda-p_2-a}B_3\rangle, \tag{5-1-8}$$

where B_1, B_2 and B_3 are appropriate polynomials. In Ref. 30) the matrix elements of $A_{zy}^{\Lambda-p_2-a}$ and A_{yz}^a with respect to Gel'fand states are calculated by the techniques of the pattern calculas[28] and by the use of the Gel'fand-Zetlin results[8] for the matrix element of the U_n generators. In our formalism this can be also easily calculated as follows. Without loss of generality, it is enough for us to calculate the matrix element

$$\langle\varphi^{(32)(\lambda\mu)}_{p'q'r',0}(R)\,|\,A_{zy}^a|\varphi^{(32)(\lambda\mu)}_{pqr,0}(R)\rangle\,. \tag{5-1-9}$$

Using the relations

$$\begin{aligned}&\varphi^{(32)(\lambda\mu)}_{p'q'r',0}(R)=\mathcal{N}(\lambda-p'+q';r')\,A_{yx}^{r'}\varphi^{(32)(\lambda\mu)}_{p'q'0,0}(R),\\ &A_{yx}^\dagger=A_{xy}\,,\\ &[A_{xy},A_{zy}^a]=0\,,\\ &A_{xy}^{r'}\varphi^{(32)(\lambda\mu)}_{pqr,0}(R)=\frac{\mathcal{N}(\lambda-p+q;r-r')}{\mathcal{N}(\lambda-p+q;r)}\varphi^{(32)(\lambda\mu)}_{p,q,r-r';0}(R),\end{aligned}$$

Eq. (5-1-9) is equal to

$$\frac{\mathcal{N}(\lambda-p'+q';r')\mathcal{N}(\lambda-p+q;r-r')}{\mathcal{N}(\lambda-p+q;r)} \times \langle \varphi^{(32)(\lambda\mu)}_{p'q'0,0}(R)\,|\,A^{a}_{zy}\,|\,\varphi^{(32)(\lambda\mu)}_{p,q,r-r';0}(R)\rangle\,. \tag{5-1-10}$$

Thus it is enough for us to calculate the matrix element Eq. (5-1-9) with $r'=0$. This matrix element is obtained by the following alternative methods, both of which we show here for later convenience. One is due to the direct integration by the use of the explicit expression Eq. (2-6-13) of the 3×2 DG polynomial, and the other is due to the generating function method by the use of the explicit expression Eq. (2-6-24) of the 3×3 DG polynomial.

The direct operation of A^{a}_{zy} on the explicit expression Eq. (2-6-13) of the 3×2 DG polynomial gives

$$\begin{aligned}
&\langle \varphi^{(32)(\lambda\mu)}_{p'q'0,0}(R)\,|\,A^{a}_{zy}\,|\,\varphi^{(32)(\lambda\mu)}_{pqr,0}(R)\rangle \\
&= N(\lambda\mu;p'q'0)\,N(\lambda\mu;pqr) \\
&\times \sum_{k=\mathrm{Max}\{0,\,p+r-\lambda\}}^{\mathrm{Min}\{r,q\}}\ \sum_{b=\mathrm{Max}\{0,\,-r+k+a\}}^{\mathrm{Min}\{a,\,\mu-q\}} (-)^{q+q'-k+b} \\
&\times r!\binom{\lambda-p}{r-k}\binom{q}{k}\binom{a}{b}\frac{(r-k)!\,(\mu-q)!}{(r-k-a+b)!\,(\mu-q-b)!} \\
&\times \langle R^{\lambda-p'}_{x1}R^{p'}_{z1}(\delta_{12})^{q'}_{y}(\delta_{12})^{\mu-q'}_{z}\,|\,R^{\lambda-p-r+k}_{x1}R^{r-k-a+b}_{y1} \\
&\times R^{p+a-b}_{z1}(\delta_{12})^{k}_{x}(\delta_{12})^{q-k+b}_{y}(\delta_{12})^{\mu-q-b}_{z}\rangle\,.
\end{aligned} \tag{5-1-11}$$

The matrix element in the right hand side of Eq. (5-1-11) is given by the the formula

$$\begin{aligned}
&\langle R^{\lambda-p'}_{x1}R^{p'}_{z1}(\delta_{12})^{q'}_{y}(\delta_{12})^{\mu-q'}_{z}\,|\,R^{A}_{x1}R^{B}_{y1}R^{C}_{z1}(\delta_{12})^{D}_{x}(\delta_{12})^{E}_{y}(\delta_{12})^{F}_{z}\rangle \\
&= (-)^{B+D}p'!\,q'!\,(\mu-q')!\,A!\,E!\,F!\binom{A+E+F+1}{F}\binom{A+C+E+1}{E-B},
\end{aligned} \tag{5-1-12a}$$

where the non-zero matrix element is given when the conditions

$$\begin{aligned}
&\lambda=A+B+C,\quad \mu=D+E+F\,, \\
&p'=B+C+D,\quad q'=E-B
\end{aligned} \tag{5-1-12b}$$

are satisfied. In order to prove this formula, we should directly integrate over $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ and take the summation by the use of the following binomial formulas;

$$\sum_{e}(-)^{c-e}\binom{a}{c-e}\binom{b+e}{e}$$

$$=\begin{cases} (-)^c\begin{pmatrix} a-b-1 \\ c \end{pmatrix} & \text{for} \quad a \geqq b+1\,, \\ \begin{pmatrix} b-a+c \\ c \end{pmatrix} & \text{for} \quad a \leqq b\,, \end{cases} \tag{5-1-13a}$$

$$\sum_{p+q=r}\begin{pmatrix} a+p \\ p \end{pmatrix}\begin{pmatrix} b+q \\ q \end{pmatrix}=\begin{pmatrix} a+b+1+r \\ r \end{pmatrix}, \tag{5-1-13b}$$

where a, b, $r \geqq 0$. After substituting the expression of the matrix element by using the formula Eq. (5-1-12a), the summations over k and b in Eq. (5-1-11) can be contracted by the formulas Eq. (5-1-13a) and

$$\sum_{p+q=r}\begin{pmatrix} a \\ p \end{pmatrix}\begin{pmatrix} b \\ q \end{pmatrix}=\begin{pmatrix} a+b \\ r \end{pmatrix}. \tag{5-1-13c}$$

Thus we obtain the following results;

$$\langle \varphi^{(32)(\lambda\mu)}_{p'q'0,0}(R)\,|\,A^a_{zy}|\varphi^{(32)(\lambda\mu)}_{pqr,0}(R)\rangle = N(\lambda\mu;p'q'0)\,N(\lambda\mu;pqr)$$
$$\times\frac{a!(\lambda-p)!(\mu-q)!p'!q'!(\lambda+q'+1)!(\lambda+\mu-p+1)!}{(\lambda+1)!(q'-q)!(\lambda-p+q'+1)!}, \tag{5-1-14a}$$

where the non-zero matrix element is given when

$$p'=p+r,\quad q'=q-r+a,\quad q'\geqq q \tag{5-1-14b}$$

are satisfied.

Another very simple method to obtain Eq. (5-1-14) is due to the use of the transformation formula Eq. (2-3-23a) and the explicit expression of the 3×3 DG polynomial Eq. (2-6-24). Namely, by taking

$$g\equiv(g_{\alpha\beta})=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tau & 0 \end{pmatrix}, \tag{5-1-15a}$$

$$G=e^{ig}=1+ig=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & i\tau & 1 \end{pmatrix}, \tag{5-1-15b}$$

$$T^L_G=\exp\{i\sum_{\alpha\beta}g_{\alpha\beta}A_{\alpha\beta}\}=\exp\{i\tau A_{zy}\}=\sum_{a=0}^{\infty}\frac{1}{a!}(i\tau)^a A^a_{zy}\,, \tag{5-1-15c}$$

we have

$$\langle \varphi^{(32)(\lambda\mu)}_{p'q'0,0}(R)\,|T^L_G|\varphi^{(32)(\lambda\mu)}_{pqr,0}(R)\rangle$$
$$=\sum_{a=0}^{\infty}\frac{1}{a!}(i\tau)^a\langle \varphi^{(32)(\lambda\mu)}_{p'q'0,0}(R)\,|\,A^a_{zy}|\varphi^{(32)(\lambda\mu)}_{pqr,0}(R)\rangle$$

$$=\frac{1}{N_H(\lambda\mu)}\varphi^{(33)(\lambda\mu)0}_{pqr,p'q'0}({}^tG)$$

$$=\mathcal{N}(\lambda\mu;pqr)\mathcal{N}(\lambda\mu;p'q'0)\,(i\tau)^{q'-q+r}$$

$$\times\frac{\lambda!\mu!(\lambda+\mu+1)!p'!q'!(\lambda+q'+1)!}{(\lambda-p')!(\mu-q')!(\lambda+\mu-p'+1)!(\lambda-p+q'+1)!(q'-q)!} \tag{5-1-16}$$

Here, summation over b in the 3×3 DG polynomial can be contracted by Eq. (5-1-13a). The term with $a=q'-q+r$ in Eq. (5-1-16) only survives and we obtain the same expression as in Eq. (5-1-14).

From the two basic expressions Eqs. (5-1-5) and (5-1-14), we obtain the following final expression;

$$\langle(\sigma\tau)\,p_1q_1(\Lambda0)\,p_2\|(\lambda\mu)\,pq\rangle$$

$$=N(\sigma\tau;p_1q_10)\,N(\lambda\mu;pqr)\,\mathcal{N}(\lambda\mu;p'q'0)$$

$$\times\frac{N_H(\lambda\mu\omega)}{N_H(\lambda\mu)}\left[\frac{(\lambda+q'+q_1-p+1)!(q'+q_1-q)!\,p_2!}{(\lambda-p+q+1)\,(\sigma-p_1+q_1)!}\right]^{1/2}$$

$$\times\frac{p_1!q_1!(\sigma+q_1+1)!(\lambda-p)!(\mu-q)!(\lambda+\mu-p+1)!r!}{(\sigma+1)!(\lambda-p+q-r)!}$$

$$\times\sum_{u=\mathrm{Max}\{0,p'+p_1-\lambda\}}^{\mathrm{Min}\{p_1,p'+p_1-p\}}\ \sum_{v=\mathrm{Max}\{0,q'+q_1-\mu\}}^{\mathrm{Min}\{q_1,q'+q_1-q,\sigma-p_1+q_1,\Lambda-p_2-u\}}(-)^{u+v}$$

$$\times\frac{(\lambda-p+q-r+\tilde{r})!}{(r-\tilde{r})!\tilde{r}!}\frac{(\lambda-\tilde{p}+\tilde{q}-\tilde{r})!}{(\lambda-\tilde{p}+\tilde{q})!}\frac{\tilde{p}!\tilde{q}!(\sigma-\bar{p})!(\tau-\bar{q})!}{\bar{p}!\bar{q}!(\lambda-\tilde{p})!(\mu-\tilde{q})!}$$

$$\times\frac{(\sigma+\tau-\bar{p}+1)!}{(\lambda+\mu-\tilde{p}+1)!}\frac{(\lambda-\tilde{p}+\tilde{q}+1)!}{(\lambda-p+\tilde{q}+1)!\,(\lambda-p'+\tilde{q}+1)!}$$

$$\times\frac{(\lambda+\tilde{q}+1)!}{(\sigma-\bar{p}+q_1+1)!(q_1-\bar{q})!(\tilde{q}-q)!}, \tag{5-1-17a}$$

where

$$\tilde{p}=p'+p_1-u,\quad \tilde{q}=q'+q_1-v,\quad \tilde{r}=u\,,$$

$$\bar{p}=\tilde{p}-p'=p_1-u,\quad \bar{q}=\tilde{q}-q'=q_1-v\,,$$

$$r=p'+p_1-p,\quad p_1+q_1+p_2=\omega+p+q \tag{5-1-17b}$$

and ω, p', q' are given by Eq. (5-1-2b).

Let us consider some special cases. The C-G coefficient of the type $(\sigma,0)\times(\Lambda,0)\rightarrow(\lambda\mu)$ will be discussed in the next subsection, since it is given by an SU_2 C-G coefficient and has a close relation to the Regge symmetry of the multiplicity-free U_3 reduced C-G coefficient. One of the cases when a simple expression is obtained from Eq. (5-1-17) is of $p=\lambda$, $q=\mu$ (the lowest weight). In this case the C-G coefficient is given by a single factor;

$$\langle(\sigma\tau)p_1q_1(\Lambda 0)p_2\|(\lambda\mu)L\rangle$$
$$=\frac{N_H(\lambda\mu\omega)}{N_H(\lambda\mu)}\cdot N(\sigma\tau;p_1q_10)N(\lambda\mu;p'q'0)(-)^{p_1+q_1+p'+q'-\lambda-\mu}$$
$$\times\left[\frac{(\lambda+\mu-p'-p_1)!\,p_2!}{(\mu+1)!(p'+p_1-\lambda)!(q'+q_1-\mu)!(q'+q_1+1)!(\sigma-p_1+q_1)!}\right]^{1/2}$$
$$\times\frac{p_1!\,q_1!\,q'!(\sigma+q_1+1)!(\mu+\omega+1)!\,\omega!}{(\sigma+1)!}, \tag{5-1-18}$$

where $p_1+q_1+p_2=\lambda+\mu+\omega$ and ω, p', q' are given by Eq. (5-1-2b). Furthermore, by setting $p_2=\Lambda$ (the lowest weight) we obtain

$$\langle(\sigma\tau)p_1q_1(\Lambda 0)L\|(\lambda\mu)L\rangle$$
$$=\frac{N_H(\lambda\mu\omega)}{N_H(\lambda\mu)}N(\sigma\tau;p_1q_10)N(\lambda\mu;p'q'0)$$
$$\times\frac{\sqrt{\Lambda!}\,p_1!\,q_1!\,q'!(\lambda+\mu-p'+1)!(\mu+\omega+1)!\,\omega!}{(\sigma+1)!(\mu+1)!}, \tag{5-1-19}$$

where $p_1=\lambda-p'$, $q_1=\mu-q'$. The expression of Eq. (5-1-19) can be also obtained from Eq. (5-1-5) by setting $p=\lambda$ and $q=\mu$.

The expression of the C-G coefficient $\langle(\sigma\tau)p_1q_1(\Lambda,0)p_2\|(\lambda\mu)H\rangle$ is very difficult to obtain in the form of a single factor from Eq. (5-1-17). So we show here only the final result obtained from the direct integration of

$$\langle\varphi^{(32)(\sigma\tau)}_{p_1q_1r_1,0}(R)\,U_{(\Lambda 0)p_20}(\boldsymbol{R}_3)\,|\varphi^{(33)(\lambda\mu)\omega}_{H,p'q'0}(R)\rangle$$

and from the use of the formulas Eqs. (5-1-12) and (5-1-13a);

$$\langle(\sigma\tau)p_1q_1(\Lambda 0)p_2\|(\lambda\mu)H\rangle$$
$$=N_H(\lambda\mu\omega)\mathcal{N}(\lambda\mu;p'q'0)N(\sigma\tau;p_1q_1r_1)(-)^{p_1+q_1}$$
$$\times\left[\frac{(\lambda+q'+q_1+1)!(p'+p_1)!}{(\lambda+1)!\,p_2!}\right]^{1/2}\frac{\lambda!\,\mu!(\lambda+\mu+1)!\,\omega!}{(\sigma+1)!(\mu-q')!(\lambda+\mu-p'+1)!}$$
$$\times\frac{(q'+q_1)!(\sigma-p_1)!(\tau-q_1)!(\sigma+\tau-p_1+1)!}{(\sigma-q'-p_1)!}, \tag{5-1-20}$$

where $r_1=q'+q_1$, $p_1+q_1+p_2=\omega$ and ω, p', q' are given by Eq. (5-1-2b). From Eq. (5-1-20) we can see that our phase convension

$$\langle(\sigma\tau)H(\Lambda 0)p_2\|(\lambda\mu)H\rangle\geqq 0, \tag{5-1-21}$$

is certainly reproduced, which is the $l=3$ case of Eq. (2-2-6c). Furthermore, by setting $p_2=\Lambda$ in Eq. (5-1-20) or $p=q=0$ in Eq. (5-1-5), we obtain $p_1=q_1=0$ (the highest weight), $\sigma=\lambda$, $\tau=\mu+\omega$, $\omega=\Lambda$ and

$$\langle(\sigma\tau)H(\Lambda 0)L\|(\lambda\mu)H\rangle=\frac{N_H(\lambda\mu\omega)}{N_H(\sigma\tau)N_H(\Lambda 0)}. \tag{5-1-22}$$

This is a $p'=q'=0$ case in

$$\langle(\sigma\tau)H(\Lambda 0)L\|(\lambda\mu)pq\rangle=\frac{N_H(\lambda\mu\omega)}{N_H(\sigma\tau)N_H(\Lambda 0)},$$

$$(\sigma,\tau)=(\lambda-p'+q',\mu+\omega-q'),\quad p=p',\quad q=q', \tag{5-1-23}$$

which is obtained by setting $p_1=q_1=0$ in Eq. (5-1-5).

The expression in Eq. (5-1-23) is also obtained from the general formula

$$\langle[q_1\cdots q_{n-1}]H[\Lambda]L\|[f_1\cdots f_n]q_1\cdots q_{n-1}\rangle_n$$
$$=\frac{N_H[f_1\cdots f_n]}{N_H[q_1\cdots q_{n-1}]N_H[\Lambda]}, \tag{5-1-24}$$

where $\Lambda=\sum_{\mu=1}^{n}f_\mu-\sum_{\mu=1}^{n-1}q_\mu$. This is obtained from the C-G series

$$\sum_{a_1a_2}\langle[q_1\cdots q_{n-1}]a_1[\Lambda]a_2|[f_1\cdots f_n]a\rangle_n\varphi^{(nn)[q_1\cdots q_{n-1}0]}_{a_1,H}(R)\varphi^{(nn)[\Lambda 0\cdots 0]}_{a_2,L}(R)$$
$$=\frac{N_H[q_1\cdots q_{n-1}]N_H[\Lambda]}{N_H[f_1\cdots f_n]}$$
$$\times\langle[q_1\cdots q_{n-1}]H[\Lambda]L|[f_1\cdots f_n]q_1\cdots q_{n-1},H\rangle_n$$
$$\times\varphi^{(nn)[f_1\cdots f_n]}_{a,q_1\cdots q_{n-1}H}(R), \tag{5-1-25a}$$

by comparing it with the vector-coupling expression Eq. (2-2-4) of $\varphi^{(nn)[f_1\cdots f_n]}_{a,q_1\cdots q_{n-1}H}(R)$. Here, we should note

$$\varphi^{(nn)[q_1\cdots q_{n-1}0]}_{a_1,H}(R)=\varphi^{(n,n-1)[q_1\cdots q_{n-1}]}_{a_1,H}(R), \tag{5-1-25b}$$

$$\varphi^{(nn)[\Lambda 0\cdots 0]}_{a_2,L}(R)=U_{[\Lambda]a_2}(\boldsymbol{R}_n). \tag{5-1-25c}$$

In Ref. 30), the fact in Eq. (5-1-24) is essentially used, if Eq. (4) of Ref. 30) is viewed from the so-called factorization lemma.

Finally we show that a special 6-$(\lambda\mu)$ coefficient is directly given by Eq. (5-1-5). We start from the SU_3 C-G series Eq. (2-4-2);

$$\sum_{a_1a_2}\langle(\lambda_1\mu_1)a_1(\lambda_2\mu_2)a_2|(\lambda\mu)a;\rho\rangle\varphi^{(33)(\lambda_1\mu_1)\omega_1}_{a_1b_1}(R)\varphi^{(33)(\lambda_2\mu_2)\omega_2}_{a_2b_2}(R)$$
$$=\frac{N_H(\lambda_1\mu_1\omega_1)N_H(\lambda_2\mu_2\omega_2)}{N_H(\lambda\mu\omega)}$$
$$\times\sum_b\langle(\lambda_1\mu_1)b_1(\lambda_2\mu_2)b_2|(\lambda\mu)b;\rho\rangle\varphi^{(33)(\lambda\mu)\omega}_{ab}(R), \tag{5-1-26}$$

where $\lambda_1+2\mu_1+3\omega_1+\lambda_2+2\mu_2+3\omega_2=\lambda+2\mu+3\omega$. Setting $b_1\equiv(p_1,\ q_1,\ r_1)=(p_1,\ q_1,\ 0)$, $\lambda_2=\Lambda_2$, $\mu_2=\omega_2=0$, $b_2\equiv(p_2,\ q_2,\ r_2)=(\Lambda_2,\ 0,\ 0)=L$ (the lowest weight) and from Eqs. (2-6-32a) and (2-6-33a), we obtain

$$\sum_{a_1a_2}\langle(\lambda_1\mu_1)a_1(\Lambda_2 0)a_2|(\lambda\mu)a\rangle$$

$$[[U_{\sigma+\tau}(\boldsymbol{R}_1)U_\tau(\boldsymbol{R}_2)]_{(\sigma\tau)}U_{\Lambda_1}(\boldsymbol{R}_3)]_{(\lambda_1\mu_1)a_1}U_{(\Lambda_2 0)a_2}(\boldsymbol{R}_3)$$
$$=\frac{N_H(\lambda_1\mu_1\omega_1)N_H(\Lambda_2 0)}{N_H(\lambda\mu\omega)}$$
$$\times\sum_{pqr}\langle(\lambda_1\mu_1)p_1q_10(\Lambda_20)\Lambda_200|(\lambda\mu)pqr\rangle\varphi^{(33)(\lambda\mu)\omega}_{a,pqr}(R), \qquad (5\text{-}1\text{-}27)$$

where $\Lambda_1\equiv\omega_1+p_1+q_1$, $(\sigma,\tau)\equiv(\lambda_1-p_1+q_1,\ \mu_1+\omega_1-q_1)$ and $\sigma+2\tau+\Lambda_1+\Lambda_2=\lambda_1+2\mu_1+3\omega_1+\Lambda_2=\lambda+2\mu+3\omega$. By writing

$$\langle(\lambda_1\mu_1)p_1q_10(\Lambda_20)\Lambda_200|(\lambda\mu)pqr\rangle$$
$$=\langle(\lambda_1\mu_1)p_1q_1(\Lambda_20)L\|(\lambda\mu)pq\rangle$$
$$\times\left\langle\frac{\lambda_1-p_1+q_1}{2}\ \frac{\lambda_1-p_1+q_1}{2}\ 0\ 0\,\middle|\,\frac{\lambda-p+q}{2}\ \frac{\lambda-p+q}{2}-r\right\rangle, \qquad (5\text{-}1\text{-}28)$$

it can be seen that non-zero contribution in the right hand side of Eq. (5-1-27) comes when $\sigma=\lambda_1-p_1+q_1=\lambda-p+q$ and $r=0$. Furthermore, from the conservation of quanta of $\boldsymbol{R}_3$ in Eq. (5-1-27), $\Lambda_1+\Lambda_2=\omega+p+q$, which completely determines p and q. On the other hand, the left hand side of Eq. (5-1-27) is rewritten, by using a 9-$(\lambda\mu)$ coefficient (which is essentially a 6-$(\lambda\mu)$ coefficient) and the relation

$$[U_{\Lambda_1}(\boldsymbol{R}_3)U_{\Lambda_2}(\boldsymbol{R}_3)]_{(lm)a}$$
$$=\delta_{(l,m),(\Lambda_1+\Lambda_2,0)}\binom{\Lambda_1+\Lambda_2}{\Lambda_1}^{1/2}U_{(\Lambda_1+\Lambda_2,0)a}(\boldsymbol{R}_3), \qquad (5\text{-}1\text{-}29)$$

to

$$[[[U_{\sigma+\tau}(\boldsymbol{R}_1)U_\tau(\boldsymbol{R}_2)]_{(\sigma\tau)}U_{\Lambda_1}(\boldsymbol{R}_3)]_{(\lambda_1\mu_1)}U_{\Lambda_2}(\boldsymbol{R}_3)]_{(\lambda\mu)a}$$
$$=\sum_{(lm)}\begin{bmatrix}(\sigma\tau) & (\Lambda_10) & (\lambda_1\mu_1)\\ (00) & (\Lambda_20) & (\Lambda_20)\\ (\sigma\tau) & (lm) & (\lambda\mu)\end{bmatrix}$$
$$\times[[U_{\sigma+\tau}(\boldsymbol{R}_1)U_\tau(\boldsymbol{R}_2)]_{(\sigma\tau)}[U_{\Lambda_1}(\boldsymbol{R}_3)U_{\Lambda_2}(\boldsymbol{R}_3)]_{(lm)}]_{(\lambda\mu)a}$$
$$=\begin{bmatrix}(\sigma\tau) & (\Lambda_10) & (\lambda_1\mu_1)\\ (00) & (\Lambda_20) & (\Lambda_20)\\ (\sigma\tau) & (\Lambda_1+\Lambda_20) & (\lambda\mu)\end{bmatrix}\cdot\binom{\Lambda_1+\Lambda_2}{\Lambda_1}^{1/2}$$
$$\times[[U_{\sigma+\tau}(\boldsymbol{R}_1)U_\tau(\boldsymbol{R}_2)]_{(\sigma\tau)}U_{\Lambda_1+\Lambda_2}(\boldsymbol{R}_3)]_{(\lambda\mu)a}. \qquad (5\text{-}1\text{-}30)$$

Thus, by comparing the coefficients of $\varphi^{(33)(\lambda\mu)\omega}_{a,pq0}(R)$, we obtain the following simple formula;

$$\begin{bmatrix}(\sigma\tau) & (\Lambda_10) & (\lambda_1\mu_1)\\ (00) & (\Lambda_20) & (\Lambda_20)\\ (\sigma\tau) & (\Lambda_1+\Lambda_20) & (\lambda\mu)\end{bmatrix}$$

$$= \binom{A_1+A_2}{A_1}^{-1/2} \frac{N_H(\lambda_1\mu_1\omega_1)\, N_H(A_2 0)}{N_H(\lambda\mu\omega)}$$

$$\times \langle (\lambda_1\mu_1)\, p_1 q_1 (A_2 0)\, L \| (\lambda\mu)\, pq \rangle\,, \tag{5-1-31}$$

where ω_1, p_1, q_1 and ω, p, q are determined from the relations

$$\sigma+2\tau+A_1=\lambda_1+2\mu_1+3\omega_1, \quad \sigma+2\tau+A_1+A_2=\lambda+2\mu+3\omega$$

$$\omega_1+p_1+q_1=A_1, \qquad \omega+p+q=A_1+A_2$$

$$\sigma=\lambda_1-p_1+q_1=\lambda-p+q\,, \tag{5-1-32}$$

namely,

$$\omega_1=\frac{1}{3}(\sigma+2\tau+A_1-\lambda_1-2\mu_1), \quad \omega=\frac{1}{3}(\sigma+2\tau+A_1+A_2-\lambda-2\mu)$$

$$p_1=\frac{1}{3}(2\lambda_1+\mu_1+A_1-2\sigma-\tau), \quad p=\frac{1}{3}(2\lambda+\mu+A_1+A_2-2\sigma-\tau),$$

$$q_1=\frac{1}{3}(-\lambda_1+\mu_1+A_1+\sigma-\tau), \quad q=\frac{1}{3}(-\lambda+\mu+A_1+A_2+\sigma-\tau). \tag{5-1-33}$$

The final expression is obtained by substituting Eq. (5-1-5);

$$\begin{bmatrix} (\sigma\tau) & (A_1 0) & (\lambda_1\mu_1) \\ (00) & (A_2 0) & (A_2 0) \\ (\sigma\tau) & (A_1+A_2 0) & (\lambda\mu) \end{bmatrix} \equiv U((\sigma\tau)\,(A_1 0)\,(\lambda\mu)\,(A_2 0);\,(\lambda_1\mu_1)\,(A_1+A_2,\,0))$$

$$= \left[\frac{A_1!\,A_2!\,(\mu_1+1)\,(\lambda_1+\mu_1+2)\,(\lambda+\mu+\omega+2)!\,(\mu+\omega+1)!}{(A_1+A_2)!\,(\lambda+\mu+\omega'+2)!\,(\lambda_1+\mu_1+\omega_1+2)!\,(\mu+\omega'+1)!} \right.$$

$$\times \frac{\omega!\,(\lambda-p'+q'+1)!\,p!\,q!\,(\lambda-p')!\,(\lambda_1-p_1)!\,(\mu-q')!\,(\mu_1-q_1)!}{(\mu_1+\omega_1+1)!\,\omega'!\,\omega_1!\,\lambda_1!\,p'!\,p_1!\,q'!\,q_1!\,(\lambda-p)!\,(\mu-q)!}$$

$$\left. \times \frac{(\lambda+q+1)!\,(\lambda+\mu-p'+1)!\,(\lambda_1+\mu_1-p_1+1)!}{(\lambda+q'+1)!\,(\lambda_1+q_1+1)!\,(\lambda+\mu-p+1)!} \right]^{1/2} \tag{5-1-34}$$

where ω_1, p_1, q_1, ω, p, q are given by Eq. (5-1-33) and ω', p', q' are

$$\omega'=\frac{1}{3}(\lambda_1+2\mu_1+A_2-\lambda-2\mu)=\omega-\omega_1\,,$$

$$p'=\frac{1}{3}(2\lambda+\mu+A_2-2\lambda_1-\mu_1)=p-p_1\,,$$

$$q'=\frac{1}{3}(-\lambda+\mu+A_2+\lambda_1-\mu_1)=q-q_1\,. \tag{5-1-35}$$

By changing the parametrization, Eq. (5-1-34) gives the same expression as that of Hecht, Eq. (A.1) of Ref. 31), which he transcribed from the result by Biedenharn, Louck, Chacón and Ciftan.[32)]

5-2. The relation of $\langle(\lambda_1 0)p_1(\lambda_2 0)p_2\|(\lambda\mu)pq\rangle$ to an SU_2 C-G coefficient viewed from the general Regge symmetry of the multiplicity-free U_n C-G coefficients

The SU_3 reduced C-G coefficient of the type $\langle(\lambda_1 0)p_1(\lambda_2 0)p_2\|(\lambda\mu)pq\rangle$ $(\lambda_1+\lambda_2=\lambda+2\mu)$ is simply given by an SU_2 C-G coefficient, namely,

$$\langle(\lambda+\mu-t,0)p_1(\mu+t,0)p_2\|(\lambda\mu)pq\rangle$$
$$=\left\langle\frac{\lambda-p+q}{2}\ \frac{\lambda-p+q}{2}-t+p-p_1\ \frac{p+q}{2}\ \frac{p+q}{2}-p_2\middle|\frac{\lambda}{2}\ \frac{\lambda}{2}-t\right\rangle, \tag{5-2-1}$$

where $p_1+p_2=p+q$ from $\varepsilon_1+\varepsilon_2=\varepsilon$. This relationship is a direct result of the two-fold vector-coupling expressions Eqs. (2-6-31a) and (2-6-31b) of the 3×2 DG polynomial. In order to prove Eq. (5-2-1), we should only compare the coefficients of monomials of $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ in Eqs. (2-6-31a) and (2-6-31b) and use the Regge symmetry Eq. (3-1-4) of the SU_2 coefficients.

The relationship Eq. (5-2-1) is essentially reduced to the Regge symmetry relation of the multiplicity-free U_3 reduced C-G coefficients Eq. (3-1-6) with $n=3$. To see this, we note here the similar relationship to Eq. (5-2-1) with respect to a general U_n case. By setting $q'_{n-1}=0$ in Eq. (3-1-6) and using the formulas

$$\langle[q_1\cdots q_{n-1}\,0]r_1\cdots r_{n-2}\,0[P_n]P_{n-1}\|[f_1\cdots f_{n-1}0]q'_1\cdots q'_{n-2}\,0\rangle_n$$
$$=\langle[q_1\cdots q_{n-1}]r_1\cdots r_{n-2}[P_n]P_{n-1}\|[f_1\cdots f_{n-1}]q'_1\cdots q'_{n-2}\rangle_{n-1}, \tag{5-2-2}$$

$$\langle[q_1\cdots q_{n-1}]r_1\cdots r_{n-2}[P_n]P_{n-1}\|[f_1\cdots f_{n-1}]q'_1\cdots q'_{n-2}\rangle_{n-1}$$
$$=\langle[q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1},0]r_1-q_{n-1},\cdots,r_{n-2}-q_{n-1}[P_n]P_{n-1}\|$$
$$[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]q'_1-q_{n-1},\cdots,q'_{n-2}-q_{n-1}\rangle_{n-1}, \tag{5-2-3}$$

we find

$$\langle[q'_1\cdots q'_{n-2}\,00]r_1\cdots r_{n-2}\,0[N_n]N_{n-1}\|[f_1\cdots f_{n-1}\,0]q_1\cdots q_{n-1}\rangle_n$$
$$=\langle[q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1},0]r_1-q_{n-1},\cdots,r_{n-2}-q_{n-1}[P_n]P_{n-1}\|$$
$$[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]q'_1-q_{n-1},\cdots,q'_{n-2}-q_{n-1}\rangle_{n-1}, \tag{5-2-4}$$

where

$$N_n=\sum_{i=1}^{n-1}f_i-\sum_{i=1}^{n-2}q'_i,\qquad N_{n-1}=\sum_{i=1}^{n-1}q_i-\sum_{i=1}^{n-2}r_i, \tag{5-2-5}$$

$$P_n=\sum_{i=1}^{n-1}(f_i-q_i),\qquad P_{n-1}=\sum_{i=1}^{n-2}(q'_i-r_i). \tag{5-2-6}$$

(The formulas Eqs. (5-2-2) and (5-2-3) are proved in Appendix E.) By setting $n=3$ in Eq. (5-2-4) and rewriting the U_3 and U_2 reduced C-G coefficients to the SU_3 and SU_2 ones, we obtain Eq. (5-2-1).

The explicit expression of $\langle(\lambda+\mu-t,0)p_1(\mu+t,0)p_2\|(\lambda\mu)pq\rangle$ can be obtained by performing the direct integration in

$$\langle U_{(\lambda+\mu-t,0)p_1r_1}(\boldsymbol{R}_1)\,U_{(\mu+t,0)p_20}(\boldsymbol{R}_2)\,|\varphi^{(32)(\lambda\mu)}_{pq0,t}(R)\rangle, \tag{5-2-7}$$

which is equivalent to the Racah's expression[17] of the SU_2 C-G coefficient in Eq. (5-2-1). The expression obtained by setting $\tau=0$ in Eq. (5-1-17) is slightly different from the Wigner's expression[33] of the SU_2 C-G coefficient. However, both expressions (the one from Eq. (5-1-17) and the Wigner's one) become the Racah's expression by the use of the following formula;

$$\begin{aligned}&\sum_t(-)^t\binom{\mu}{t}\binom{M_1+M_2-t}{M_2}\binom{M_3+M_4-t}{M_4}\\&=(-)^\mu\sum_t(-)^t\binom{\mu}{t}\binom{M_1+M_2-\mu}{M_2-t}\binom{M_3+M_4-\mu}{M_3-t},\end{aligned} \tag{5-2-8}$$

where M_1, M_2, M_3, M_4, $\mu\geqq 0$, $\mathrm{Min}\{M_1+M_2,\ M_3+M_4\}\geqq\mu$ and the index t runs over all integers such that none of the factorial arguments are negative. (The formula Eq. (5-2-8) is obtained by considering the generating function $G(x,y)\equiv(x-y)^\mu(1+x)^{M_1+M_2-\mu}(1+y)^{M_3+M_4-\mu}$. See Appendix 3 of Ref. 34).)

5-3. The two-row 9-($\lambda\mu$) coefficient and its relation to a 9-j coefficient

In this subsection, we show that the so-called two-row 9-($\lambda\mu$) coefficient is given by a 9-j coefficient and furthermore that the stretched two-row 9-($\lambda\mu$) coefficient (which is equal to a stretched 9-j coefficient) is essentially given by an SU_2 C-G coefficient.

The two-row condition in the SU_3 coupling $(\lambda_1\mu_1)\times(\lambda_2\mu_2)\to(\lambda\mu)$ is stated as

$$\lambda_1+2\mu_1+\lambda_2+2\mu_2=\lambda+2\mu\,. \tag{5-3-1}$$

In this case the SU_3 C-G coefficient

$$\begin{aligned}&\langle(\lambda_1\mu_1)\,\varepsilon_1\Lambda_1\nu_1(\lambda_2\mu_2)\,\varepsilon_2\Lambda_2\nu_2|\,(\lambda\mu)\,\varepsilon\Lambda\nu\rangle\\&=\langle(\lambda_1\mu_1)\,\varepsilon_1\Lambda_1(\lambda_2\mu_2)\,\varepsilon_2\Lambda_2\|\,(\lambda\mu)\,\varepsilon\Lambda\rangle\left\langle\Lambda_1\frac{\nu_1}{2}\Lambda_2\frac{\nu_2}{2}\middle|\Lambda\frac{\nu}{2}\right\rangle\end{aligned} \tag{5-3-2}$$

with ε, $\Lambda=\varepsilon_H$, Λ_H (the highest weight) simply becomes an SU_2 C-G coefficient under an appropriate phase convention. In fact, the relation $\varepsilon_1+\varepsilon_2=\varepsilon$ written as $3(p_1+q_1)-\lambda_1-2\mu_1+3(p_2+q_2)-\lambda_2-2\mu_2=3(p+q)-\lambda-2\mu$ gives $p_1=q_1=p_2=q_2=0$ when $p=q=0$ and Eq. (5-3-1) is satisfied, which means that ε_1, Λ_1 and ε_2, Λ_2 should have also the highest weights and $|\langle(\lambda_1\mu_1)\,H(\lambda_2\mu_2)\,H\|\,(\lambda\mu)\,H\rangle|=1$. Therefore, if we adopt the phase convention*

* This phase convention does not conflict with that used in the preceding sections. See Eq. (5-1-21).

$$\langle(\lambda_1\mu_1)H(\lambda_2\mu_2)H\|(\lambda\mu)H\rangle=1$$
$$\text{for}\quad \lambda_1+2\mu_1+\lambda_2+2\mu_2=\lambda+2\mu\,, \tag{5-3-3}$$

the two-row SU_3 C-G coefficient Eq. (5-3-2) with ε, $\Lambda=\varepsilon_H$, Λ_H and ε_1, $\Lambda_1=(\varepsilon_1)_H$, $(\Lambda_1)_H$ and ε_2, $\Lambda_2=(\varepsilon_2)_H$, $(\Lambda_2)_H$ is simply $\langle(\lambda_1/2)\,(\lambda_1/2)-r_1\ (\lambda_2/2)\,(\lambda_2/2)-r_2|\,(\lambda/2)\,(\lambda/2)-r\rangle$. The same discussion is also possible when we take ε_1, $\Lambda_1=(\varepsilon_1)_H$, $(\Lambda_1)_H$ and ε_2, $\Lambda_2=(\varepsilon_2)_H$, $(\Lambda_2)_H$.

We start from the definition of 9-$(\lambda\mu)$ coefficients expressed by the 3×2 DG polynomials (with the highest weight in the SU_2 part) as

$$\begin{aligned}
&\sum_{\substack{a_1a_2a_3a_4\\ a_{12}\ a_{34}}}\langle(\lambda_{12}\mu_{12})a_{12}(\lambda_{34}\mu_{34})a_{34}|(\lambda\mu)a\rangle\\
&\quad\times\langle(\lambda_1\mu_1)a_1(\lambda_2\mu_2)a_2|(\lambda_{12}\mu_{12})a_{12}\rangle\langle(\lambda_3\mu_3)a_3(\lambda_4\mu_4)a_4|(\lambda_{34}\mu_{34})a_{34}\rangle\\
&\quad\times\varphi^{(32)(\lambda_1\mu_1)}_{a_1,0}(R_{(15)})\varphi^{(32)(\lambda_2\mu_2)}_{a_2,0}(R_{(26)})\varphi^{(32)(\lambda_3\mu_3)}_{a_3,0}(R_{(37)})\varphi^{(32)(\lambda_4\mu_4)}_{a_4,0}(R_{(48)})\\
&=\sum_{(\lambda_{13}\mu_{13})(\lambda_{24}\mu_{24})}\begin{bmatrix}(\lambda_1\mu_1) & (\lambda_2\mu_2) & (\lambda_{12}\mu_{12})\\ (\lambda_3\mu_3) & (\lambda_4\mu_4) & (\lambda_{34}\mu_{34})\\ (\lambda_{13}\mu_{13}) & (\lambda_{24}\mu_{24}) & (\lambda\mu)\end{bmatrix}\\
&\quad\times\sum_{\substack{a_1a_2a_3a_4\\ a_{13}\ a_{24}}}\langle(\lambda_{13}\mu_{13})a_{13}(\lambda_{24}\mu_{24})a_{24}|(\lambda\mu)a)\\
&\quad\times\langle(\lambda_1\mu_1)a_1(\lambda_3\mu_3)a_3|(\lambda_{13}\mu_{13})a_{13}\rangle\langle(\lambda_2\mu_2)a_2(\lambda_4\mu_4)a_4|(\lambda_{24}\mu_{24})a_{24}\rangle\\
&\quad\times\varphi^{(32)(\lambda_1\mu_1)}_{a_1,0}(R_{(15)})\varphi^{(32)(\lambda_2\mu_2)}_{a_2,0}(R_{(26)})\varphi^{(32)(\lambda_3\mu_3)}_{a_3,0}(R_{(37)})\varphi^{(32)(\lambda_4\mu_4)}_{a_4,0}(R_{(48)})\,,
\end{aligned}\tag{5-3-4}$$

where $R_{(ij)}=(\boldsymbol{R}_i,\boldsymbol{R}_j)$ is a 3×2 matrix. Here, we assume the following two-row condition

$$\begin{aligned}
&\lambda_1+2\mu_1+\lambda_2+2\mu_2=\lambda_{12}+2\mu_{12}\,,\\
&\lambda_3+2\mu_3+\lambda_4+2\mu_4=\lambda_{34}+2\mu_{34}\,,\\
&\lambda_{12}+2\mu_{12}+\lambda_{34}+2\mu_{34}=\lambda+2\mu\,,
\end{aligned}\tag{5-3-5}$$

from which we can easily find

$$\begin{aligned}
&\lambda_1+2\mu_1+\lambda_3+2\mu_3=\lambda_{13}+2\mu_{13}\,,\\
&\lambda_2+2\mu_2+\lambda_4+2\mu_4=\lambda_{24}+2\mu_{24}\,,\\
&\lambda_{13}+2\mu_{13}+\lambda_{24}+2\mu_{24}=\lambda+2\mu\,,
\end{aligned}\tag{5-3-6}$$

in the right hand side of Eq. (5-3-4). By taking $a\equiv(p,q,r)=(0,0,r)$ in Eq. (5-3-4), we find that only the terms with $a_{ij}\equiv(p_{ij},q_{ij},r_{ij})=(0,0,r_{ij})$ and $a_i\equiv(p_i,q_i,r_i)=(0,0,r_i)$ survive due to the two row condition Eqs. (5-3-5) and (5-3-6). Thus all the SU_3 C-G coefficients in Eq. (5-3-4) reduce to the SU_2 C-G coefficients. On the other hand, the 3×2 DG polynomials are reduced, from Eqs. (2-6-26), (4-4-2) and (2-6-30a), to

$$\varphi^{(32)(\lambda_i\mu_i)}_{00r_i,0}(R_{(ij)}) = \varphi^{(22)(\lambda_i)\mu_i}_{r_i,0}(R_{(ij)})$$
$$= \frac{N_H(\lambda_i\mu_i)}{N_H(\lambda_i 0)} |R_{(ij)}|^{\mu_i} \varphi^{(22)(\lambda_i)0}_{r_i,0}(R_{(ij)})$$
$$= \frac{N_H(\lambda_i\mu_i)}{N_H(\lambda_i 0)} |R_{(ij)}|^{\mu_i} v_{\lambda_i/2\ \lambda_i/2-r_i}(\boldsymbol{R}_i), \tag{5-3-7}$$

where $R_{(ij)}$ except in the 3×2 DG polynomial should be read as

$$R_{(ij)} = (\boldsymbol{R}_i, \boldsymbol{R}_j) = \begin{pmatrix} R_{xi} R_{xj} \\ R_{yi} R_{yj} \end{pmatrix}.$$

As a result, Eq. (5-3-4) gives a simple formula of the angular momentum recoupling, from which we obtain

$$\begin{bmatrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) & (\lambda_{12}\mu_{12}) \\ (\lambda_3\mu_3) & (\lambda_4\mu_4) & (\lambda_{34}\mu_{34}) \\ (\lambda_{13}\mu_{13}) & (\lambda_{24}\mu_{24}) & (\lambda\mu) \end{bmatrix}$$
$$= [(\lambda_{12}+1)(\lambda_{34}+1)(\lambda_{13}+1)(\lambda_{24}+1)]^{1/2}$$
$$\times \begin{Bmatrix} \frac{\lambda_1}{2} & \frac{\lambda_2}{2} & \frac{\lambda_{12}}{2} \\ \frac{\lambda_3}{2} & \frac{\lambda_4}{2} & \frac{\lambda_{34}}{2} \\ \frac{\lambda_{13}}{2} & \frac{\lambda_{24}}{2} & \frac{\lambda}{2} \end{Bmatrix}, \tag{5-3-8}$$

when Eq. (5-3-5) or (5-3-6) is satisfied.[35]

In order to obtain the relation of a stretched two-row 9-$(\lambda\mu)$ coefficient to an SU_2 C-G coefficient, we can use the SU_2 C-G series of the 2×2 DG polynomials;

$$\sum_{r_1 r_2} \left\langle \frac{\lambda_1}{2}\, \frac{\lambda_1}{2}-r_1\, \frac{\lambda_2}{2}\, \frac{\lambda_2}{2}-r_2 \,\middle|\, \frac{\lambda}{2}\, \frac{\lambda}{2}-r \right\rangle \varphi^{(22)(\lambda_1)\mu_1}_{r_1 t_1}(R)\, \varphi^{(22)(\lambda_2)\mu_2}_{r_2 t_2}(R)$$
$$= \frac{N_H(\lambda_1\mu_1)\, N_H(\lambda_2\mu_2)}{N_H(\lambda\mu)}$$
$$\times \sum_t \left\langle \frac{\lambda_1}{2}\, \frac{\lambda_1}{2}-t_1\, \frac{\lambda_2}{2}\, \frac{\lambda_2}{2}-t_2 \,\middle|\, \frac{\lambda}{2}\, \frac{\lambda}{2}-t \right\rangle \varphi^{(22)(\lambda)\mu}_{rt}(R). \tag{5-3-9}$$

By using the vector-coupling expression of the 2×2 DG polynomial Eq. (2-6-30a), the left hand side of Eq. (5-3-9) becomes

$$[[v_{(\lambda_1+\mu_1-t_1)/2}(\boldsymbol{R}_1)\, v_{(\mu_1+t_1)/2}(\boldsymbol{R}_2)]_{\lambda_1/2} [v_{(\lambda_2+\mu_2-t_2)/2}(\boldsymbol{R}_1)\, v_{(\mu_2+t_2)/2}(\boldsymbol{R}_2)]_{\lambda_2/2}]_{\lambda/2,\lambda/2-r}$$
$$= \sum_{J_1 J_2} [(\lambda_1+1)(\lambda_2+1)(2J_1+1)(2J_2+1)]^{1/2}$$

$$\times \begin{Bmatrix} \frac{\lambda_1+\mu_1-t_1}{2} & \frac{\mu_1+t_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2+\mu_2-t_2}{2} & \frac{\mu_2+t_2}{2} & \frac{\lambda_2}{2} \\ J_1 & J_2 & \frac{\lambda}{2} \end{Bmatrix}$$

$$\times [[v_{(\lambda_1+\mu_1-t_1)/2}(\boldsymbol{R}_1)\, v_{(\lambda_2+n_2-t_2)/2}(\boldsymbol{R}_1)]_{J_1} [v_{(\mu_1+t_1)/2}(\boldsymbol{R}_2)\, v_{(\mu_2+t_2)/2}(\boldsymbol{R}_2)]_{J_2}]_{\lambda/2,\lambda/2-r}$$

$$= [(\lambda_1+1)(\lambda_2+1)(\lambda_1+\mu_1-t_1+\lambda_2+\mu_2-t_2+1)(\mu_1+t_1+\mu_2+t_2+1)]^{1/2}$$

$$\times \begin{Bmatrix} \frac{\lambda_1+\mu_1-t_1}{2} & \frac{\mu_1+t_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2+\mu_2-t_2}{2} & \frac{\mu_2+t_2}{2} & \frac{\lambda_2}{2} \\ \frac{\lambda_1+\mu_1-t_1+\lambda_2+\mu_2-t_2}{2} & \frac{\mu_1+t_1+\mu_2+t_2}{2} & \frac{\lambda}{2} \end{Bmatrix}$$

$$\times \left[\binom{\lambda_1+\mu_1-t_1+\lambda_2+\mu_2-t_2}{\lambda_1+\mu_1-t_1} \binom{\mu_1+t_1+\mu_2+t_2}{\mu_1+t_1} \right]^{1/2} \cdot \varphi_{rt}^{(22)(\lambda)\mu}(R), \quad \text{(5-3-10)}$$

where $t=\frac{\lambda-\lambda_1-\lambda_2}{2}+t_1+t_2$ and the relation

$$[v_{j_1}(\boldsymbol{R})\, v_{j_2}(\boldsymbol{R})]_J = \delta_{J,j_1+j_2} \binom{2j_1+2j_2}{2j_1}^{1/2} v_{j_1+j_2}(\boldsymbol{R}), \qquad \text{(5-3-11)}$$

is used. Thus we obtain

$$[(\lambda_1+1)(\lambda_2+1)(\lambda_1+\mu_1-t_1+\lambda_2+\mu_2-t_2+1)(\mu_1+t_1+\mu_2+t_2+1)]^{1/2}$$

$$\times \begin{Bmatrix} \frac{\lambda_1+\mu_1-t_1}{2} & \frac{\mu_1+t_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2+\mu_2-t_2}{2} & \frac{\mu_2+t_2}{2} & \frac{\lambda_2}{2} \\ \frac{\lambda_1+\mu_1-t_1+\lambda_2+\mu_2-t_2}{2} & \frac{\mu_1+t_1+\mu_2+t_2}{2} & \frac{\lambda}{2} \end{Bmatrix}$$

$$= \frac{N_H(\lambda_1\mu_1)\, N_H(\lambda_2\mu_2)}{N_H(\lambda\mu)} \left[\binom{\lambda_1+\mu_1-t_1+\lambda_2+\mu_2-t_2}{\lambda_1+\mu_1-t_1} \binom{\mu_1+t_1+\mu_2+t_2}{\mu_1+t_1} \right]^{-1/2}$$

$$\times \left\langle \frac{\lambda_1}{2}\, \frac{\lambda_1}{2}-t_1\, \frac{\lambda_2}{2}\, \frac{\lambda_2}{2}-t_2 \,\middle|\, \frac{\lambda}{2}\, \frac{\lambda}{2}-t \right\rangle, \qquad \text{(5-3-12)}$$

where $t=(\lambda-\lambda_1-\lambda_2)/2+t_1+t_2$ and $\lambda_1+2\mu_1+\lambda_2+2\mu_2=\lambda+2\mu$.[36),24),37)] Combining Eq. (5-3-12) with Eq. (5-3-8), we find the following simple relation;

$$\begin{bmatrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) & (\lambda_{12}\mu_{12}) \\ (\lambda_3\mu_3) & (\lambda_4\mu_4) & (\lambda_{34}\mu_{34}) \\ (\lambda_1+\lambda_3,\ \mu_1+\mu_3) & (\lambda_2+\lambda_4,\ \mu_2+\mu_4) & (\lambda\mu) \end{bmatrix}$$

$$= [(\lambda_{12}+1)(\lambda_{34}+1)(\lambda_1+\lambda_3+1)(\lambda_2+\lambda_4+1)]^{1/2}$$

$$\times \begin{Bmatrix} \frac{\lambda_1}{2} & \frac{\lambda_2}{2} & \frac{\lambda_{12}}{2} \\ \frac{\lambda_3}{2} & \frac{\lambda_4}{2} & \frac{\lambda_{34}}{2} \\ \frac{\lambda_1+\lambda_3}{2} & \frac{\lambda_2+\lambda_4}{2} & \frac{\lambda}{2} \end{Bmatrix}$$

$$= \frac{N_H\left(\lambda_{12}, \frac{\lambda_1+\lambda_2-\lambda_{12}}{2}\right) N_H\left(\lambda_{34}, \frac{\lambda_3+\lambda_4-\lambda_{34}}{2}\right)}{N_H\left(\lambda, \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda}{2}\right)}$$

$$\times \left[\binom{\lambda_1+\lambda_3}{\lambda_1}\binom{\lambda_2+\lambda_4}{\lambda_2}\right]^{-1/2}$$

$$\times \left\langle \frac{\lambda_{12}}{2}\ \frac{\lambda_1-\lambda_2}{2}\ \frac{\lambda_{34}}{2}\ \frac{\lambda_3-\lambda_4}{2} \middle| \frac{\lambda}{2}\ \frac{\lambda_1-\lambda_2+\lambda_3-\lambda_4}{2} \right\rangle \qquad (5\text{-}3\text{-}13)$$

where $\lambda_1+\lambda_2+\lambda_3+\lambda_4+2(\mu_1+\mu_2+\mu_3+\mu_4)=\lambda+2\mu$ should be satisfied.

5-4. A special 6-(λμ) coefficient and its relation to a 6-*j* coefficient

In this subsection we show another simple relation between a special 6-$(\lambda\mu)$ coefficient and a 6-j coefficient which cannot be reduced to the two row case in general. From the transformation formula Eq. (2-3-20) of the 3×3 DG polynomial, we can write

$$\varphi^{(33)(\lambda\mu)\omega}_{pqr,a}(AR) = \frac{1}{N_H(\lambda\mu\omega)} \sum_{p'q'r'} \varphi^{(33)(\lambda\mu)\omega}_{pqr,p'q'r'}(A)\,\varphi^{(33)(\lambda\mu)\omega}_{p'q'r',a}(R), \qquad (5\text{-}4\text{-}1)$$

where A is given by Eq. (3-2-2). The left hand side of Eq. (5-4-1) is rewritten, by Eq. (2-6-32b), as

$$\varphi^{(33)(\lambda\mu)\omega}_{pqr,a}(AR) = [[U_{P_x}(\boldsymbol{R}^r_x)U_{P_y}(\boldsymbol{R}^r_z)]_{(lm)}U_{P_z}(\boldsymbol{R}^r_y)]_{(\lambda\mu)a}$$

$$= \sum_{(l'm')} \begin{bmatrix} (P_x\ 0) & (P_y 0) & (lm) \\ (P_z\ 0) & (0\ 0) & (P_z 0) \\ (l'm') & (P_y 0) & (\lambda\mu) \end{bmatrix}$$

$$\times [[U_{P_x}(\boldsymbol{R}^r_x)U_{P_z}(\boldsymbol{R}^r_y)]_{(l'm')}U_{P_y}(\boldsymbol{R}^r_z)]_{(\lambda\mu)a},$$

$$= \sum_{p'q'r} \begin{bmatrix} (P_x\ 0) & (P_y 0) & (lm) \\ (P_z\ 0) & (0\ 0) & (P_z 0) \\ (l'm') & (P_y 0) & (\lambda\mu) \end{bmatrix} \varphi^{(33)(\lambda\mu)\omega}_{p'q'r',a}(R), \qquad (5\text{-}4\text{-}2)$$

where P_x, P_y, P_z, (lm) are related with p, q, r by Eq. (2-6-33b), the sum over p', q', r' should be restricted as $p'+r'=p+r$, $p'+q'=\mu-q+r$ and $(l', m')=(\lambda-p'+q', \mu+\omega-q')$. By comparing Eqs. (5-4-1) and (5-4-2) and using the relation

$$\varphi^{(33)(\lambda\mu)\omega}_{pqr,p'q'r'}(A)=(-)^{\lambda+\omega}\varphi^{(33)(\lambda\mu)\omega}_{pqr,p'q'r'}(-A)$$
$$=(-)^{\lambda+\omega}\varphi^{(33)(\lambda\mu)0}_{pqr,p'q'r'}(-A), \qquad (5\text{-}4\text{-}3)$$

we obtain

$$\begin{bmatrix} (P_x\ 0) & (P_y 0) & (lm) \\ (P_z\ 0) & (0\ \ 0) & (P_z 0) \\ (l'm') & (P_y 0) & (\lambda\mu) \end{bmatrix}=\frac{(-)^{\lambda+\omega}}{N_H(\lambda\mu\omega)}\varphi^{(33)(\lambda\mu)0}_{pqr,p'q'r'}(-A), \qquad (5\text{-}4\text{-}4a)$$

where $P_x+P_y=l+2m$, $P_x+P_z=l'+2m'$ and

$$\omega=\frac{1}{3}(P_x+P_y+P_z-\lambda-2\mu),$$
$$p=\lambda+\mu+\omega-l-m, \qquad p'=\lambda+\mu+\omega-l'-m',$$
$$q=\mu+\omega-m, \qquad q'=\mu+\omega-m',$$
$$r=P_y-m, \qquad r'=P_z-m'. \qquad (5\text{-}4\text{-}4b)$$

From Eqs. (5-4-4) and (3-2-4), we obtain the following relation;

$$\begin{bmatrix} (P_x\ 0) & (P_y 0) & (lm) \\ (P_z\ 0) & (0\ \ 0) & (P_z 0) \\ (l'm') & (P_y 0) & (\lambda\mu) \end{bmatrix}$$
$$=\frac{N_H(\lambda\mu)}{N_H(\lambda\mu\omega)}(-)^{\lambda+\omega+m+m'}[(l+1)(l'+1)]^{1/2}$$
$$\times\begin{Bmatrix} \frac{P_y-\omega}{2} & \frac{P_x-\omega}{2} & \frac{l}{2} \\ \frac{P_z-\omega}{2} & \frac{\lambda}{2} & \frac{l'}{2} \end{Bmatrix}$$
$$=\frac{N_H(\lambda\mu)}{N_H(\lambda\mu\omega)}(-)^{\omega}[(l+1)(l'+1)(P_y-\omega+1)(P_z-\omega+1)]^{1/2}$$
$$\times\begin{Bmatrix} \frac{P_x-\omega}{2} & \frac{P_y-\omega}{2} & \frac{l}{2} \\ \frac{P_z-\omega}{2} & 0 & \frac{P_z-\omega}{2} \\ \frac{l'}{2} & \frac{P_y-\omega}{2} & \frac{\lambda}{2} \end{Bmatrix}, \qquad (5\text{-}4\text{-}5)$$

where $P_x+P_y=l+2m$, $P_x+P_z=l'+2m'$ and $\omega=(P_x+P_y+P_z-\lambda-2\mu)/3$.

§ 6. Application to Talmi-Moshinsky-Smirnov coefficients of the n-body systems

Kaufman and Noack[38] have shown that the Talmi-Moshinsky-Smirnov (abbreviated to TMS) coefficient of the three body system is given by the simple rotation matrix when the relative harmonic oscillator (h.o.) wave functions associated with the two Jacobi-coordinates are coupled into the IR of the SU_3 group. In that case, the TMS transformation can be considered to be a rotation in the quasi-spin space. In our terminology, the TMS transformation is nothing but the right transformation Eq. (2-3-23b) of the 3×2 DG polynomials. The transformation coefficients are composed of the representation matrix of $GL(2;C)$ with a specific orthogonal argument matrix, which depends on the mass of each subunit and on the way of choice of the Jacobi coordinates. The equivalence of the representation matrix of $GL(2;C)$ with a unimodular unitary argument matrix to the usual angular momentum rotation matrix is already shown in Eq. (2-6-8).

The above discussion is easily extended to the n-body system. In that case, we should consider the $3\times(n-1)$ DG polynomial and the transformation coefficients are given by the representation matrix of the type $(n-1)\times(n-1)$. Let us consider the n-body system of structureless particles located at $\boldsymbol{X}_i$ and having the mass number A_i $(i=1\sim n)$. It is convenient to consider the coordinates scaled by the mass number; $\hat{\boldsymbol{X}}_i=\sqrt{A_i}\boldsymbol{X}_i$. One type of the Jacobi coordinates is defined by $\boldsymbol{\xi}_\mu=\boldsymbol{X}_{\mu+1}-\sum_{j=1}^{\mu}A_j\boldsymbol{X}_j/\bar{A}_\mu$ $(\mu=1\sim n-1)$, where $\bar{A}_i=A_1+A_2+\cdots+A_i$ is a mass number of the subunit system up to the i-th particle. The scaled Jacobi coordinates are $\hat{\boldsymbol{\xi}}_\mu=\sqrt{\bar{A}_\mu A_{\mu+1}/\bar{A}_{\mu+1}}\cdot\boldsymbol{\xi}_\mu$. By writing $\hat{\xi}=(\hat{\boldsymbol{\xi}}_1\cdots\hat{\boldsymbol{\xi}}_{n-1})$ and $\hat{X}=(\hat{\boldsymbol{X}}_1\cdots\hat{\boldsymbol{X}}_n)$ in the $3\times(n-1)$ matrix form and $3\times n$ one, respectively, we obtain the transformation

$$(\hat{\xi},\hat{\boldsymbol{X}}_G)=\hat{X}(B,a)=(\hat{X}B,\hat{X}a),\tag{6-1}$$

where $\hat{\boldsymbol{X}}_G$ is a scaled coordinate of center of mass, $B=(B_{i\mu})$ is an $n\times(n-1)$ matrix, a is an n-dimensional vector, and their explicit forms are

$$a=\begin{pmatrix}\sqrt{A_1/A}\\ \vdots\\ \sqrt{A_n/A}\end{pmatrix},\quad A=A_1+\cdots+A_n=\bar{A}_n,\tag{6-2}$$

$$B_{i\mu}=\begin{cases}-\sqrt{A_{\mu+1}A_i/(\bar{A}_\mu\bar{A}_{\mu+1})} & i\leqq\mu\\ \sqrt{\bar{A}_\mu/\bar{A}_{\mu+1}} \quad \text{for} & i=\mu+1\\ 0 & i\geqq\mu+2\end{cases}\tag{6-3}$$

It is easily proved from Eqs. (6-2) and (6-3) that

$$ {}^tBB = E_{n-1}, \quad {}^taB = 0, \quad {}^taa = 1 \,, \tag{6-4}$$

where E_{n-1} is the $(n-1)\times(n-1)$ unit matrix. Since the inverse transformation matrix is

$$(B,\, a)^{-1} = \begin{pmatrix} {}^tB \\ {}^ta \end{pmatrix} \tag{6-5}$$

we find

$$(B,\, a)\begin{pmatrix} {}^tB \\ {}^ta \end{pmatrix} = B\,{}^tB + a\,{}^ta = E_n \,. \tag{6-6}$$

Next, we introduce the permutation

$$P \equiv \begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ p_1 & p_2 & \cdots\cdots & p_n \end{pmatrix} \in S_n \tag{6-7}$$

of n particles. By operating P on $\hat{X}$ and a,

$$\begin{aligned} P\hat{X} &= P(\hat{\boldsymbol{X}}_1 \cdots \hat{\boldsymbol{X}}_n) = (\hat{\boldsymbol{X}}_{p_1} \cdots \hat{\boldsymbol{X}}_{p_n}) \equiv (\hat{\boldsymbol{X}}_1 \cdots \hat{\boldsymbol{X}}_n)\, M_P \\ &= \hat{X} M_P \quad \text{and} \quad Pa = {}^tM_P a \,, \end{aligned} \tag{6-8}$$

where M_P is an orthogonal $n\times n$ matrix (${}^tM_P M_P = M_P {}^tM_P = E_n$) with $(M_P)_{ij}$ δ_{i,p_j}. In order to define another type of the Jacobi coordinates $\hat{\xi}_P = ((\hat{\boldsymbol{\xi}}_P)_1, \cdots, (\hat{\boldsymbol{\xi}}_P)_{n-1})$ associated with the permutation P, we should define a new transformation matrix $PB \equiv B_P$ which can be defined from Eq. (6-3) by using a new set of mass numbers $(A_1' \cdots A_n') \equiv (A_{p_1} \cdots A_{p_n}) = P(A_1 \cdots A_n)$ instead of $(A_1 \cdots A_n)$. By using Eqs. (6-1), (6-8) and (6-5), we find

$$\begin{aligned} (\hat{\xi}_P,\, \hat{\boldsymbol{X}}_G) &\equiv P(\hat{\xi},\, \hat{\boldsymbol{X}}_G) = P\hat{X}(B,\, a) \\ &= \hat{X} M_P (B_P,\, {}^tM_P a) = \hat{X}(M_P B_P,\, a) \\ &= (\hat{\xi},\, \hat{\boldsymbol{X}}_G)\begin{pmatrix} {}^tB \\ {}^ta \end{pmatrix}(M_P B_P,\, a) \\ &= (\hat{\xi},\, \hat{\boldsymbol{X}}_G)\begin{pmatrix} {}^tBM_PB_P & {}^tBa \\ {}^taM_PB_P & {}^taa \end{pmatrix}. \end{aligned} \tag{6-9}$$

Here, by using Eq. (6-4) and the relation ${}^taM_PB_P = {}^t(Pa)\,PB = P({}^taB) = 0$, we obtain the following general TMS transformation;

$$\hat{\xi}_P = \hat{\xi}\,\Omega_P, \quad \Omega_P = {}^tBM_PB_P \,, \tag{6-10}$$

where Ω_P is a orthogonal $(n-1)\times(n-1)$ matrix since, from Eqs. (6-6) and (6-4),

$$\begin{aligned} {}^t\Omega_P \Omega_P &= {}^tB_P {}^tM_P B\,{}^tB M_P B_P \\ &= {}^tB_P B_P - {}^tB_P {}^tM_P a\,{}^ta M_P B_P = E_{n-1} \,. \end{aligned}$$

Let us consider a $3\times(n-1)$ complex matrix $R=(R_{\alpha i})$ composed of 3-dimensional complex generator coordinate vectors $\boldsymbol{R}_1,\cdots,\boldsymbol{R}_n$. By defining $R_P \equiv R\cdot\Omega_P$, we can write the transformation formula Eq. (2-3-23b) as

$$\varphi_{ab}^{(3,n-1)(\lambda\mu)\omega}(R_P)=\sum_c \varphi_{ac}^{(3,n-1)(\lambda\mu)\omega}(R)\,D_{cb}^{(n-1,n-1)(\lambda\mu)\omega}(\Omega_P),$$

$$D_{cb}^{(n-1,n-1)(\lambda\mu)\omega}(\Omega_P)\equiv\frac{1}{N_H(\lambda\mu\omega)}\varphi_{cb}^{(n-1,n-1)[\lambda+\mu+\omega,\mu+\omega,\omega,0,\cdots,0]}(\Omega_P). \tag{6-11}$$

Then we should recall that the complex generator coordinate in the Bargmann space plays the role of the boson creation operator;[6] namely, by using the expression in Eq. (2-2-1) with $n=3$,

$$U_{(N0)\boldsymbol{N}}(\boldsymbol{a}_i^\dagger)\phi_0(i)=X_{\boldsymbol{N}}(\hat{\boldsymbol{\xi}}_i,\nu),$$

$$\phi_0(i)\equiv\left(\frac{2\nu}{\pi}\right)^{3/4}\exp\{-\nu\hat{\boldsymbol{\xi}}_i^2\},$$

$$a_{\alpha i}^\dagger\equiv\sqrt{\nu}\,\hat{\xi}_{\alpha i}-\frac{1}{2\sqrt{\nu}}\frac{\partial}{\partial\hat{\xi}_{\alpha i}},\quad \alpha=x,y,z, \tag{6-12}$$

where $X_N(\hat{\boldsymbol{\xi}}_i;\nu)$ is a 3-dimensional h.o. wave function in the rectangular representation with the quanta $\boldsymbol{N}\equiv(N_x,N_y,N_z)$ $(N=|\boldsymbol{N}|=N_x+N_y+N_z)$.[39),40)] Thus by replacing $R_{\alpha i}$ and $(R_P)_{\alpha i}$ in Eq. (6-11) with $a_{\alpha i}^\dagger$ and $(a_P)_{\alpha i}^\dagger$, respectively, and operating on the n-body relative wave function of the ground state $\phi_0\equiv\phi_0(1)\cdots\phi_0(n-1)$, we obtain the following TMS transformation of the h.o. wave functions in the vector-coupling expression;

$$X_{ab}^{N(\lambda\mu)}(\hat{\xi}_P;\nu)=\sum_c X_{ac}^{N(\lambda\mu)}(\hat{\xi};\nu)\,D_{cb}^{(n-1,n-1)(\lambda\mu)\omega}(\Omega_P), \tag{6-13}$$

where, from Eq. (2-2-4),

$$X_{ac}^{N(\lambda\mu)}(\hat{\xi};\nu)\equiv[\cdots[[X_{N_1}(\hat{\boldsymbol{\xi}}_1;\nu)X_{N_2}(\hat{\boldsymbol{\xi}}_2;\nu)]_{(\sigma_2,\tau_2)}$$
$$X_{N_3}(\hat{\boldsymbol{\xi}}_3;\nu)]_{(\sigma_3,\tau_3)}\cdots X_{N_{n-1}}(\hat{\boldsymbol{\xi}}_{n-1};\nu)]_{(\lambda\mu)a},$$

$$\left|\begin{matrix}(\lambda\mu)\,\omega\\ c\end{matrix}\right\rangle\equiv|[f_{\mu\nu}]\rangle$$

$$=\left|\begin{matrix} f_{1,n-1} & & f_{2,n-1} & & f_{3,n-1} & & 0\cdots\cdots\cdots\cdots 0\\ & f_{1,n-2} & & f_{2,n-2} & & f_{3,n-2} & 0\cdots\cdots 0\\ & & \ddots & & \ddots & & \ddots \quad 0\\ & & & f_{13} & & f_{23} & & f_{33}\\ & & & & f_{12} & & f_{22}\\ & & & & & f_{11} \end{matrix}\right\rangle,$$

$$[f_{1,n-1},f_{2,n-1},f_{3,n-1}]=[\lambda+\mu+\omega,\mu+\omega,\omega],$$

$$N_i\equiv\sum_{j=1}^{\mathrm{Min}\{i,3\}}f_{ji}-\sum_{j=1}^{\mathrm{Min}\{i-1,3\}}f_{j,i-1}\ \ (i=2\sim n-1),\quad N_1=f_{11},$$

$$(\sigma_i, \tau_i) \equiv (f_{1i}-f_{2i}, f_{2i}-f_{3i}) \quad (i=2\sim n-2),$$
$$N=\lambda+2\mu+3\omega\,, \tag{6-14}$$

and $X_{ab}^{N(\lambda\mu)}(\hat{\xi}_P;\nu)$ is obtained by replacing $\hat{\xi}_i \to (\hat{\xi}_P)_i$ and $c\to b$. Furthermore, the TMS transformation in Eqs. (6-10) and (6-13) is easily generalized to that from Jacobi coordinates $\hat{\xi}_Q$ associated with the permutation

$$Q=\begin{pmatrix}1 & 2\cdots\cdots & n\\ q_1 & q_2\cdots\cdots & q_n\end{pmatrix} \tag{6-15}$$

to $\hat{\xi}_P$; namely,

$$\hat{\xi}_P=\hat{\xi}_Q\cdot\Omega(Q\to P),$$
$$\Omega(Q\to P)={}^tB_Q M_{PQ^{-1}}B_P\,,$$
$$(M_{PQ^{-1}})_{ij}=\delta_{q_i,p_j}\,,$$
$$X_{ab}^{N(\lambda\mu)}(\hat{\xi}_P)=\sum_c X_{ac}^{N(\lambda\mu)}(\hat{\xi}_Q)\,D_{cb}^{(n-1,n-1)(\lambda\mu)\omega}(\Omega(Q\to P))\,. \tag{6-16}$$

Appendix A: Normalization constant of the $n\times n$ DG polynomial with the highest weight

In Refs. 4), 41) and 46), it is stated that the normalization constant $N_H[f_1\cdots f_n]$ in Eq. (2-1-9) is closely connected with the dimension of the IR of the symmetric group $S_N(N=f_1+\cdots+f_n)$, which we write $|f|_{S_N}$. Indeed, we can prove

$$N_H[f_1\cdots f_n]=\left[\frac{|f|_{S_N}}{N!}\right]^{1/2},$$

by considering the special $N\times N$ DG polynomials[47),48)] for the application to the symmetric group. In Ref. 42), this normalization constant is calculated by the transformation of the integral variables in the Bargmann space. We think, however, that it is useful to give here another proof by the straightforward integration in the Bargmann space and by the construction of transfer operators composed of the U_n generators.

The formula which we prove here is a recursion formula of $N_H[f_1\cdots f_n]$;

$$\left[\frac{N_H[f_1\cdots f_n]}{N_H[f_1\cdots f_{n-1}]}\right]^2=\frac{1}{f_n!}\prod_{\mu=1}^{n-1}\left(\frac{f_\mu-f_n+n-\mu}{f_\mu+n-\mu}\right), \tag{A-1}$$

from which we obtain Eq. (2-1-9) without a phase factor.

Starting from the expression

$$N_H[f_1\cdots f_n]^{-2}=\langle(\Delta_1^1)^{f_1-f_2}(\Delta_{12}^{12})^{f_2-f_3}\cdots(\Delta_{12\cdots n}^{12\cdots n})^{f_n}|$$
$$\times(\Delta_1^1)^{f_1-f_2}(\Delta_{12}^{12})^{f_2-f_3}\cdots(\Delta_{12\cdots n}^{12\cdots n})^{f_n}\rangle\,, \tag{A-2}$$

we first expand $\Delta^{12\cdots n}_{12\cdots n}$ as

$$\Delta^{12\cdots n}_{12\cdots n}=\sum_{\alpha=1}^{n}\Delta^{\alpha}_{n}\nabla^{\alpha}_{n}=\sum_{\alpha=1}^{n}R_{\alpha n}\nabla^{\alpha}_{n}\,, \tag{A-3}$$

where ∇^{α}_{β} is a (α,β) cofactor of $\Delta^{12\cdots n}_{12\cdots n}$. The integration over the variables $R_{\alpha n}(\alpha=1\sim n)$ in Eq. (A-2) gives

$$N_H[f_1\cdots f_n]^{-2}=\sum_{r_1+\cdots+r_n=f_n}\frac{(f_n!)^2}{r_1!\,r_2!\cdots r_n!}I(r_1,r_2,\cdots,r_n), \tag{A-4}$$

$$I(r_1,r_2,\cdots,r_n)\equiv\langle r_1,r_2,\cdots,r_n|r_1,r_2,\cdots,r_n\rangle\,, \tag{A-5}$$

$$\begin{aligned}|r_1,r_2,\cdots,r_n\rangle&\equiv(\Delta^1_1)^{f_1-f_2}(\Delta^{12}_{12})^{f_2-f_3}\cdots(\Delta^{12\cdots n-1}_{12\cdots n-1})^{f_{n-1}-f_n}\\&\quad\times(\nabla^1_n)^{r_1}(\nabla^2_n)^{r_2}\cdots(\nabla^n_n)^{r_n}\\&=\prod_{\mu=1}^{n-1}(\Delta^{12\cdots\mu}_{12\cdots\mu})^{f_\mu-f_{\mu+1}}\cdot\prod_{\nu=1}^{n}(\nabla^{\nu}_{n})^{r_\nu}\,,\end{aligned} \tag{A$\cdot$6}$$

where we always consider the case $r_1+r_2+\cdots+r_n=f_n$ and $I(0,0,\cdots,f_n)=N_H[f_1\cdots f_{n-1}]^{-2}$.

In order to find a recursion formula of $I(r_1,r_2,\cdots,r_n)$, we use the following transfer operator with respect to r_m and r_{m-1};

$$G^{m-1}_{m}\equiv C^{m-1}_{m}(C^{m}_{m}-f_m)+\sum_{l=m+1}^{n}C^{m-1}_{l}C^{l}_{m}\quad\text{for}\quad m=2\sim n\,, \tag{A-7}$$

where we write $A_{\alpha\beta}$ as C^{β}_{α}. Noting the basic relation

$$C^{l}_{m}\nabla^{\nu}_{n}=-\delta_{\nu,m}\nabla^{l}_{n} \tag{A-8}$$

for $m\neq l$ $(1\leqq m,l\leqq n)$, we easily find

$$\begin{aligned}&G^{m-1}_{m}|r_1,r_2,\cdots,r_n\rangle\\&\quad=(-r_m)\,[C^{m-1}_{m}|r_1,\cdots,r_n\rangle+\sum_{l=m+1}^{n}C^{m-1}_{l}|r_m-1,r_l+1\rangle],\end{aligned} \tag{A-9}$$

where we have used the shorthand notation $|r_m-1,\ r_l+1\rangle\equiv|r_1,\cdots,r_{m-1}r_m-1,r_{m+1},\cdots,r_{l-1},r_l+1,r_{l+1},\cdots,r_n\rangle$. Each term in Eq. (A-9) is calculated to be

$$\begin{aligned}C^{m-1}_{m}|r_1,\cdots,r_n\rangle&=|r_{m-1}+1,r_m-1\rangle\\&\quad\times\left[(f_{m-1}-f_m)\frac{\Delta^{1\cdots m}_{1\cdots m-1}\cdot\nabla^{m}_{n}}{\Delta^{1\cdots m-1}_{1\cdots m-1}\cdot\nabla^{m-1}_{n}}-r_m\right],\end{aligned} \tag{A-10}$$

$$\begin{aligned}C^{m-1}_{l}|r_m-1,r_l+1\rangle&=|r_{m-1}+1,r_m-1\rangle\\&\quad\times\left[\sum_{\mu=m-1}^{l-1}(f_\mu-f_{\mu+1})\frac{\Delta^{1\cdots l\cdots\mu}_{1\cdots m-1\cdots\mu}\cdot\nabla^{l}_{n}}{\Delta^{1\cdots\mu}_{1\cdots\mu}\cdot\nabla^{m-1}_{n}}-(r_l+1)\right],\end{aligned} \tag{A-11}$$

for $l=m+1\sim n$. From Eqs. (A-9) $\sim$ (A-11) we have

$$G^{m-1}_{m}|r_1,\cdots,r_n\rangle=(-r_m)\,|r_{m-1}+1,r_m-1\rangle$$

$$\times\left[\sum_{l=m}^{n}\sum_{\mu=m-1}^{l-1}(f_\mu-f_{\mu+1})\frac{\Delta^{1\cdots l\cdots\mu}_{1\cdots m-1\cdots\mu}\cdot\nabla^l_n}{\Delta^{1\cdots\mu}_{1\cdots\mu}\cdot\nabla^{m-1}_n}-\sum_{l=m}^{n}r_l-n+m\right]$$

$$=(-r_m)\,|r_{m-1}+1,r_m-1\rangle$$

$$\times\left[\sum_{\mu=m-1}^{n-1}(f_\mu-f_{\mu+1})\frac{1}{\Delta^{1\cdots\mu}_{1\cdots\mu}\cdot\nabla^{m-1}_n}\sum_{l=\mu+1}^{n}\Delta^{1\cdots l\cdots\mu}_{1\cdots m-1\cdots\mu}\cdot\nabla^l_n-\sum_{l=m}^{n}r_l-n+m\right]. \tag{A·12}$$

The sum over l in Eq. (A-12) is taken by the following formula;

$$\sum_{l=\mu+1}^{n}\Delta^{1\cdots l\cdots\mu}_{1\cdots m-1\cdots\mu}\cdot\nabla^l_n=-\Delta^{1\cdots\mu}_{1\cdots\mu}\cdot\nabla^{m-1}_n \tag{A·13}$$

where $m-1\leqq\mu\leqq n-1$. This formula is easily proved by cofactor expansion of $\Delta^{1\cdots l\cdots\mu}_{1\cdots m\cdots\mu}=\sum_{\alpha=1}^{\mu}\Delta^l_\alpha\cdot\bar{\nabla}^m_\alpha$. ($\bar{\nabla}^\alpha_\beta$ is a (α,β) cofactor of $\Delta^{12\cdots\mu}_{12\cdots\mu}$.) Thus we find the effect of G^{m-1}_m on $|r_1,\cdots,r_n\rangle$ is

$$G^{m-1}_m|r_1,\cdots,r_{m-1},r_m,\cdots,r_n\rangle=r_m(f_{m-1}-\sum_{l=1}^{m-1}r_l+n-m)$$

$$\times|r_1,\cdots,r_{m-1}+1,r_m-1,\cdots,r_n\rangle\,. \tag{A-14}$$

In a similar way, the effect of $(G^{m-1}_m)^\dagger$ on $|r_1,\cdots,r_n\rangle$ is more easily found and is

$$(G^{m-1}_m)^\dagger|r_1,\cdots,r_{m-1},r_m,\cdots,r_n\rangle=r_{m-1}(f_m-\sum_{l=1}^{m-1}r_l+n-m+1)$$

$$\times|r_1,\cdots,r_{m-1}-1,r_m+1,\cdots,r_n\rangle\,. \tag{A-15}$$

From Eqs. (A-14) and (A-15) we obtain the following recursion formula;

$$I(r_1,\cdots,r_{m-1},r_m,\cdots,r_n)$$

$$=\frac{r_{m-1}(f_m-\sum_{l=1}^{m-1}r_l+n-m+1)}{(r_m+1)\,(f_{m-1}-\sum_{l=1}^{m-1}r_l+n-m+1)}$$

$$\times I(r_1,\cdots,r_{m-1}-1,r_m+1,\cdots,r_n)\,. \tag{A·16}$$

From this equation we find

$$I(r_1,\cdots,r_{n-1},r_n)=\frac{1}{f_n!}\prod_{\mu=1}^{n}r_\mu!$$

$$\times\prod_{\mu=1}^{n-1}\left\{\frac{(f_\mu-\sum_{l=1}^{\mu}r_l+n-\mu-1)!\,(f_{\mu+1}+n-\mu-1)!}{(f_{\mu+1}-\sum_{l=1}^{\mu}r_l+n-\mu-1)!\,(f_\mu+n-\mu-1)!}\right\}$$

$$\times I(0,\cdots,0,f_n), \tag{A-17}$$

where $r_1+\cdots+r_n=f_n$. Finally, by substituting Eq. (A-17) to Eq. (A-4) the summation over $r_1, r_2, \cdots, r_n$ is taken iteratively by the following formula;

$$A_k \equiv \sum_{r_1+\cdots+r_k=0}^{f_n} \frac{1}{(f_n-\sum_{l=1}^{k} r_l)!} \prod_{\mu=1}^{k} \frac{(f_\mu-\sum_{l=1}^{\mu} r_l+n-\mu-1)!}{(f_\mu-\sum_{l=1}^{\mu-1} r_l+n-\mu)!},$$

$$A_k = \frac{1}{(f_k-f_n+n-k)} \cdot A_{k-1} \quad (k=1\sim n-1),$$

$$A_0 \equiv \frac{1}{f_n!}, \tag{A-18}$$

which can be proved by

$$\sum_{r=0}^{b} \frac{(a+r)!}{r!} = \frac{1}{(a+1)} \cdot \frac{(a+b+1)!}{b!}. \tag{A-19}$$

Appendix B: Proof of the completeness of the $n\times m$ DG polynomials

In this appendix we prove the completeness of the $n\times m$ DG polynomials by directly counting the number of the independent basis states. Here we assume $n\leqq m$ without loss of generality. The number of independent monomials of $R_{\alpha i}$ $(\alpha=1\sim n,\ i=1\sim m)$ with degree N is $\binom{N+nm-1}{nm-1}=\binom{N+nm-1}{N}$. On the other hand the number of the independent $n\times m$ DG polynomials with a partition $[f_1\cdots f_n]$ is a product of the dimensionalities of the IR of U_n and U_m; $d_n[f_1\cdots f_n]\cdot d_m[f_1\cdots f_n\ 0\cdots 0]$. Thus for the proof of the completeness we should only verify[43),44)]

$$\sum_{\substack{f_1\geq\cdots\geq f_n\geq 0\\ f_1+\cdots+f_n=N}} d_n[f_1\cdots f_n]\cdot d_m[f_1\cdots f_n 0\cdots 0] = \binom{N+nm-1}{N}. \tag{B-1}$$

By usnig the expansion

$$\frac{1}{(1-x)^{nm}} = \sum_{N=0}^{\infty} \binom{N+nm-1}{N} x^N \tag{B-2}$$

and the Weyl's formula[45)] of the dimensionality of U_n or U_m, we can convert Eq. (B-1) into the following relation of the generating function;

$$\sum_{f_1\geq\cdots\geq f_n\geq 0}^{\infty} \frac{D(h_1,\cdots,h_n)}{D(n-1,\cdots,0)} \cdot \frac{D(h'_1,\cdots,h'_m)}{D(m-1,\cdots,0)} \cdot x^{f_1+\cdots f_n}$$

$$= \frac{1}{(1-x)^{nm}}, \tag{B-3}$$

where $h_i \equiv f_i+n-i$, $h'_i \equiv f_i+m-i$ for $i=1\sim n$ and $h'_i \equiv m-i$ for $i=n+1\sim m$, and

$$D(x_1, \cdots, x_n) \equiv \prod_{i<j}^{n} (x_i - x_j)$$
$$= \begin{vmatrix} x_1^{n-1} & x_1^{n-2} \cdots x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} \cdots x_2 & 1 \\ \vdots & \vdots \quad \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} \cdots x_n & 1 \end{vmatrix} . \tag{B-4}$$

In the following we sometimes use the shorthand notation $D(x_1, \cdots, x_n) \equiv |x^{n-1}, x^{n-2}, \cdots, x, 1|$.

The proof of Eq. (B-3) is as follows. First we define

$$F^{(nm)}(x) \equiv \sum_{f_1 \geq \cdots \geq f_n \geq 0}^{\infty} D(h_1, \cdots, h_n) D(h_1', \cdots, h_m') x^{f_1 + \cdots + f_n} . \tag{B-5}$$

By converting the summations over $f_1, \cdots, f_n$ to those over $h_1, \cdots, h_n$ and using the antisymmetry of D and $D(x_1 + a, x_2 + a, \cdots, x_n + a) = D(x_1, x_2, \cdots, x_n)$ for arbitrary a, we can write

$$F^{(nm)}(x) = x^{-n(n-1)/2} \sum_{h_1 > \cdots > h_n \geq 0}^{\infty} D(h_1, \cdots, h_n)$$
$$\times D(h_1 + m - n, \cdots, h_n + m - n, m - n - 1, \cdots, 0) x^{h_1 + \cdots + h_n}$$
$$= \frac{1}{n!\, x^{n(n-1)/2}} \sum_{h_1=0}^{\infty} \cdots \sum_{h_n=0}^{\infty} D(h_1, \cdots, h_n)$$
$$\times D(h_1, \cdots, h_n, -1, \cdots, -(m-n)) x^{h_1 + \cdots + h_n} . \tag{B-6}$$

Therefore, if we define a new function

$$G^{(nm)}(x_1, \cdots, x_n) \equiv \sum_{h_1=0}^{\infty} \cdots \sum_{h_n=0}^{\infty} D(h_1, \cdots, h_n)$$
$$\times D(h_1, \cdots, h_n, -1, \cdots, -(m-n)) x_1^{h_1} \cdots x_n^{h_n} , \tag{B-7}$$

we can find

$$F^{(nm)}(x) = \frac{1}{n!\, x^{n(n-1)/2}} G^{(nm)}(x, \cdots, x) . \tag{B-8}$$

In order to perform the summations in Eq. (B-7) we introduce number operators $C_i \equiv x_i \cdot (\partial / \partial x_i)$ $(i = 1 \sim n)$ which satisfy $C_i x_i^{h_i} = h_i x_i^{h_i}$, and substitute C_i for h_i in the determinants. Thus we obtain

$$G^{(nm)}(x_1, \cdots, x_n) = D(C_1, \cdots, C_n) D(C_1, \cdots, C_n, -1, \cdots, -(m-n))$$
$$\times \frac{1}{(1-x_1) \cdots (1-x_n)} . \tag{B-9}$$

Here we can show

$$D(C_1, \cdots, C_n, -1, \cdots, -(m-n))\frac{1}{(1-x_1)\cdots(1-x_n)}$$

$$=D(m-1, \cdots, 0)\ \frac{D(x_1, \cdots, x_n)}{\{(1-x_1)\cdots(1-x_n)\}^m}. \tag{B-10}$$

In fact, if we define $C_i \equiv -(i-n)$ and $x_i \equiv 0$ for $i=n+1 \sim m$, we can write the left hand side of Eq. (B-10) as

$$\begin{vmatrix} C_1^{m-1} & C_1^{m-2} \cdots C_1 & 1 \\ C_2^{m-1} & C_2^{m-2} \cdots C_2 & 1 \\ \vdots & \vdots \quad \vdots & \vdots \\ C_m^{m-1} & C_m^{m-2} \cdots C_m & 1 \end{vmatrix} \cdot \frac{1}{(1-x_1)\cdots(1-x_m)}$$

$$= \left| C^{m-1}\frac{1}{1-x}, C^{m-2}\frac{1}{1-x}, \cdots, C\frac{1}{1-x}, \frac{1}{1-x} \right|$$

$$= \left| (C+m-1)(C+m-2)\cdots(C+1)\frac{1}{1-x}, (C+m-2)\cdots(C+1)\frac{1}{1-x}, \right.$$

$$\left. \cdots, (C+1)\frac{1}{1-x}, \frac{1}{1-x} \right|$$

$$= \begin{vmatrix} \frac{(m-1)!}{(1-x_1)^m} \cdots \frac{(m-n)!}{(1-x_1)^{m-n+1}} & \frac{(m-n-1)!}{(1-x_1)^{m-n}} \cdots \frac{k!}{(1-x_1)^{k+1}} \cdots \frac{0!}{1-x_1} \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \frac{(m-1)!}{(1-x_n)^m} \cdots \frac{(m-n)!}{(1-x_n)^{m-n+1}} & \frac{(m-n-1)!}{(1-x_n)^{m-n}} \cdots \frac{k!}{(1-x_n)^{k+1}} \cdots \frac{0!}{1-x_n} \\ 0 \cdots\cdots 0 & 0 \cdots\cdots 0 \cdots (-)^0 \cdot 0! \\ \vdots \quad 0 \quad \vdots & \vdots \quad 0 \quad (-)^k \cdot k! \quad * \quad \vdots \\ 0 \cdots\cdots 0 & (-)^{m-n-1} \cdot (m-n-1)! \cdots\cdots * \end{vmatrix} \begin{matrix} \uparrow \\ n \\ \downarrow \\ \uparrow \\ m-n \\ \downarrow \end{matrix}$$

$$\longleftarrow n \longrightarrow \qquad \longleftarrow m-n \longrightarrow$$

$$= (m-1)!\cdots 0!\frac{1}{\{(1-x_1)\cdots(1-x_n)\}^m}|1, 1-x, (1-x)^2, \cdots, (1-x)^{n-1}|$$

$$=D(m-1, \cdots, 0)\ \frac{D(x_1, \cdots, x_n)}{\{(1-x_1)\cdots(1-x_n)\}^m}. \tag{B-11}$$

Therefore,

$$G^{(nm)}(x, \cdots, x) = D(m-1, \cdots, 0)$$

$$\times \left[D(C_1, \cdots, C_n)\frac{D(x_1, \cdots, x_n)}{\{(1-x_1)\cdots(1-x_n)\}^m} \right]_{x_1=\cdots=x_n=x}. \tag{B-12}$$

Here we can prove that the non-zero term after setting $x_1 = \cdots = x_n = x$ comes only when all the differential operators C_i operate to the function $D(x_1, \cdots, x_n)$ and that

$$[D(C_1, \cdots, C_n) D(x_1, \cdots, x_n)]_{x_1=\cdots=x_n=x} = D(n-1, \cdots, 0)\, n!\, x^{n(n-1)/2}\,. \tag{B-13}$$

Thus we obtain

$$G^{(nm)}(x, \cdots, x) = n!\, x^{n(n-1)/2} D(n-1, \cdots, 0)\, D(m-1, \cdots, 0) \times \frac{1}{(1-x)^{nm}}\,, \tag{B-14}$$

which proves Eq. (B-3) from Eqs. (B-8) and (B-5).

We prove the rest of the proof in the following form. We expand the determinent of operators C_i as

$$D(C_1, \cdots, C_n) = \sum_{p \in S_n} \delta_p C_1^{p(n-1)} C_2^{p(n-2)} \cdots C_n^{p(0)}\,. \tag{B-15}$$

For any arbitrary $p \in S_n$ we introduce integers $r_1, \cdots, r_n$ which safty the condition

$$p(n-1) \geqq r_1 \geqq 0,\ p(n-2) \geqq r_2 \geqq 0, \cdots, p(0) \geqq r_n \geqq 0\,. \tag{B-16}$$

There holds

$$\delta_p C_1^{p(n-1)-r_1} C_2^{p(n-2)-r_2} \cdots C_n^{p(0)-r_n} D(x_1 \cdots x_n)\,|_{x_1=\cdots=x_n=x} = 0 \quad \text{unless} \quad r_1 = \cdots = r_n = 0\,. \tag{B-17}$$

In fact, by setting $x_1 = \cdots = x_n = x$ in

$$\begin{aligned} &\delta_p C_1^{p(n-1)-r_1} C_2^{p(n-2)-r_2} \cdots C_n^{p(0)-r_n} D(x_1, \cdots, x_n) \\ &= \delta_p \begin{vmatrix} C_1^{p(n-1)-r_1} x_1^{n-1}, & C_1^{p(n-1)-r_1} x_1^{n-2}, & \cdots\cdots, & C_1^{p(n-1)-r_1} \cdot 1 \\ \vdots & \cdots & & \vdots \\ C_n^{p(0)-r_n} x_n^{n-1}, & C_n^{p(0)-r_n} x_n^{n-2}, & \cdots\cdots, & C_n^{p(0)-r_n} \cdot 1 \end{vmatrix} \\ &= \delta_p \begin{vmatrix} (n-1)^{p(n-1)-r_1} x_1^{n-1}, & (n-2)^{p(n-1)-r_1} x_1^{n-2}, & \cdots\cdots, & 0^{p(n-1)-r_1} \cdot 1 \\ \vdots & \vdots & & \vdots \\ (n-1)^{p(0)-r_n} x_n^{n-1}, & (n-2)^{p(0)-r_n} x_n^{n-2}, & \cdots\cdots, & 0^{p(0)-r_n} \cdot 1 \end{vmatrix} \end{aligned} \tag{B-18}$$

and permuting the rows, we find that the left-hand side of Eq. (B-18) is

$$x^{n(n-1)/2} \begin{vmatrix} (n-1)^{(n-1)-r_1'}, & (n-2)^{(n-1)-r_1'}, & \cdots\cdots, & 0^{(n-1)-r_1'} \\ \vdots & \vdots & & \vdots \\ (n-1)^{0-r_n'}, & (n-2)^{0-r_n'}, & \cdots\cdots, & 0^{0-r_n'} \end{vmatrix}, \tag{B-19}$$

where $p^{-1}(r_1, \cdots, r_n) \equiv (r_1', \cdots, r_n')$. Here $(n-1) - r_1'$, $(n-2) - r_2', \cdots, 0 - r_n'$ are one of the integers from zero to $n-1$. Therefore we find that Eq. (B-19) is zero unless $r_n' = r_{n-1}' = \cdots = r_1' = 0$, which leads to $r_1 = r_2 = \cdots = r_n = 0$. Especially, in the case of $r_1 = \cdots = r_n = 0$ we can take the sum over all the permutations $p \in S_n$ in (B-19), which leads us to Eq. (B-13).

Appendix C: Proof of Eq. (2-5-4)

We start from the relation of the matrix elements of the U_n generators $C_m^{m'}$ with respect to Gel'fand bases and their conjugate ones;

$$\langle f'_{\mu\nu}|C_m^n|f_{\mu\nu}\rangle = (-)^{n-m}\langle \bar{f}'_{\mu\nu}|-C_n^m|\bar{f}_{\mu\nu}\rangle\,, \tag{C-1}$$

which can be proved by direct examination of the generator matrix elements.[14),5),8),4),10)] We only prove Eq. (2-5-4) with respect to the linear term of ε by transforming $u_{\alpha\beta}$ to $\varepsilon u_{\alpha\beta}$, where ε is a real infinitesimal number, because we can extend ε to 1. Since, from Eq. (2-5-3),

$$D_{\bar{c}\bar{a}}^{(\bar{\lambda})}(U)^* = D_{\bar{c}\bar{a}}^{(\bar{\lambda})}(U^*) = \langle \varphi_{\bar{c}\bar{b}}^{N(\bar{\lambda})}(R)\,|\exp\{i\sum_{\alpha\beta}(-u_{\beta\alpha})A_{\alpha\beta}\}\,|\varphi_{\bar{a}\bar{b}}^{N(\bar{\lambda})}(R)\rangle \tag{C-2}$$

for an appropriate N and an arbitrary b, we find, from Eq. (C-1),

$$\begin{aligned}
&\text{linear term of } \varepsilon \text{ of } D_{\bar{c}\bar{a}}^{(\bar{\lambda})}(U(u_{\alpha\beta}\to\varepsilon u_{\alpha\beta}))^* \\
&= i\varepsilon \sum_{\alpha\beta} u_{\alpha\beta}\langle \varphi_{\bar{c}\bar{b}}^{N(\bar{\lambda})}(R)\,|-A_{\beta\alpha}|\varphi_{\bar{a}\bar{b}}^{N(\bar{\lambda})}(R)\rangle \\
&= i\varepsilon \sum_{\alpha\beta}(-)^{\alpha-\beta}u_{\alpha\beta}\langle \varphi_{cb}^{N(\lambda)}(R)\,|A_{\alpha\beta}|\varphi_{ab}^{N(\lambda)}(R)\rangle \\
&= \text{linear term of } \langle \varphi_{cb}^{N(\lambda)}(R)\,|\exp\{i\varepsilon\sum_{\alpha\beta}(-)^{\alpha-\beta}u_{\alpha\beta}A_{\alpha\beta}\}\,|\varphi_{ab}^{N(\lambda)}(R)\rangle\,.
\end{aligned} \tag{C-3}$$

By using the relation

$$\exp\{i\varepsilon\sum_{\alpha\beta}(-)^{\alpha-\beta}u_{\alpha\beta}A_{\alpha\beta}\} = (-)^{\sum_\gamma \gamma\cdot A_{\gamma\gamma}}\exp\{i\varepsilon\sum_{\alpha\beta}u_{\alpha\beta}A_{\alpha\beta}\}\cdot(-)^{-\sum_\gamma \gamma\cdot A_{\gamma\gamma}}, \tag{C-4}$$

we can calculate the phase parts; for instance,

$$\begin{aligned}
\sum_\gamma \gamma A_{\gamma\gamma}\varphi_{cb}^{N(\lambda)}(R) &= \{\sum_\gamma \gamma(\sum_{\mu=1}^{\gamma} f_{\mu\gamma} - \sum_{\mu=1}^{\gamma-1} f_{\mu,\gamma-1})\}\varphi_{cb}^{N(\lambda)}(R) \\
&= (nN - \sum_{\nu=1}^{n-1}\sum_{\mu=1}^{\nu} f_{\mu\nu})\varphi_{cb}^{N(\lambda)}(R) \equiv (nN - \psi^{(\lambda)}(c))\varphi_{cb}^{N(\lambda)}(R),
\end{aligned} \tag{C-5}$$

when $N(\lambda)c \equiv [f_{\mu\nu}]$. Thus Eq. (C-3) becomes

$$\begin{aligned}
&\text{linear term of } \varepsilon \text{ of } (-)^{-\psi^{(\lambda)}(c)+\psi^{(\lambda)}(a)} \\
&\qquad \times\langle \varphi_{cb}^{N(\lambda)}(R)\,|\exp\{i\varepsilon\sum_{\alpha\beta}u_{\alpha\beta}A_{\alpha\beta}\}\,|\varphi_{ab}^{N(\lambda)}(R)\rangle \\
&= \text{linear term of } \varepsilon \text{ of } (-)^{\psi^{(\lambda)}(a)-\psi^{(\lambda)}(c)}D_{ca}^{(\lambda)}(U(u_{\alpha\beta}\to\varepsilon u_{\alpha\beta})),
\end{aligned} \tag{C-6}$$

which is the desired relation.

Appendix D: A derivation method of Eq. (2-6-22)

In this appendix we show a method to derive Eq. (2-6-22). When p

or p' is zero, the lowering operation is very easy and the result is

$$O_{zx}^{p}R_{x1}^{\lambda}\delta_{z3}^{\mu}=\frac{\lambda!(\lambda+\mu+1)!}{(\lambda-p)!(\lambda+\mu+1-p)!}R_{x1}^{\lambda-p}R_{z1}^{p}\delta_{z3}^{\mu}\,, \tag{D-1a}$$

$$Q_{31}^{p'}R_{x1}^{\lambda}\delta_{z3}^{\mu}=\frac{\lambda!(\lambda+\mu+1)!}{(\lambda-p')!(\lambda+\mu+1-p')!}R_{x1}^{\lambda-p'}R_{x3}^{p'}\delta_{z3}^{\mu}\,. \tag{D-1b}$$

When both p and p' are non-zero, we may assume the following form from the consideration of polynomial degrees concerning $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$ and $\boldsymbol{R}_x^r$, $\boldsymbol{R}_y^r$, $\boldsymbol{R}_z^r$;

$$O_{zx}^{p}Q_{31}^{p'}R_{x1}^{\lambda}\delta_{z3}^{\mu}=\sum_{a=\mathrm{Max}\{0,\,p+p'-\lambda\}}^{\mathrm{Min}\{p,\,p'\}}\ \sum_{b=\mathrm{Max}\{0,\,a-\mu\}}^{a} I_{p,p'}^{(\lambda\mu)}(a,b)$$
$$\times R_{x1}^{\lambda-p-p'+a}R_{x3}^{p'-a}R_{z1}^{p-a}R_{z3}^{b}\delta_{z3}^{\mu-a+b}|R|^{a-b}\,. \tag{D-2}$$

Here $I_{p,p'}^{(\lambda,\mu)}(a,b)$ are the coefficients to be determined and should satisfy $I_{p,p'}^{(\lambda,\mu)}(a,b)=I_{p',p}^{(\lambda,\mu)}(a,b)$. Especially, from Eqs. (D-1a) and (D-1b),

$$I_{0,p'}^{(\lambda\mu)}(0,0)=\frac{\lambda!(\lambda+\mu+1)!}{(\lambda-p')!(\lambda+\mu-p'+1)!}\,, \tag{D-3a}$$

$$I_{p,0}^{(\lambda\mu)}(0,0)=\frac{\lambda!(\lambda+\mu+1)!}{(\lambda-p)!(\lambda+\mu-p+1)!}\,. \tag{D-3b}$$

By operating O_{zx} to Eq. (D-2) we obtain the following recursion formula for $I_{p,p'}^{(\lambda,\mu)}(a,b)$;

$$\begin{aligned}I_{p+1,p'}^{(\lambda\mu)}(a,b)&=(\lambda+\mu-p-a+b+1)(\lambda-p-p'+a)I_{p,p'}^{(\lambda\mu)}(a,b)\\&+(\lambda+\mu-p-a+b+1)(p'-a+1)I_{p,p'}^{(\lambda\mu)}(a-1,b-1)\\&-(\mu-a+b+1)(p'-a+1)I_{p,p'}^{(\lambda\mu)}(a-1,b)\,.\end{aligned} \tag{D-4}$$

In the case of $a=b=0$ we obtain, from Eq. (D-3a),

$$\begin{aligned}I_{p,p'}^{(\lambda\mu)}(0,0)&=(\lambda+\mu-p+2)(\lambda-p-p'+1)I_{p-1,p'}^{(\lambda\mu)}(0,0)\\&=\frac{(\lambda+\mu+1)!(\lambda-p')!}{(\lambda+\mu+1-p)!(\lambda-p-p')!}I_{0,p'}^{(\lambda\mu)}(0,0)\\&=\frac{(\lambda+\mu-a+b+1)!}{(\lambda+\mu-p+1)!(\lambda+\mu-p'+1)!(\lambda-p-p')!}\,.\end{aligned} \tag{D-5}$$

On the other hand, the dependence of $I_{p,p'}^{(\lambda,\mu)}(a,b)$ on a and b is given by

$$\begin{aligned}&I_{p,p'}^{(\lambda\mu)}(a,b)\\&=(-)^{a+b}\frac{(\lambda+\mu-a+b+1)!}{(\lambda-p-p'+a)!(p'-a)!(p-a)!b!(\mu-a+b)!(a-b)!}\\&\quad\times F_{p,p'}^{(\lambda\mu)}\,.\end{aligned} \tag{D-6}$$

In fact, by substituting Eq. (D-6) into Eq. (D-4) we obtain

$$F^{(\lambda\mu)}_{p+1,p'}=(p+1)(\lambda+\mu-p+1)F^{(\lambda\mu)}_{p,p'}\,. \tag{D-7}$$

Setting $a=b=0$ in Eq. (D-6) and comparing it with Eq. (D-5), we obtain

$$F^{(\lambda\mu)}_{p,p'}=\frac{\lambda!\,\mu!\,(\lambda+\mu+1)!\,p!\,p'!}{(\lambda+\mu-p+1)!\,(\lambda+\mu-p'+1)!}\,, \tag{D-8}$$

which of course satisfies Eq. (D-7). The Eqs. (D-2), (D-6) and (D-8) prove Eq. (2-6-22).

Appendix E: Proof of the Regge symmetry and the reduction formulas of the multiplicity-free U_n reduced C-G coefficients

In this Appendix, we show the outline of the proof of Eqs. (3-1-6), (5-2-2) and (5-2-3). The proof of Eq. (3-1-6) is almost the same as that in Ref. 16); namely, we should only compare the expansion coefficients of the $n\times n$ DG polynomial by the $(n-1)\times(n-1)$ DG polynomials in two different processes. We first expand the $n\times n$ DG polynomial by the $n\times(n-1)$ DG ones based on Eq. (2-2-4a) and next expand the $n\times(n-1)$ DG polynomials by the $(n-1)\times(n-1)$ DG ones based on Eq. (2-2-15);

$$\begin{aligned}
&\varphi^{(nn)[f_1\cdots f_n]}_{\substack{q_1\cdots q_{n-1},\ q_1'\cdots q_{n-1}'\\ c\qquad\quad c'}}(R)\\
&\equiv\left|\begin{matrix} f_1\cdots\cdots\cdots f_n & f_1\cdots\cdots\cdots f_n\\ q_1\cdots\cdots q_{n-1} & q_1'\cdots\cdots q_{n-1}'\\ c & c'\end{matrix}\right\rangle\\
&=\sum_{[r_1\cdots r_{n-1}]a}\ \sum_{|\boldsymbol{N}_n|=N_n}\langle[q_1'\cdots q_{n-1}']r_1\cdots r_{n-1},\ a[N_n]\boldsymbol{N}_n|\\
&\quad\times[f_1\cdots f_n]q_1\cdots q_{n-1},\ c\rangle_n\,\varphi^{(n,n-1)[q_1'\cdots q'_{n-1}]}_{\substack{r_1\cdots r_{n-1},\ \\ a\qquad c'}}(R)\,U^{(n)}_{[N_n]\boldsymbol{N}_n}(\boldsymbol{R}_n)\\
&=\sum_{[r_1\cdots r_{n-1}]a,b}\ \sum_{|\boldsymbol{N}_n|=N_n}\ \sum_{|\boldsymbol{P}_{n-1}|=P_{n-1}}\langle[q_1'\cdots q_{n-1}']r_1\cdots r_{n-1},\ a[N_n]\boldsymbol{N}_n|\\
&\quad\times[f_1\cdots f_n]q_1\cdots q_{n-1},\ c\rangle_n\langle[r_1\cdots r_{n-1}]b[P_{n-1}]\boldsymbol{P}_{n-1}|[q_1'\cdots q_{n-1}']c'\rangle_{n-1}\\
&\quad\varphi^{(n-1,n-1)[r_1\cdots r_{n-1}]}_{a,b}(R)\,U^{(n-1)}_{[P_{n-1}]\boldsymbol{P}_{n-1}}(\boldsymbol{R}^r_n)\,U^{(n)}_{[N_n]\boldsymbol{N}_n}(\boldsymbol{R}_n),
\end{aligned} \tag{E-1}$$

where

$$N_n=\sum_{i=1}^{n}f_i-\sum_{i=1}^{n-1}q_i',\quad P_{n-1}=\sum_{i=1}^{n-1}(q_i'-r_i), \tag{E-2}$$

$$\boldsymbol{N}_n\equiv(N_{1n},\cdots,N_{nn}), \tag{E-3a}$$

$$U^{(n)}_{[N_n]\boldsymbol{N}_n}(\boldsymbol{R}_n)=\frac{R_{1n}^{N_{1n}}\cdots R_{nn}^{N_{nn}}}{\sqrt{N_{1n}!\cdots N_{nn}!}}\,, \tag{E-3b}$$

$$\boldsymbol{P}_{n-1}\equiv(P_{n1},\cdots,P_{n,n-1}), \tag{E-4a}$$

$$U^{(n-1)}_{[P_{n-1}]\boldsymbol{P}_{n-1}}(\boldsymbol{R}^r_n)=\frac{R_{n1}^{P_{n1}}\cdots R_{n,n-1}^{P_{n,n-1}}}{\sqrt{P_{n1}!\cdots P_{n,n-1}!}}\,. \tag{E-4b}$$

By separating the U_n C-G coefficients in Eq. (E-1) into the U_n reduced C-G coefficients and the U_{n-1} C-G coefficients, we can rewrite Eq. (E-1) as

$$\sum_{[r_1\cdots r_{n-1}]a,b}\ \sum_{|\boldsymbol{N}_{n-1}|=N_{n-1}}\ \sum_{|\boldsymbol{P}_{n-1}|=P_{n-1}}$$
$$\times\langle[q_1'\cdots q_{n-1}']r_1\cdots r_{n-1}[N_n]N_{n-1}\|[f_1\cdots f_n]q_1\cdots q_{n-1}\rangle_n$$
$$\times\langle[r_1\cdots r_{n-1}]a[N_{n-1}]\boldsymbol{N}_{n-1}|[q_1\cdots q_{n-1}]c\rangle_{n-1}$$
$$\times\langle[r_1\cdots r_{n-1}]b[P_{n-1}]\boldsymbol{P}_{n-1}|[q_1'\cdots q_{n-1}']c'\rangle_{n-1}$$
$$\times\varphi_{a,b}^{(n-1,n-1)[r_1\cdots r_{n-1}]}(R)\,U^{(n-1)}_{[P_{n-1}]\boldsymbol{P}_{n-1}}(\boldsymbol{R}_n^r)\,U^{(n-1)}_{[N_{n-1}]\boldsymbol{N}_{n-1}}(\boldsymbol{R}_n)\frac{R_{nn}^{N_{nn}}}{\sqrt{N_{nn}!}}, \qquad \text{(E-5)}$$

where $\boldsymbol{N}_n=(\boldsymbol{N}_{n-1}, N_{nn})$ and

$$N_{n-1}=|\boldsymbol{N}_{n-1}|=\sum_{i=1}^{n-1}(q_i-r_i),$$
$$N_{nn}=N_n-N_{n-1}=\sum_{i=1}^{n}f_i-\sum_{i=1}^{n-1}q_i-\sum_{i=1}^{n-1}q_i'+\sum_{i=1}^{n-1}r_i. \qquad \text{(E-7)}$$

On the other hand, we can perform the similar expansion of the $n\times n$ DG polynomial, first by the $(n-1)\times n$ DG ones and secondly by the $(n-1)\times(n-1)$ DG ones;

$$\varphi^{(nn)[f_1\cdots f_n]}_{\substack{q_1\cdots q_{n-1},\ q_1'\cdots q_{n-1}'\\ c\qquad\ \ c'}}(R)$$
$$=\sum_{[r_1\cdots r_{n-1}]b,}\ \sum_{|\boldsymbol{P}_n|=P_n}\langle[q_1\cdots q_{n-1}]r_1\cdots r_{n-1},\ b[P_n]\boldsymbol{P}_n|$$
$$\times[f_1\cdots f_n]q_1'\cdots q_{n-1}',\ c'\rangle_n\,\varphi^{(n-1,n)[q_1\cdots q_{n-1}]}_{c,\ \substack{r_1\cdots r_{n-1}\\ b}}(R)\,U^{(n)}_{[P_n]\boldsymbol{P}_n}(\boldsymbol{R}_n^r)$$
$$=\sum_{[r_1\cdots r_{n-1}]a,b}\ \sum_{|\boldsymbol{P}_{n-1}|=P_{n-1}}\ \sum_{|\boldsymbol{N}_{n-1}|=N_{n-1}}$$
$$\times\langle[q_1\cdots q_{n-1}]r_1\cdots r_{n-1}[P_n]P_{n-1}\|[f_1\cdots f_n]q_1'\cdots q_{n-1}'\rangle_n$$
$$\times\langle[r_1\cdots r_{n-1}]b[P_{n-1}]\boldsymbol{P}_{n-1}|[q_1'\cdots q_{n-1}']c'\rangle_{n-1}$$
$$\times\langle[r_1\cdots r_{n-1}]a[N_{n-1}]\boldsymbol{N}_{n-1}|[q_1\cdots q_{n-1}]c\rangle_{n-1}$$
$$\times\varphi_{ab}^{(n-1,n-1)[r_1\cdots r_{n-1}]}(R)\,U^{(n-1)}_{[N_{n-1}]\boldsymbol{N}_{n-1}}(\boldsymbol{R}_n)\,U^{(n-1)}_{[P_{n-1}]\boldsymbol{P}_{n-1}}(\boldsymbol{R}_n^r)\frac{R_{nn}^{P_{nn}}}{\sqrt{N_{nn}!}}, \qquad \text{(E-8)}$$

where

$$P_n=\sum_{i=1}^{n}f_i-\sum_{i=1}^{n-1}q_i, \qquad \text{(E-9)}$$
$$\boldsymbol{P}_n=(\boldsymbol{P}_{n-1}, P_{nn}), \qquad \text{(E-10)}$$

and $P_{n-1}=|\boldsymbol{P}_{n-1}|$, $\boldsymbol{P}_{n-1}$ are given by Eqs. (E-2), (E-4a) and $P_{nn}=P_n-P_{n-1}=N_{nn}$ in Eq. (E-7). Comparing Eqs. (E-5) and (E-8) we obtain Eq. (3-1-6).

The basic equation in order to prove Eq. (5-2-2) is

$$\varphi^{(nn)[f_1\cdots f_{n-1}0]}_{a,\ q_1'\cdots q'_{n-2}\,0\ L}(R)=\varphi^{(n,n-1)[f_1\cdots f_{n-1}]}_{a,\ q_1'\cdots q'_{n-2}\ L}(\boldsymbol{R}_2,\cdots,\boldsymbol{R}_n), \qquad \text{(E-11)}$$

where L denotes the lowest weight and the argument matrix of the $n\times(n-1)$ DG polynomial in the right hand side is $(\boldsymbol{R}_2,\cdots,\boldsymbol{R}_n)$ instead of $(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_{n-1})$. Equation (E-11) is easily proved by expressing those DG polynomials in the vector-coupling form of Eq. (2-2-4a). By taking $a=\begin{bmatrix} q_1\cdots q_{n-1}\\ H\end{bmatrix}$ in Eq. (E-11), the other type of the vector-coupling expression Eq. (2-2-15) gives

$$\sum_{c,|\boldsymbol{P}_n|=P_n}\langle[q_1\cdots q_{n-1}\,0]c[P_n]\boldsymbol{P}_n|[[f_1\cdots f_{n-1}\,0]q_1'\cdots q_{n-2}'\,0,\,L\rangle_n$$
$$\times\varphi^{(n-1,n)[q_1\cdots q_{n-1}]}_{H,c}(R)\,U^{(n)}_{[P_n]\boldsymbol{P}_n}(\boldsymbol{R}_n^r)$$
$$=\sum_{c',|\boldsymbol{P}_n'|=P_n}\langle[q_1\cdots q_{n-1}]c'[P_n]\boldsymbol{P}_n'|[f_1\cdots f_{n-1}]q_1'\cdots q_{n-2}',\,L\rangle_{n-1}$$
$$\times\varphi^{(n-1,n-1)[q_1\cdots q_{n-1}]}_{H,c'}(\underset{\sim}{R})\,U^{(n-1)}_{[P_n]\boldsymbol{P}_n'}(\underset{\sim}{\boldsymbol{R}}_n^r), \qquad \text{(E-12)}$$

where $P_n=\sum_{i=1}^{n-1}(f_i-q_i)$ and

$$R=\begin{pmatrix}R_{11}\cdots\cdots R_{1n}\\ \vdots \qquad \vdots\\ R_{n-1,1}\cdots R_{n-1,n}\end{pmatrix},\quad \boldsymbol{R}_n^r=(R_{n1}\cdots R_{nn}), \qquad \text{(E-13a)}$$

$$\underset{\sim}{R}=\begin{pmatrix}R_{12}\cdots\cdots R_{1n}\\ \vdots \qquad \vdots\\ R_{n-1,2}\cdots R_{n-1,n}\end{pmatrix},\quad \underset{\sim}{\boldsymbol{R}}_n^r=(R_{n2}\cdots R_{nn}). \qquad \text{(E-13b)}$$

It should be noted that the left hand side of Eq. (E-12) does not contain the variables $R_{11},\cdots,R_{n1}$ and is a function only of the variables Eq. (E-13b). Now, by setting $a=H$ in Eq. (E-11) and changing the notation of quantum numbers we have, as in Eq. (5-1-25b),

$$\varphi^{(n-1,n)[q_1\cdots q_{n-1}]}_{H,\ r_1'\cdots r'_{n-2}\,0\ L}(R)=\varphi^{(n-1,n-1)[q_1\cdots q_{n-1}]}_{H,\ r_1'\cdots r'_{n-2}\ L}(\underset{\sim}{R}). \qquad \text{(E-14)}$$

Integrating both sides of Eq. (E-12) by eq. (E-14) over the variables Eq. (E-13b) (or Eq. (E-13a)) we obtain

$$\langle[q_1\cdots q_{n-1}\,0]r_1'\cdots r_{n-2}'\,0,\,L[P_n]\boldsymbol{P}_n|[f_1\cdots f_{n-1}\,0]q_1'\cdots q_{n-2}'\,0,\,L\rangle_n$$
$$=\langle[q_1\cdots q_{n-1}]r_1'\cdots r_{n-2}',\,L[P_n]\boldsymbol{P}_n'|[f_1\cdots f_{n-1}]q_1'\cdots q_{n-2}',\,L\rangle_{n-1}, \qquad \text{(E-15)}$$

where $P_n=\sum_{i=1}^{n-1}(f_i-q_i)$, $\boldsymbol{P}_n$, $\boldsymbol{P}_n'$ are uniquely determined from the consideration of the weight. By separating the reduced C-G coefficients in Eq. (E-15) and using Eq. (E-15) with $n-1$ instead of n ($\langle[r_1'\cdots r_{n-2}'0]L[P_{n-1}]\boldsymbol{P}_{n-1}|[q_1'\cdots q_{n-2}'0]L\rangle_{n-1}=\langle[r_1'\cdots r_{n-2}']L[P_{n-1}]\boldsymbol{P}_{n-1}'|[q_1'\cdots q_{n-2}']L\rangle_{n-2}$) we obtain the reducttion formula Eq. (5-2-2) from a special U_n reduced C-G coefficient to a U_{n-1} reduced C-G coefficient.

For the proof of Eq. (5-2-3), the following relation is essential;

$$\varphi^{(n-1,n)[f_1\cdots f_{n-1}]}_{H,\ q_1\cdots q_{n-1}\atop H}(R)$$
$$=\left| \begin{matrix} f_1\cdots\cdots f_{n-1} \\ H \end{matrix} \quad \begin{matrix} f_1\cdots\cdots\cdots f_{n-1}\ 0 \\ q_1\cdots\cdots\cdots q_{n-1} \\ H \end{matrix} \right\rangle$$
$$=\frac{N_H[q_1\cdots q_{n-1}]}{N_H[q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}]}\cdot(\Delta^{1\cdots n-1}_{1\cdots n-1})^{q_{n-1}}$$
$$\times\varphi^{(n-1,n-1)[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]}_{H,\ q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\atop H}(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_{n-2},\boldsymbol{R}_n), \tag{E-16}$$

where the argument matrix of the $(n-1)\times(n-1)$ DG polynomial in the right-hand side is $(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_{n-2},\boldsymbol{R}_n)$ instead of $(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_{n-2},\boldsymbol{R}_{n-1})$. Equation (E-16) is proved as follows. By using the expression Eq. (2-1-20) of the semi-maximum weight state, we have

$$\varphi^{(n-1,n)[f_1\cdots f_{n-1}]}_{H,\ q_1\cdots q_{n-1}\atop H}(R)=\varphi^{(nn)[f_1\cdots f_{n-1}\ 0]}_{H,\ q_1\cdots q_{n-1}\atop H}(R)$$
$$=N_H[f_1\cdots f_{n-1}\ 0]\,N\begin{bmatrix} f_1\cdots f_{n-1} \\ f_1\cdots f_{n-1}\ 0 \\ q_1\cdots q_{n-1}\end{bmatrix}^{-1}$$
$$\times\left\{\prod_{\nu>m=1}^{n-1}\frac{(f_m-f_\nu+\nu-m-1)!}{(q_m-f_\nu+\nu-m-1)!}\right\}\left\{\prod_{m=1}^{n-1}\frac{(f_m+n-m-1)!}{(q_m+n-m-1)!}\right\}$$
$$\times\prod_{m=1}^{n-2}\{(\Delta^{1\cdots m}_{1\cdots m})^{q_m-f_{m+1}}(\Delta^{1\cdots m}_{1\cdots n})^{f_m-q_m}\}(\Delta^{1\cdots n-1}_{1\cdots n-1})^{q_{n-1}}(\Delta^{1\cdots n-1}_{1\cdots n})^{f_{n-1}-q_{n-1}}, \tag{E-17}$$

$$\varphi^{(n-1,n-1)[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]}_{H,\ q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\atop H}(R)$$
$$=N_H[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]\,N\begin{bmatrix} f_1-q_{n-1},\cdots,f_{n-2}-q_{n-1} \\ f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1} \\ q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\end{bmatrix}^{-1}$$
$$\times\left\{\prod_{\nu>m=1}^{n-1}\frac{(f_m-f_\nu+\nu-m-1)!}{(q_m-f_\nu+\nu-m-1)!}\right\}$$
$$\times\prod_{m=1}^{n-2}\{(\Delta^{1\cdots m}_{1\cdots m})^{q_m-f_{m+1}}(\Delta^{1\cdots m}_{1\cdots n-1})^{f_m-q_m}\}(\Delta^{1\cdots n-1}_{1\cdots n-1})^{f_{n-1}-q_{n-1}}. \tag{E-18}$$

From Eqs. (E-17), (E-18) and

$$N_H[f_1\cdots f_{n-1}\ 0]=N_H[f_1\cdots f_{n-1}], \tag{E-19a}$$

$$N\begin{bmatrix} f_1-q_{n-1},\cdots,f_{n-2}-q_{n-1} \\ f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1} \\ q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\end{bmatrix}^{-1}=N\begin{bmatrix} f_1\cdots f_{n-2} \\ f_1\cdots f_{n-1} \\ q_1\cdots q_{n-2}\end{bmatrix}^{-1}, \tag{E-19b}$$

we obtain

$$\varphi^{(n-1,n)[f_1\cdots f_{n-1}]}_{H,\ q_1\cdots q_{n-1}\ H}(R)=\frac{N_H[f_1\cdots f_{n-1}]}{N_H[f_1-q_{n-1},\ \cdots,\ f_{n-1}-q_{n-1}]}$$

$$\times N\begin{bmatrix} f_1\cdots f_{n-1} \\ f_1\cdots f_{n-1}\,0 \\ q_1\cdots q_{n-1}\end{bmatrix}^{-1} N\begin{bmatrix} f_1\cdots f_{n-2} \\ f_1\cdots f_{n-1} \\ q_1\cdots q_{n-2}\end{bmatrix}\cdot\prod_{m=1}^{n-1}\frac{(f_m+n-m-1)!}{(q_m+n-m-1)!}$$

$$\times(\Delta^{1\cdots n-1}_{1\cdots n-1})^{q_{n-1}}\varphi^{(n-1,n-1)[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]}_{H,\ q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\ H}(\boldsymbol{R}_1,\ \cdots,\ \boldsymbol{R}_{n-2},\ \boldsymbol{R}_n). \tag{E-20}$$

By using the relation

$$\frac{N_H[f_1\cdots f_{n-1}]}{N_H[f_1-q_{n-1},\ \cdots,\ f_{n-1}-q_{n-1}]}=\left[\prod_{m=1}^{n-1}\frac{(f_m-q_{n-1}+n-1-m)!}{(f_m+n-1-m)!}\right]^{1/2},$$

$$N\begin{bmatrix} f_1\cdots f_{n-1} \\ f_1\cdots f_{n-1}\,0 \\ q_1\cdots q_{n-1}\end{bmatrix}^{-1}=N\begin{bmatrix} f_1\cdots f_{n-2} \\ f_1\cdots f_{n-1} \\ q_1\cdots q_{n-2}\end{bmatrix}^{-1}$$

$$\times\left[\prod_{m=1}^{n-1}\left\{\frac{(q_m-q_{n-1}+n-1-m)!(q_m+n-1-m)!}{(f_m-q_{n-1}+n-1-m)!(f_m+n-1-m)!}\right\}\right]^{1/2},$$

which can be proved from their explicit expressions Eqs. (2-1-9) and (2-1-19), we find that the coefficient in Eq. (E-20) is equal to that in Eq. (E-16).

Let us proceed to the proof of Eq. (5-2-3). From the property Eq. (E-19b), the lowering operation on Eq. (E-16) with respect to the U_{n-1} generator algebras $\{A_{\alpha\beta};\alpha,\beta=1\sim n-1\}$ of the $(n-1)\times n$ DG polynomial gives

$$\varphi^{(n-1,n)[f_1\cdots f_{n-1}]}_{a,\ q_1\cdots q_{n-1}\ H}(R)=\frac{N_H[q_1,\ \cdots,\ q_{n-1}]}{N_H[q_1-q_{n-1},\ \cdots,\ q_{n-2}-q_{n-1}]}\cdot(\Delta^{1\cdots n-1}_{1\cdots n-1})^{q_{n-1}}$$

$$\times\varphi^{(n-1,n-1)[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]}_{a-q_{n-1};q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\ H}(\boldsymbol{R}_1,\ \cdots,\ \boldsymbol{R}_{n-2},\ \boldsymbol{R}_n), \tag{E-21}$$

where $a-q_{n-1}$ denotes the Gel'fand pattern which is obtained by subtracting the common number q_{n-1} from all the partition numbers of the Gel'fand pattern a. On the other hand we can write the vector-coupling expression Eq. (2-2-4) as

$$\varphi^{(n-1,n)[f_1\cdots f_{n-1}]}_{a,\ q_1\cdots q_{n-1}\ H}(R)=\sum_{b,|\boldsymbol{N}_n|=N_n}\langle[q_1\cdots q_{n-1}]b[N_n]\boldsymbol{N}_n|[f_1\cdots f_{n-1}]a\rangle_{n-1}$$

$$\times\varphi^{(n-1\,n-1)[q_1\cdots q_{n-1}]}_{b,H}(R)\,U^{(n-1)}_{[N_n]\boldsymbol{N}_n}(\boldsymbol{R}_n), \tag{E-22}$$

where $N_n=\sum_{i=1}^{n-1}(f_i-q_i)$. By substituting Eq. (E-21) and the expression

$$\varphi^{(n-1,n-1)[q_1\cdots q_{n-1}]}_{b,H}(R)=\frac{N_H[q_1\cdots q_{n-1}]}{N_H[q_1-q_{n-1},\ \cdots,q_{n-2}-q_{n-1}]}\cdot(\Delta^{1\cdots n-1}_{1\cdots n-1})^{q_{n-1}}$$

$$\times\varphi^{(n-1,n-1)[q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1},0]}_{b-q_{n-1},H}(R)$$

$$
= \frac{N_H[q_1 \cdots q_{n-1}]}{N_H[q_1 - q_{n-1}, \cdots, q_{n-2} - q_{n-1}]} \cdot (\Delta^{1\cdots n-1}_{1\cdots n-1})^{q_{n-1}}
$$
$$
\times \varphi^{(n-1,n-2)[q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}]}_{b-q_{n-1},H}(\boldsymbol{R}_1, \cdots, \boldsymbol{R}_{n-2}), \tag{E-23}
$$

in Eq. (E-22), we obtain

$$
\varphi^{(n-1,n-1)[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]}_{a-q_{n-1};\ q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\ H}(\boldsymbol{R}_1, \cdots, \boldsymbol{R}_{n-2}, \boldsymbol{R}_n)
$$
$$
= \sum_{b,|\boldsymbol{N}_n|=N_n} \langle [q_1 \cdots q_{n-1}] b [N_n] \boldsymbol{N}_n | [f_1 \cdots f_{n-1}] a \rangle_{n-1}
$$
$$
\times \varphi^{(n-1,n-2)[q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}]}_{b-q_{n-1},H}(\boldsymbol{R}_1, \cdots, \boldsymbol{R}_{n-2}) U^{(n-1)}_{[N_n]\boldsymbol{N}_n}(\boldsymbol{R}_n). \tag{E-24}
$$

Replacing $\boldsymbol{R}_n$ in Eq. (E-24) by $\boldsymbol{R}_{n-1}$ and comparing it with the vector-coupling expression

$$
\varphi^{(n-1,n-1)[f_1-q_{n-1},\cdots,f_{n-1}-q_{n-1}]}_{a-q_{n-1};\ q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}\ H}(R)
$$
$$
= \sum_{b,|\boldsymbol{N}_n|=N_n} \langle [q_1 - q_{n-1}, \cdots, q_{n-2} - q_{n-1}] b - q_{n-1} [N_n] \boldsymbol{N}_n |
$$
$$
\times [f_1 - q_{n-1}, \cdots, f_{n-1} - q_{n-1}] a - q_{n-1} \rangle_{n-1}
$$
$$
\times \varphi^{(n-1,n-2)[q_1-q_{n-1},\cdots,q_{n-2}-q_{n-1}]}_{b-q_{n-1},H}(R) U^{(n-1)}_{[N_n]\boldsymbol{N}_n}(\boldsymbol{R}_{n-1}), \tag{E-25}
$$

we find

$$
\langle [q_1 \cdots q_{n-1}] b [N_n] \boldsymbol{N}_n | [f_1 \cdots f_{n-1}] a \rangle_{n-1}
$$
$$
= \langle [q_1 - q_{n-1}, \cdots, q_{n-2} - q_{n-1}] b - q_{n-1} [N_n] \boldsymbol{N}_n |
$$
$$
\times [f_1 - q_{n-1}, \cdots, f_{n-1} - q_{n-1}] a - q_{n-1} \rangle_{n-1}, \tag{E-26}
$$

where $N_n = \sum_{i=1}^{n-1} (f_i - q_i)$. By separating the reduced C-G coefficients in Eq. (E-26) and using Eq. (E-26) with $n-2$ instead of $n-1$, we obtain the formula Eq. (5-2-3), which connects many mutually equivalent U_{n-1} reduced C-G coefficients.

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