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Fractional Derivative Models of Damped Oscillations

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1 Introduction

The history of fractional differentiation is quite old. According to the historical survey in [1], L'Hôpital asked Leibniz in 1695 about the meaning of $\mathrm{d}^n y/\mathrm{d}x^n$, when n is a fraction. Since then, fractional differentiation attracted attention of many mathematicians, including Euler, Laplace, Fourier, Abel, Liouville, Riemann and Laurent, though the subject is not very much popular today. The fractional derivative was defined in terms of its action on an exponential function by Liouville in 1832, and Riemann gave a definition in terms of integration in the publication of his earlier works in 1892. This has opened modern treatment of fractional differentiation.

Due to the needs of industrial application of oil related materials, the study of viscoelastic fluids and bodies has begun since early twentieth century. When a rod immersed in viscous (Newtonian) fluid is pulled out, the strain (distance moved) ϵ is proportional to the time, $\epsilon \propto t$, while with elastic material, the strain stays constant in time, $\epsilon \sim$ const. In 1921, Nutting [2, 3] has observed that in the rod experiment with viscoelastic fluids, the strain increases with time as $\epsilon \propto t^\nu$, $0 < \nu < 1$. The idea of using fractional derivatives to describe viscoelasticity is due to Gemant [4], and Scott Blair [5, 6]. In 1936, based on his own experiment, Gemant noted that fractional derivative defined in terms of fractional power series would show good fit with his data. Scott Blair suggested in 1947 that the Nutting behavior would be accounted for by the use of fractional derivatives in the relation between strain and stress, the so-called constitutive equation.

The most typical formulation of the problem is to use the fractional derivative term in place of the viscous damper term in the usual damped oscillation. In the light of modern treatment of fractional differentiation, the problem becomes a study of differential equations with fractional derivatives. Such models have been extensively studied by Bagley and Torvik in a series of papers [7, 8, 9]; see also references in [10, 11]. An extensive review of fractional differential equations is given in Podlybny [10], and a survey of applications of fractional calculus in physics including viscoelasticity is given in Hilfer [11].

Denoting the amount of the shrinkage of the body under compression by $x(t)$, the restoring force is proportional to $x(t)$. In terms of Fourier transform, the term corresponds to a real coefficient. On the other hand, the damping force due to viscous damper is proportional to the velocity $\mathrm{d}x/\mathrm{d}t$, which corresponds to an imaginary coefficient in the Fourier transform. In the proposed model, the viscoelastic damping is assumed to be proportional to its fractional derivative $\mathrm{d}^\nu x/\mathrm{d}t^\nu$, $\nu \in \mathbb{R}$, $\nu > 0$. Its Fourier transform is proportional to $(i\omega)^\nu$, which has both a real part and an imaginary part, thereby accounting for elasticity and viscousity by a single term.

In most experiments carried out today, response of the system under applied sinusoidal force is studied. Thus, the solutions are usually studied in the frequency domain, and closed form solutions of the oscillation in the time domain are not always derived. To fill this gap, Sakakibara [12] has studied the damped oscillation with a viscoelastic damping corresponding to a fractional derivative of order 1/2 in detail. This paper is intended to generalize the study to an arbitrary order. To make the presentation self contained, we first give a brief review of the fractional derivative and the method of solving fractional differential equation, in the following section.

2 Fractional Derivatives

Let $D^{-1}f(t)$ be the integral

$$D^{-1}f(t) = \int_c^t f(t_1)\mathrm{d}t_1, \quad c \in \mathbb{R}$$

of a Riemann integrable function $f(t)$, and similarly $D^{-n}f(t)$ be the n times integral

$$D^{-n}f(t) = \int_c^t \mathrm{d}t_1 \int_c^{t_1} \mathrm{d}t_2 \int_c^{t_2} \mathrm{d}t_3 \cdots \int_c^{t_{n-1}} f(t_n)\mathrm{d}t_n. \tag{1}$$

Proposition 1 *The integral* (1) *can be rewritten as*

$$D^{-n}f(t) = \frac{1}{(n-1)!}\int_c^t (t-t_n)^{n-1} f(t_n)\mathrm{d}t_n. \tag{2}$$

Proof For $n = 2$, by exchanging the order of integration,

$$D^{-2}f(t) = \int_c^t \mathrm{d}t_1 \int_c^{t_1} f(t_2)\mathrm{d}t_2 = \int_c^t \mathrm{d}t_2 \int_{t_2}^t f(t_2)\mathrm{d}t_1 = \int_c^t (t-t_2)f(t_2)\mathrm{d}t_2.$$

Next, we assume

$$D^{-n+1}f(t_1) = \frac{1}{(n-2)!}\int_c^{t_1} (t_1 - t_n)^{n-2} f(t_n)\mathrm{d}t_n$$

Then, by the exchange of integration order,

$$\begin{aligned} D^{-n}f(t) &= \int_c^t \mathrm{d}t_1 D^{-n+1}f(t_1) = \int_c^t \mathrm{d}t_1 \frac{1}{(n-2)!}\int_c^{t_1} (t_1 - t_n)^{n-2} f(t_n)\mathrm{d}t_n \\ &= \int_c^t \mathrm{d}t_n \int_{t_n}^t \frac{(t_1 - t_n)^{n-2}}{(n-2)!} f(t_n)\mathrm{d}t_1 = \int_c^t \frac{(t-t_n)^{n-1}}{(n-1)!} f(t_n)\mathrm{d}t_n. \end{aligned}$$

By simple induction, (2) is valid for $n \in \mathbb{N}$. □

By the shift of variable $t \to t - c$, the lower limit c may be set equal to 0, without loss of generality. By extending n in (2) to a real number μ, we define the fractional integral of a Riemann integrable function defined on $[0, \infty)$.

Definition The Riemann-Liouville fractional integral of $f(t)$ is defined by

$$D^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)\, \mathrm{d}\tau, \quad \operatorname{Re}\mu > 0, \tag{3}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t \tag{4}$$

is the Euler gamma function. It satisfies the identity

$$\Gamma(z+1) = z\,\Gamma(z)$$

and $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. The Riemann-Liouville fractional derivative of $f(t)$ is defined by

$$D^{\nu} f(t) = D^n D^{-n+\nu} f(t), \quad n = \lceil \nu \rceil, \quad \nu \neq n, \tag{5}$$

where $D^n f(t) = \mathrm{d}^n f(t)/\mathrm{d}t^n$ is the usual n-th derivative.

A typical example would be $f(t) = t^n$ for which the fractional integral is

$$D^{-\mu} t^n = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \tau^n \mathrm{d}\tau = \frac{1}{\Gamma(\mu)} t^{n+\mu} \int_0^1 (1-u)^{\mu-1} u^n \mathrm{d}u = \frac{\Gamma(n+1)}{\Gamma(n+\mu+1)} t^{n+\mu}, \tag{6}$$

where

$$B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\, du$$

is the Beta function, and the well known identity

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

has been used. For $\mu = 1$, it reduces to the familiar result

$$D^{-1} t^n = \frac{\Gamma(n+1)}{\Gamma(n+2)} t^{n+1} = \frac{1}{n+1} t^{n+1}.$$

The fractional derivative of t^n is

$$D^{\nu} t^n = D D^{-1+\nu} t^n = D \frac{\Gamma(n+1)}{\Gamma(n+2-\nu)} t^{n+1-\nu} = \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} t^{n-\nu}. \tag{7}$$

However, note that for $n = 0$,

$$D^{\nu} t^0 = D D^{-1+\nu} t^0 = \frac{1}{\Gamma(1-\nu)} t^{-\nu}$$

but

$$D^{-1+\nu} D t^0 = 0.$$

In contrast to the usual derivatives of integer order, we have

$$D^{\nu} D^{\mu} f(t) \neq D^{\mu} D^{\nu} f(t) \neq D^{\mu+\nu} f(t) \tag{8}$$

in general.

3 The Eigenfunction

In order to find solutions to linear differential equations involving fractional derivatives, we need the eigenfunction of D^ν, $\nu > 0$, i.e., the solution of

$$D^\nu f(t) = a\, f(t).$$

Recalling that the eigenfunction of D, or D^{-1}, is e^{at}, define the Miller-Ross function

$$E_t(\nu, a) = D^{-\nu} e^{at} = t^\nu e^{at} \gamma^*(\nu, at), \tag{9}$$

where

$$\gamma^*(\nu, t) = \frac{t^{-\nu}}{\Gamma(\nu)} \int_0^\infty \xi^{\nu-1} e^{-\xi} d\xi$$

is the incomplete gamma function [1].

Theorem 2 *Let $E_t(\nu, a)$ be the Miller-Ross function defined by* (9). *Then*

$$E_t(\nu, a) = t^\nu E_{1,1+\nu}(at), \tag{10}$$

where

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)} \tag{11}$$

is the Mittag-Leffler function [10].

Proof Using (6),

$$E_t(\nu, a) = D^{-\nu} e^{at} = D^{-\nu} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k+1+\nu)} t^{k+\nu} = t^\nu \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1+\nu)} (at)^k$$

which verifies (10). □

Some of the special cases are:

$$E_t(0, a) = e^{at} \tag{12}$$

$$E_0(\nu, a) = 0, \quad \operatorname{Re} \nu > 0,$$

$$E_t(\nu, 0) = \frac{t^\nu}{\Gamma(1+\nu)},$$

$$E_t(\tfrac{1}{2}, a) = e^{at} \frac{\operatorname{erf} \sqrt{at}}{\sqrt{a}} \tag{13}$$

where

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\xi^2} d\xi$$

is the error function. By direct calculations, we have

Theorem 3 *The Miller-Ross function* (9) *satisfies the following identities:*

$$\begin{aligned} E_t(\nu, a) &= a\,E_t(\nu+1, a) + \frac{t^\nu}{\Gamma(\nu+1)}, \\ E_t(-p, a) &= a^p\,E_t(0, a), \quad p \ge 0, \\ D^\mu E_t(\nu, a) &= E_t(\nu-\mu, a), \quad \mu \in \mathbb{R}. \end{aligned} \tag{14}$$

From the above properties, it can be shown that

$$D\,E_t(0, a) = E_t(-1, a) = a\,E_t(0, a)$$

which means that $E_t(0, a) = e^{at}$ is the eigenfunction of D with eigenvalue a. Similarly, we see that

$$\begin{aligned} D^{1/2}\,E_t(0, a) &= E_t(-\tfrac{1}{2}, a) \\ D^{1/2}\,E_t(-\tfrac{1}{2}, a) &= E_t(-1, a) = a\,E_t(0, a) \end{aligned}$$

or

$$D^{1/2}\big(E_t(-\tfrac{1}{2}, a^2) + a\,E_t(0, a^2)\big) = a\big(E_t(-\tfrac{1}{2}, a^2) + a\,E_t(0, a^2)\big).$$

Thus, the sum

$$e_2(t, a) = E_t(-\tfrac{1}{2}, a^2) + a\,E_t(0, a^2)$$

is the eigenfunction of $D^{1/2}$ with eigenvalue a.

Theorem 4 *Let $E_t(\nu, a)$ be the Miller-Ross function defined by* (9). *Then*

$$e_n(t, a) = \sum_{k=0}^{n-1} a^k E_t\Big(-\frac{n-k-1}{n}, a^n\Big), \quad n \in \mathbb{N}, \tag{15}$$

is the eigenfunction of $D^{1/n}$ with eigenvalue a, i.e., it satisfies

$$D^{1/n} e_n(t, a) = a\,e_n(t, a). \tag{16}$$

Note that the eigenfunction $e_n(t, a)$ is singular at the origin.

4 The Laplace transform

The Laplace transform of a function $f(t)$ is defined as

$$f(t) \circ\!\!-\!\!\bullet\; F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

where the Doetch symbol $f \circ\!\!-\!\!\bullet F$ shows the correspondence between the function and its Laplace transform. It is well known that

$$f(t) = e^{at} \circ\!\!-\!\!\bullet\; \frac{1}{s-a}$$

which is readily obtained by definition as

$$\int_0^\infty e^{-st} e^{at} \mathrm{d}t = \frac{e^{-s+a}}{-s+a}\bigg|_0^\infty = \frac{1}{s-a}.$$

Note that the same result may be obtained by the series expansion. To this end, we first note that

Lemma 5

$$\frac{t^k}{k!} \circ\!\!-\!\!\bullet \frac{1}{s^{k+1}} \tag{17}$$

Proof Using ,

$$\int_0^{\infty} e^{-st} \frac{t^k}{k!} dt = \frac{1}{s^{k+1}\, k!} \int_0^{\infty} e^{-st} (st)^k d(st) = \frac{1}{s^{k+1}\, k!} \Gamma(k+1) = \frac{1}{s^{k+1}}$$

□

Then by applying Lemma 5, we have

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} \circ\!\!-\!\!\bullet \sum_{k=0}^{\infty} \frac{a^k}{s^{k+1}} = \frac{1}{s} \sum_{k=0}^{\infty} \frac{a^k}{s^k} = \frac{1}{s(1 - a/s)} = \frac{1}{s-a}.$$

Similarly, we can show that

Lemma 6

$$E_t(\nu, a) \circ\!\!-\!\!\bullet \frac{1}{s^{\nu}(s-a)}. \tag{18}$$

Proof Using Lemma 5,

$$E_t(\nu, a) = t^{\nu} E_{1,1+\nu}(at) = \sum_{k=0}^{\infty} \frac{a^k t^{k+\nu}}{\Gamma(k+\nu+1)} \circ\!\!-\!\!\bullet \sum_{k=0}^{\infty} \frac{a^k}{s^{k+\nu+1}} = \frac{1}{s} \sum_{k=0}^{\infty} \frac{a^k}{s^k} = \frac{1}{s^{\nu+1}(1 - a/s)} = \frac{1}{s^{\nu}(s-a)}.$$

□

For the eigenfunction $e_2(t, a)$ of $D^{1/2}$ given by (15) with $n = 2$, we have

$$e_2(t, a) = E_t(-\tfrac{1}{2}, a^2) + a\, E_t(0, a^2) \circ\!\!-\!\!\bullet \frac{1}{s^{-1/2}(s-a^2)} + \frac{a}{s-a^2} = \frac{s^{1/2}+a}{s-a^2} = \frac{1}{s^{1/2}-a}.$$

Similarly, we can show

Corollary 7 *The Laplace transform of the eigenfunction $e_n(t, a)$ of $D^{1/n}$, $n \in \mathbb{N}$ is given by*

$$e_n(t, a) \circ\!\!-\!\!\bullet \frac{1}{s^{1/n} - a}. \tag{19}$$

5 Oscillation Equation with Fractional Derivative Damping

We consider the fractional differential equation

$$D^2 x(t) + a_{2n-1} D^{2-1/n} x(t) + \ldots + a_1 D^{1/n} x(t) + x(t) = 0, \quad n \in \mathbb{N}. \tag{20}$$

When $a_n = 0$, $n = 1, 2, \ldots, 2n-1$, the equation reduces to the harmonic oscillation equation $D^2x(t) + x(t) = 0$. Actually, the equation reads $m D^2 x(t) + k x(t) = 0$, where m is the mass and k the spring constant of the oscillator. Defining $\omega_0 = \sqrt{k/m}$ and rescaling time $t \to t/\omega_0$, we normalized the coefficients to unity. We will be interested in the impulse response of the system.

Let

$$P(z) = z^N + a_{N-1}z^{N-1} + \ldots + a_2 z^2 + a_1 z + 1 \tag{21}$$

be a polynomial of degree N, and α_j be N roots of $P(z) = 0$. Then

$$P(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_N) = \prod_{j=1}^{N}(z - \alpha_j)$$

Note that

$$\frac{1}{P(z)} = \sum_{j=1}^{N} \frac{1}{P'(\alpha_j)} \frac{1}{z - \alpha_j}. \tag{22}$$

By multiplying $P(z)$ on both sides, we obtain the Lagrange interpolation formula

$$\sum_{j=1}^{N} \frac{1}{P'(\alpha_j)} \prod_{\substack{k=1 \\ k\neq j}}^{N} (z - \alpha_k) = 1 \tag{23}$$

Furthermore, by comparing the coefficients on both sides of this equation, and combining the terms, we find that

Theorem 8 *Let $P(z)$ be the polynomial of degree N defined in* (21), *and α_j, $j = 1, 2, \ldots, N$, be the roots of $P(z) = 0$. All roots are assumed to be distinct. Then*

$$\sum_{j=1}^{N} \frac{\alpha_j{}^m}{P'(\alpha_j)} = \begin{cases} 0, & m = 0, 1, 2, \ldots, N-2, \\ 1, & m = N-1, \\ \sum_{j=1}^{N} \alpha_j = -a_{N-1}, & m = N. \end{cases} \tag{24}$$

The fractional differential equation (20) may be written in terms of $P(z)$, with $N = 2n$,

$$P(D^{1/n})x(t) = \left(D^2 + a_{2n-1}D^{2-1/n} + \ldots + a_1 D^{1/n} + 1\right)x(t) = \prod_{j=1}^{2n}(D^{1/n} - \alpha_j)x(t) = 0. \tag{25}$$

The solution $x(t)$ of the equation (25) is obtained by the analogy to the method of solution of the usual differential equation

$$Dx(t) - a\,x(t) = 0, \quad x(0) = 1.$$

By taking the Laplace transform, we find

$$sX(s) - x(0) - aX(s) = 0,$$

or

$$X(s) = \frac{1}{s - a}.$$

The inverse Laplace transform gives

$$x(t) = e^{at}.$$

Thus, formally considering the Laplace transform of $D^{1/n}x(t)$ as $s^{1/n}X(s)$, the equation (25) becomes

$$P(s^{1/n})\, X(s) = 0.$$

If we want $x(t)$ to represent the impulse response, then 0 on the right-hand side of (25) would be replaced by $\delta(t)$. Since the Laplace transform of $\delta(t)$ is 1, we have

$$X(s) = \frac{1}{P(s^{1/n})} = \sum_{j=1}^{2n} \frac{1}{P'(\alpha_j)} \frac{1}{s^{1/n} - \alpha_j}.$$

By taking the inverse Laplace transform,

$$x(t) = \sum_{j=1}^{2n} \frac{e_n(t, \alpha_j)}{P'(\alpha_j)}.$$

Note that the correspondence $D^{1/n} f(t) \circ\!\!-\!\!\bullet\; s^{1/n}F(s)$ may not be guaranteed for an arbitrary $f(t)$.

Theorem 9 *Let $x(t)$ be given by*

$$x(t) = \sum_{j=1}^{2n} \frac{e_n(t, \alpha_j)}{P'(\alpha_j)}. \tag{26}$$

Then, it is a solution of the fractional differential equation (25), *with* $x(0) = 0$ *and* $Dx(0) = 1$.

Proof Due to the eigenequation (16), the equation (25) is verified by direct substitution of (26). into , the equality is seen to hold. Note that $x(t)$ may be expressed in several different forms,

$$\begin{aligned}
x(t) &= \sum_{j=1}^{2n} \frac{e_n(t, \alpha_j)}{P'(\alpha_j)} \\
&= \sum_{j=1}^{2n} \frac{1}{P'(\alpha_j)} \sum_{k=0}^{n-1} \alpha_j^{\,k} E_t\Big(-\frac{n-1-k}{n}, \alpha_j^{\,n}\Big) &(27)\\
&= \sum_{j=1}^{2n} \frac{1}{P'(\alpha_j)} \sum_{k=0}^{n-1} \alpha_j^{\,k} t^{-(n-1-k)/n} E_{1,1-(n-1-k)/n}(\alpha_j^{\,n} t) \\
&= \sum_{j=1}^{2n} \frac{1}{P'(\alpha_j)} \sum_{k=0}^{n-1} \sum_{l=0}^{\infty} \frac{\alpha_j^{\,ln+k} t^{l-1+(1+k)/n}}{\Gamma\big(l + (1+k)/n\big)}. &(28)
\end{aligned}$$

The terms of $l = 0$ in (28) vanishes due to Theorem 8. The terms of $l = 1$ and $k < n - 1$ also vanishes. Thus, for small t, keeping only the terms of $l = 1, k = n - 1$, and $l = 2, k = 0$, we have

$$\begin{aligned}
x(t) &= \sum_{j=1}^{2n} \frac{\alpha_j^{\,2n-1}}{P'(\alpha_j)} t + \sum_{j=1}^{2n} \frac{\alpha_j^{\,2n}}{P'(\alpha_j)} \frac{t^{1+1/n}}{\Gamma(2+1/n)} + O(t^{1+2/n}) &(29)\\
&= t - \frac{a_{2n-1}}{\Gamma(2+1/n)} t^{1+1/n} + O(t^{1+2/n}). &(30)
\end{aligned}$$

Thus we prove that $x(0) = 0$. Furthermore, by taking the first derivative,

$$Dx(t) = 1 - \frac{a_{2n-1}}{\Gamma(1 + 1/n)} t^{1/n} + O(t^{2/n}).$$

Hence $Dx(0) = 1$, which completes the proof. □

The asymtotic expansion of the solution $x(t)$ is obtained by expressing the Mittag-Leffler function in terms of the confluent hypergeometric function, which is defined by

$$M(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is the Pochhammer symbol. The asymptotic expansion is

$$M(a; b; z) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a} + O(|z|^{-a-1}),$$

for large $|z|$; see for example [13]. Noting that $(1)_n = n!$, we have the relation

$$M(1, b; z) = \sum_{n=0}^{\infty} \frac{\Gamma(b)}{\Gamma(b+n)} z^n = \Gamma(b) E_{1,b}(z).$$

Thus

$$E_t(v, a) = \frac{t^v(-at)^{-1}}{\Gamma(v)}, \quad |z| \to \infty. \tag{31}$$

Theorem 10 *The asymptotic expansion of the solution* (26) *is given by*

$$x(t) = \frac{a_1}{n\Gamma\left(1 - \frac{1}{n}\right)} t^{-1/n-1} + O(|t|^{-1/n-2}), \quad |t| \to \infty. \tag{32}$$

where a_1 is the coefficient in (20).

Proof The leading term in the asymptotic expansion of $x(t)$ in (27) arises from the term of $k = n - 2$ in (28). Using (31), we find

$$x(t) = \sum_{j=1}^{2n} \frac{\alpha_j^{\,n-2}}{P'(\alpha_j)} \frac{t^{-1/n}(-\alpha_j^{\,n} t)^{-1}}{\Gamma(-1/n)} + O(|t|^{-1/n-2}) = \sum_{j=1}^{2n} \frac{\alpha_j^{\,-2}}{P'(\alpha_j)} \frac{t^{-1/n-1}}{n\Gamma(1-1/n)} + O(|t|^{-1/n-2})$$

By differentiating both sides of (22), we obtain

$$\frac{P'(z)}{P(z)^2} = \sum_{j=1}^{2n} \frac{1}{P'(\alpha_j)(z - \alpha_j)^2},$$

and, by setting $z = 0$,

$$a_1 = \frac{P'(0)}{P(0)^2} = \sum_{j=1}^{2n} \frac{1}{P'(\alpha_j)\alpha_j^{\,2}}.$$

Subtituting this result in the above expression, we complete the proof. □

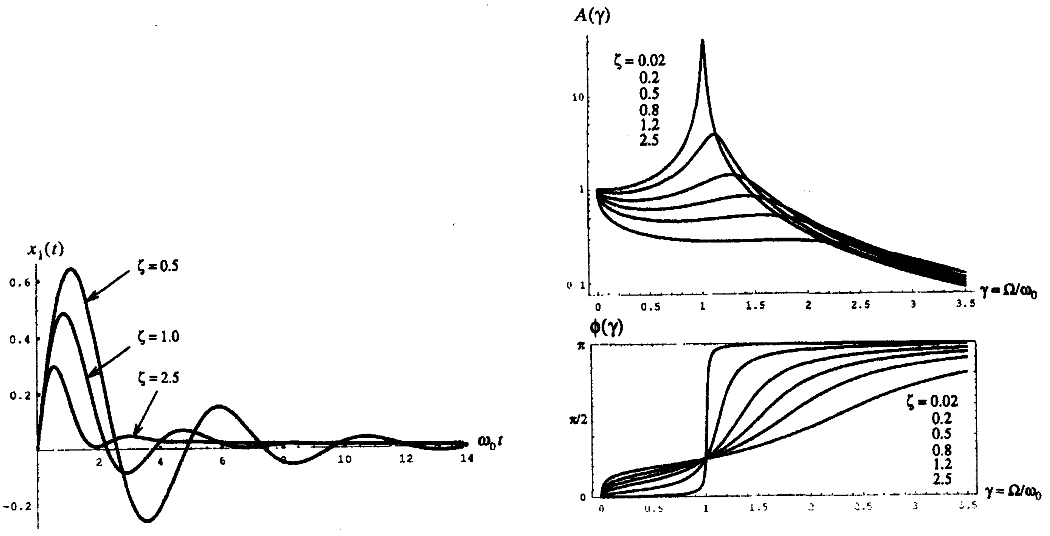

Figure 1: The impulse responce $x(t)$ (left), and the resonance curve (right)

6 An Example

One of the most simple example is the case with $n = 2$, and $a_1 \neq 0, a_2 = a_3 = 0$,

$$D^2x(t) + a_1 D^{1/2}x(t) + x(t) = 0,$$

assuming that a_1 is a positive real number. This is studied in [12] in some detail. Due to the relation (13), the eigenfunction becomes

$$e_2(t, a) = ae^{a^2t}\left(1 + \mathrm{erf}(a\sqrt{t})\right) + \frac{1}{\sqrt{\pi t}}.$$

The solution of the equation is given by

$$x(t) = \sum_{j=1}^{4} \frac{\alpha_j}{P'(\alpha_j)} e^{\alpha_j^2 t}\left(1 + \mathrm{erf}(\alpha_j\sqrt{t})\right),$$

where α_j are four roots of

$$P(z) = z^4 + a_1 z + 1.$$

Since the coeffients are all real, the roots arise in complex conjugate pairs, or as real numbers, which guarantees that the solution $x(t)$ is real valued. The solutions are plotted in Figure 1 (left) for some values of $\zeta = a_1 3^{3/4}/4$. Also noted is the resonance curve in Figure 1 (right). When the oscillator is subject to the external force $\sin \gamma t$, the stationary response $x(t) = A\sin(\gamma t - \phi)$ is obtained. The amplification A and phase delay ϕ are plotted against the frequency γ of the applied force. As ζ increases, not only the damping but also the frequency is increase, as implied by viscoelasticity. The asymptotic behavior is given by

$$x(t) = \frac{a_1}{2\sqrt{\pi}} t^{-3/2} + O(|t|^{-5/2}), \quad |t| \to \infty,$$

implying slow decay as compared with exponential decay of the usual viscous damping.

7 Conclusions

The solution of the fractional differential equations describing oscillation with viscoelastic damping has been given in a closed form. This allows us to study the behavior of the solution precisely. In particular, the asymptotic power-law decay is determined by the lowest order derivative $D^{1/n}x(t)$ whose coefficient is a_1. On the other hand, the initial decrease in the velocity $Dx(t)$ is determined by the therm $D^{2-1/n}x(t)$ with the coefficient a_{2n-1}. Thus, if $a_{2n-1} = 0$, then it will depend on the next term $D^{2-2/n}x(t)$. As is clear from the method of solution, n is a natural number.

Apart from the phenomenological reasoning of fractional derivatives given in Introduction, one may be concerned about underlying physics. The asymptotic power law decay indicates the lack of characteristic scale, implying fractal structure. In order to identify such a correspondence, Schiessel and Blumen [14, 11] have shown that by forming a nested ladder of the usual spring-dashpot combinations, one can obtain a mechanical model which has fractal like properties. This has been shown in the Laplace transform $X(s)$ of $x(t)$, for some special choice of parameters. Sakakibara [15, 16] has shown that their result can be cast into a closed form for $x(t)$, which shows the power-law decay $x(t) \sim t^{-\gamma}$. The exponent $-\gamma$ may be an arbitrary real positive number. In the models discussed here, the exponent must be of the form $-1 - 1/n$, $n \in \mathbb{N}$. Thus, our models might well be considered as phenomenological, effective theory of oscillations with viscoelastic damping.

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