数理解析研究所講究録 1385 巻 2004 年 105-116

A talk at RIMS on 8 July 2003

WHY, EVEN A TINY LITTLE OLD MATHEMATICIAN DOES HAVE AN INTEREST IN WAVELETS?

ATSUSHI INOUE; (ALIAS ATLOM)

Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152-0033, Japan, e-mail:inoue@math.titech.ac.jp

ABSTRACT. : As a tiny little old mathematician, I dare to say the following for young people: For analysis in 21-th century, we should find out something-like "PDE on infinite dimensional phase space" which governs all Green functions (alias n-point functions) of quantum field theory, and the star product below with respect to $\hbar$ would be "asymptotic" in some sense relating to these Green functions expanded in $\hbar$. In the following, I enumerate what I think interesting and relating the final goal to **analysis on infinite dimensional configuration manifolds with photons and electrons on equal footing**.

1. INTRODUCTION

The first reason why I have an interest in wavelets is that wavelets analysis stems from searching petroleum by analyzing wave pulses caused by explosion of dynamite. We need to localize the information where the source is.

This is like fishing. We know there are fishes in the sea, but where and how to find them out and how to get them? Remember "Project-X" report on developping the sonnar system for searching fishes.

In any way, to recognize some thing, we should "localize" what we want to know. Localization is to recognize the position where it is, or in which part of the scene the object is situated?

I have a dogmatic intuition that wavelets analysis has something to do with localization.

2. CONTINUUM MECHANICS, ESPECIALLY FLUID MECHANICS

Some thirty years before, I want to touch on the Navier-Stokes equation, because I have nothing special to do at that time. Before touching, I need to persuade myself whether that equation is meaningful to do research?

My conclusion: The Navier-Stokes equation has seemingly a good reason to be studied!

The derivation of the Navier-Stokes equation is nicely explained, for example, in Serrin [46] (see also, Truesdell [51]). Moreover, it is **believed** in general that the Navier-Stokes equation represents not only the motion of the laminar flow but also that of the turbulent flow. For example, Hopf wrote in p.88 of [28] as follows:

It is known that in many applications the size of the smallest eddies for highly turbulent flow is still very large in comparison to the molecular scale. In other words, for most purposes of statistical hydrodynamics a fluid may be regarded as a strictly continuous medium. Statistical hydrodynamics is the theory of the "typical" solutions of the Navier-Stokes equations.

Even after such a claim, we want to ask: Why so believed ?

There exist at least three reasons which make us pose such a very primitive question.

First of all, the meaning of the viscosity is not so clear. Because, in spite of the fundamental concept of the continuous mechanics which is to ignore the particle like feature of the flow, it is considered that the viscosity is produced by the collision of the flow particles. We ask whether the viscosity stems from before or after the so-called coarse-graining of the motion of almost infinite particles. In other words, is there any derivation of the viscosity term of the Navier-stokes equation in the frame work of the continuum mechanics? Though, on the other hand, by the Hilbert-Chapman-Enskogs expansion, we have the Euler and the Navier-Stokes equation from the Boltzman equation.

Secondly, the Navier-Stokes equation should be introduced after observing the regular motion of the laminar flow. Because, it seems rather unnatural to consider that in those days someone dared to describe the chaotic motion of the turbulent flow by using there existed mathematical language. (Historically, Reynolds is the first person who recognized the turbulent flow 'scientifically' and he derived the so-called Reynolds equation **from** the Navier-Stokes equation.) Moreover, it is doubtful to believe without hesitation that the **mathematical description** used to represent the behavior of the laminar flow is so good to do also that of the turbulent flow.

Thirdly, the so-called adhesion condition of the flow at the boundary has been the origin of the debate. We are not sure that this problem is exactly settled or not.

On the other hand, the Euler equation may be considered as a geodesic equation of $\mathrm{Diff}_\sigma(M)$, as was shown by Arnold [1], where $\mathrm{Diff}_\sigma(M)$ stands for the group of diffeomorphisms which preserve the volume element of the configuration manifold M. In other word, the Euler equation has a purely mathematical foundation. So, we try to give another derivation of the Navier-Stokes equation from the Euler equation which makes a bypass of the problems posed above except for the adhesion problem. And by this, we confirm the raison-d'être of the Navier-Stokes equation without mentioning eddies or other physical reasonings.

In any way, we were very lucky if we could use innocently the Navier-Stokes equation to treat approximately the motion of the turbulent flow.

To the Euler equation, we add "by hand" the random force caused by the white noise. This gives us the Brownian affected Euler equation, whose average and fluctuation satisfy the Reynolds equation. (This idea was explained in Inoue & Funaki [34] when the configuration manifold is $\mathbb{R}^n$.) Reinterpreting above result, we may construct a solution of the Hopf equation. (Here, we use the idea of Foiąs [21].) Using the argument in Inoue [31] we get the Navier-Stokes equation as the characteristic equation of the Hopf equation. Moreover, a special solution of the Hopf equation with some subsidary conditions satisfies the Navier-Stokes equation. In this sense, we may rederive the Navier-Stokes equation from the Euler equation.

2.1. The derivation of the Navier-Stokes equation.

2.1.1. **The variational derivation of the Euler equation.** Let (M,g) be a d-dimensional compact smooth manifold without boundary. We denote by $\mathcal{X}(M)$ (or $\mathcal{X}_\sigma(M)$) the space of all (solenoidal) vector fields on M. Let $\{\Phi_t\}_{0\le t\le 1}$, be a one parameter family of $\mathrm{Diff}_\sigma(M)$. whose jacobian are equal to 1. (A subscripted variable will never stand for the differentiation in this section.)

We consider the following variational problem (VP):

(VP) *Let $F=\{F_t(\cdot)\}_{0\le t\le 1}$, $F_t\in\mathcal{X}(M)$ be given such that $F_0=F_1=0$. Given Φ^0 and $\Phi^1\in\mathrm{Diff}_\sigma(M)$, find a one parameter family of diffeomorphisms $\Phi=\{\Phi_t\}_{0\le t\le 1}$ such that*

- $\Phi_0=\Phi^0$, $\Phi_1=\Phi^1$ *and*
- Φ *attains a stationary point of the following energy functional:*

$$J(\Phi;F)=\int_0^1\int_M g\Big(\big(\frac{\partial}{\partial t}\Phi_t^i(x)+\tilde F_t^i(x)\big)\frac{\partial}{\partial x^i},\big(\frac{\partial}{\partial t}\Phi_t^j(x)+\tilde F_t^j(x)\big)\frac{\partial}{\partial x^j}\Big)d_gxdt \tag{2.1}$$

where $\Phi_t(x)=(\Phi_t^1(x),\cdots,\Phi_t^d(x))$, $F_t(x)=F_t^j(x)\frac{\partial}{\partial x^j}$ and $\tilde F_t^j(x)=F_t^j(\Phi_t(x))$.

Here and what follows, we use Einstein's convention for contracting indeces.

We have the following result due to Arnold [1]:

Theorem 2.1 (Arnold). *Let Φ be a desired solution of the above variational problem. Then, the vector field u defined by $u(x,t)=u^j(x,t)\frac{\partial}{\partial x^j}$ with $u^j(x,t)=\frac{\partial}{\partial t}\Phi_t^j(x)$, satisfies the Euler equation in (M,g) with the exterior force F. That is, u satisfies the following equations:*

$$\begin{cases}\dfrac{\partial}{\partial t}u(x,t)+\nabla_{u(x,t)}u(x,t)+\nabla p(x,t)=f(x,t),\\ u(x,0)=u_0(x).\end{cases}\tag{2.2}$$

where $f(x,t)=\frac{\partial}{\partial t}F_t(x)$.

2.1.2. **The derivation of the Brownian affected Euler equation.** We don't describe here this procedure because we need to explain too many things.

2.1.3. **From B.A.E.E to the Reynolds equation and the Hopf equations.** Without explaining terminologies needed to understand the statements, we enumerate some results:

Definition 2.1. *A weak solution of the Reynolds equation on $(0,T)$ $(T\le\infty)$ is a vector field $u(\cdot,\cdot)$ belonging to $L^\infty(0,T;\mathbb{H})\cap L^2(0,T;\mathbb{V})$ and a $\mathbb{H}$ valued stochastic process $\{\delta u(t,\omega)\}_{0<t<T}$ such that*

$$E_\omega(\delta u(t,\omega))=0,\qquad E_\omega(\|\delta(t,\omega)\|^2)\in L^1(0,T)$$

and that

$$\begin{aligned}(2.3)\qquad &\int_0^T \left[-(\overline{u(t)}, v'(t)) + \nu((\overline{u(t)}, v(t))) + b(\overline{u(t)}, \overline{u(t)}, v(t))\right] dt \\ &= (\overline{u_0}, v(0)) + \int_0^T \big(f(t) - E_\omega\big(B(\delta u(t,\omega), \delta u(t,\omega))\big), v(t)\big)\, dt\end{aligned}$$

for some $\overline{u_0} \in \mathbb{H}$ *and all* $v \in C_0([0,T); \mathcal{X}_{0,\sigma}(M))$.

Definition 2.2. *A weak solution of the Reynolds equation will be called real if moreover the family of probabilities* $\{\mu'_t\}_{0<t<\infty}$ *defined on* $\mathbb{H}$ *by*

$$\mu'_t(A) = Prob(\omega; \delta u(t,\omega) + \overline{u(t)} \in A) \qquad for\ A \in \{Borel\ subsets\ of\ \mathbb{H}\}$$

is a weak solution of the Hopf equation with some initial data μ_0 *satisfying*

$$\int_{\mathbb{H}} |u|^2 \mu_0(du) < \infty$$

and $\int_{\mathbb{H}} u\mu'_0(du) = \overline{u_0}$.

Following theorem is due to Foiaş [21, 22].

Theorem 2.2. *Let* $\{\mu_t\}_{0<t<\infty}$ *be a weak solution of the Hopf equation with initial data* μ_0 *satsfying above condition. Define* $\overline{u(t)}$ *and* $\overline{u_0}$ *by*

$$\overline{u_0} = \int_{\mathbb{H}} u\mu_0(du), \qquad \overline{u(t)} = \int_{\mathbb{H}} u\mu_t(du).$$

Then, there exists a stochastic process $\{\delta u(t,\omega)\}_{0<t<\infty}$ *such that* $\left\{\overline{u(\cdot,\cdot)}, \delta u(\cdot,\cdot;\omega)\right\}$ *is a real solution of the Reynolds equation satisfying*

$$\mu'_t = \mu_t \qquad for \quad 0 < t < \infty.$$

Theorem 2.3. *Let* μ_0 *be a Borel probability measure on* $\mathbb{H}$ *satisfying above condition. Among all weak solutions* $\{\mu_t\}_{0<t<\infty}$ *of the Hopf equation satisfying the energy inequality of the strong form with initial data* μ_0*, there exists at least one* $\{\mu_t^{min}\}_{0<t<\infty}$ *minimizing*

$$\int_0^T \| \int_{\mathbb{H}} u\mu_t(du)\|^2 dt.$$

If $\left\{\overline{u(\cdot,\cdot)^{min}}, \delta u(\cdot,\cdot;\omega)^{min}\right\}$ *denotes the solution of the Reynolds equation corresponding to* $\{\mu_t^{min}\}_{0<t<\infty}$ *, then* $u^{min}(t)$ *and* $\overline{B(\delta u(t,\omega)^{min}, \delta u(t,\omega)^{min})}$ *are uniquely determined a.e. on (0,T).*

2.1.4. **The Hopf equation to the Navier-Stokes equation** *(Foiaş's and Inoue's arguments).* We construct a mathematical link between the Navier-Stokes equation and the Euler equation via the Hopf equation. Adding a random force to the Euler equation, we derive a vector field satisfying the Reynolds equation which gives a Borel measure being a strong solution of the Hopf equation. From this, we deduce a solution of the Navier-Stokes equation.

2.2. **Problems.**

2.2.1. **Taylor's problem with bottom and ceiling.** There is a mathematical theory when concetric rotating cylinders has infinite length. What occurs mathematically, when there are bounds of cylinder? See, Benjamin [5, 6]

Donoho et al [19] seemingly discuss the observation limitation? and the observation support?

2.2.2. **Save miserable gold fishes.** Can we have a weak solution for the Navier-Stokes equation in a Lipshitz domain in space-time? See, Inoue & Wakimoto [35].

2.2.3. **Small another dimension, Hale's idea of the domain perturbation.** In 2-dimension, the Navier-Stokes equation has an attractor with finite dimensions. Hale in ODE school, use these information to construct a good solution for the Navier-Stokes equation on a thin domain!

2.2.4. **Far from obstacle, have we ideal fluid flow?** Physisists say that we may consider flow as an ideal one when we are far from obstacle. How do we justify this statement?

2.2.5. **Find another construction of suitably weak solution of CKN.** Use wavelets as Galerkin basis for constructing weak solution of Hopf type. Seek some relation for generalized energy integral. See, Caffarelli, Kohn and Nirenberg [10], Lin [39].

3. Feynman's problem and related ones

3.1. **Feynman's problem.** Feynman introduced the expression

$$E(t,s:q,q') = \int_{C_{t,s;q,q'}} d_F\gamma \exp\left[i\hbar^{-1}\int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))d\tau\right], \quad L(t,\gamma,\dot\gamma) = \frac{1}{2}|\dot\gamma|^2 - V(t,\gamma)$$

$$\text{where} \quad C_{t,s;q,q'} \sim \{\gamma(\cdot)\in C([s,t]:\mathbb{R}^m)\,|\,\gamma(s)=q',\ \gamma(t)=q\},$$

and rederived the Schrödinger equation, not by substituting $-i\hbar\partial_q$ into $H(t,q,p) = \frac{1}{2}|p|^2 + V(t,q)$.

(A) This expression contains the notorious Feynman measure $d_F\gamma$. That is, it is proved unfortunately that there exists no non-trivial 'Feynman measure' on ∞-dimensional spaces $C_{t,q,q'}$. Therefore, one of our main concern is how to 'justify' the results obtained by using such a notorious measure.

Why it is necessary to do so? Because, even if the usage of the Feynman measure is prohibited in mathematics, they get new insights in "quantum area" by 'using' it, for example, works done by E. Witten and other physicists.

Problem 1: Without using measure theory, how should one define path-integrals? There exists at least two candidates. A candidate of this direction would be to develop a theory of functional derivative equations, see, Inoue [29, 30, 31], or Gaussian functional integral such as Simon [48, 49], Glim and Jaffe [25]. Another one would be the generalized Riemann integral developped by Henstock [26], Pfeffer [44], Bartle [3], etc, which proceed without measure theoretic preparation.

Remark : It is well-known that the so-called Feynman measure does not exist.

(B) On the other hand, Feynman's derivation is efficiently used to construct a fundamental solution of the Schrödinger equation for suitable potentials. That is, a Fourier Integral Operator

$$U(t,s)u(q) = (2\pi\hbar)^{-m/2}\int_{\mathbb{R}^m} dq'\, D^{1/2}(t,s;q,q')e^{i\hbar^{-1}S(t,s;q,q')}u(q')$$

gives a "good parametrix" of the Schrödinger equation (shown by Fujiwara [23]). Here, $S(t,s;q,q')$ satisfies **the Hamilton-Jacobi equation** and $D(t,s;q,q')$, **the van Vleck determinant** of $S(t,s;q,q')$, satisfies the continuity equation. (**Good parametrix** means that not only it gives a parametrix but also its dependence of $\hbar$ is explicit, in other word, **Bohr's correspondence principle** is easily recovered via **stationary phase method** making $\hbar \to 0$. Feynman's main idea is to apply the stationary phase method to the path-integral formulation of the quantum mechanics when $\hbar \to 0$, by that we may recover the classical mechanics rather transparently.)

This formula is reformulated (by Inoue [32]) in the Hamiltonian form as

$$\begin{aligned} U_H(t,s)\,u(q) &= (2\pi\hbar)^{-m/2}\int_{\mathbb{R}^m} dp\, D_H^{1/2}(t,s\,;q,p)e^{i\hbar^{-1}S_H(t,s\,;q,p)}\hat{u}(p) \\ &= (2\pi\hbar)^{-m}\int_{\mathbb{R}^m\times\mathbb{R}^m} dp\,dq'\, D_H^{1/2}(t,s\,;q,p)e^{i\hbar^{-1}(S_H(t,s\,;q,p)-q'p)}u(q'). \end{aligned}$$

On the other hand, Feynman posed the following problem:

> ...... **path integrals suffer grievously from a serious defect**. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

3.2. **A partial answer of Feynman's problem.** Now, we give a partial answer of this problem by taking the Weyl equation as the simplest model with spin. That is, we rederive the Weyl equation from the Hamiltonian mechanics on superspace (called pseudo classical mechanics). More precisely speaking, introducing fermion variables to decompose the matrix structure, we define a Hamiltonian function on the superspace from which we construct solutions of the superspace version of the Hamilton-Jacobi and the continuity equations, respectively. (The fermion variables are assumed to have the inner structure represented by a countable number of Grassmann generators with the Fréchet topology.) Defining a Fourier Integral Operator with phase and amplitude given by these solutions, we may define the good parametrix for the (super) Weyl equation. This means, back to the ordinary matrix-valued representation, that we rederive the Weyl equation and therefore we give a partial solution of Feynman's problem ("partial" because we have not yet constructed an explicit integral representation of the fundametal solution itself using superanalysis. The desired explicit expression contains, for example, Maslov index and Berry's phase etc.).

We reformulate the above problem in mathematical language as follows:

Problem 2: Find a "**good representation**" of $\psi(t,q):\mathbb{R}\times\mathbb{R}^3\to\mathbb{C}^2$ satisfying

$$\text{(W)}\qquad \begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi(t,q)=\mathbb{H}(t)\psi(t,q),\\ \psi(\underline{t},q)=\underline{\psi}(q). \end{cases}$$

Here, $\underline{t}$ is arbitrarily fixed and

$$\text{(3.1)}\qquad \mathbb{H}(t)=\mathbb{H}\Big(t,q,\frac{\hbar}{i}\frac{\partial}{\partial q}\Big)=\sum_{k=1}^{3}c\boldsymbol{\sigma}_k\Big(\frac{\hbar}{i}\frac{\partial}{\partial q_k}-\frac{\varepsilon}{c}A_k(t,q)\Big)+\varepsilon A_0(t,q)$$

with the Pauli matrices $\{\boldsymbol{\sigma}_j\}$.

In order to get a good parametrix, we transform the Weyl equation (W) on the Euclidian space $\mathbb{R}^3$ with value $\mathbb{C}^2$ to the super Weyl equation (SW) on the superspace $\mathfrak{R}^{3|2}$ with value $\mathfrak{C}$:

$$\text{(SW)}\qquad \begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t,x,\theta)=\mathcal{H}\Big(t,x,\dfrac{\hbar}{i}\dfrac{\partial}{\partial x},\theta,\dfrac{\partial}{\partial\theta}\Big)u(t,x,\theta),\\ u(\underline{t},x,\theta)=\underline{u}(x,\theta). \end{cases}$$

Remark. For example, the operators

$$\sigma_1\Big(\theta,\frac{\partial}{\partial\theta}\Big)=\theta_1\theta_2-\frac{\partial^2}{\partial\theta_1\partial\theta_2},\quad \sigma_2\Big(\theta,\frac{\partial}{\partial\theta}\Big)=i\Big(\theta_1\theta_2+\frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big),$$
$$\sigma_3\Big(\theta,\frac{\partial}{\partial\theta}\Big)=1-\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2},$$

act on $u(\theta_1,\theta_2)=u_0+u_1\theta_1\theta_2$ as $\boldsymbol{\sigma}_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\boldsymbol{\sigma}_2=\begin{pmatrix}0&i\\-i&0\end{pmatrix}$, $\boldsymbol{\sigma}_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, respectively.

Theorem 3.1. *Let* $\{A_j(t,q)\}_{j=0}^3\in C^\infty(\mathbb{R}\times\mathbb{R}^3:\mathbb{R})$ *satisfy, for any* $k=0,1,2,\cdots$,

$$\text{(3.2)}\qquad |\!|\!|A_j|\!|\!|_{k,\infty}=\sup_{t,q,|\gamma|=k}|(1+|q|)^{|\gamma|-1}\partial_q^\gamma A_j(t,q)|<\infty\quad \textit{for}\quad j=0,\cdots,3.$$

We have a "good parametrix" for (SW) *represented by*

$$\mathcal{U}(t,\underline{t})\,u(x,\theta)=(2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}}d\xi\,d\pi\,\mathcal{D}^{1/2}(t,\underline{t}\,;x,\theta,\xi,\pi)e^{i\hbar^{-1}\mathcal{S}(t,\underline{t}\,;x,\theta,\xi,\pi)}\mathcal{F}\underline{u}(\xi,\pi).$$

Here, $\mathcal{S}(t,\underline{t}\,;x,\theta,\xi,\pi)$ *and* $\mathcal{D}(t,\underline{t}\,;x,\theta,\xi,\pi)$ *satisfy the Hamilton-Jacobi equation and the continuity equation, respectively:*

$$\text{(H−J)}\quad \begin{cases} \dfrac{\partial}{\partial t}\mathcal{S}+\mathcal{H}\Big(t,x,\dfrac{\partial\mathcal{S}}{\partial x},\theta,\dfrac{\partial\mathcal{S}}{\partial\theta}\Big)=0,\\ \mathcal{S}(\underline{t},\underline{t}\,;x,\theta,\xi,\pi)=\langle x|\xi\rangle+\langle\theta|\pi\rangle, \end{cases}$$

and

$$\text{(C)}\quad \begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}+\dfrac{\partial}{\partial x}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\xi}\Big)+\dfrac{\partial}{\partial\theta}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\pi}\Big)=0,\\ \mathcal{D}(\underline{t},\underline{t}\,;x,\theta,\xi,\pi)=1. \end{cases}$$

Here, for $u(x,\theta)=u_0(x)+u_1(x)\theta_1\theta_2$, Fourier transformation $\mathcal{F}$ is defined by

$$\mathcal{F}u(\xi,\pi)=(2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}}dxd\theta\,e^{i\hbar^{-1}(\langle x|\xi\rangle+\langle\theta|\pi\rangle)}u(x,\theta)=\hat{u}_1(\xi)+\hat{u}_0(\xi)\pi_1\pi_2.$$

Using the identification maps

$$\sharp : L^2(\mathbb{R}^3 : \mathbb{C}^2) \to \mathfrak{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2}) \quad \text{and} \quad \flat : \mathfrak{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2}) \to L^2(\mathbb{R}^3 : \mathbb{C}^2),$$

$$\sharp \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (x, \theta) = u_0(x) + u_1(x)\theta_1\theta_2 \quad \text{with} \quad u_j(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_q^\alpha \psi_{j+1}(x_{\mathrm{B}}) x_{\mathrm{S}}^\alpha$$

for $x = x_{\mathrm{B}} + x_{\mathrm{S}}$, $j = 0, 1$,

$$(\flat u)(q) = \begin{pmatrix} \psi_1(q) \\ \psi_2(q) \end{pmatrix} \quad \text{with} \quad \psi_1(q) = u(x,\theta)\big|_{x=q,\theta=0}, \ \psi_2(q) = \frac{\partial^2}{\partial\theta_2\partial\theta_1} u(x,\theta)\bigg|_{x=q,\theta=0},$$

we get

Corollary 3.2. *Let* $\{A_j(t,q)\}_{j=0}^3 \in C^\infty(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})$ *satisfy* (3.2). *We have a good parametrix for* (W) *represented by*

$$\mathbb{U}(t,\underline{t})\underline{\psi}(q) = \flat(2\pi\hbar)^{-3/2}\hbar \int_{\mathfrak{R}^{3|2}} d\xi \, d\pi \, \mathcal{D}^{1/2}(t,\underline{t}; x,\theta,\xi,\pi) e^{i\hbar^{-1}\mathcal{S}(t,\underline{t};x,\theta,\xi,\pi)} \mathcal{F}(\sharp\underline{\psi})(\xi,\pi)\bigg|_{x_{\mathrm{B}}=q}.$$

Remark: Feynman's problem should be considered also as the question of obtaining explicit solution formula for a certain PDE. For example, Strichartz estimate for a linear Schrödinger equation is derived by using the explicit representation of the solution. See, Taylor [50], Burq et al [9].

For example in [50], the fundamental solution $S(t,x) = e^{-it\Delta}\delta(x)$ of

$$i\frac{\partial u}{\partial t} = \Delta u \quad \text{with} \quad u(0,x) = \delta(x) \quad \text{on } S^d$$

has the special form at time $t = 2\pi m/n$. That is, when $d = 1$, we have

$$S(2\pi m/n, x) = \frac{1}{n} \sum_{j=0}^{n-1} G(m,n,j)\delta_{2\pi j/n} \quad \text{with} \quad G(m,n,j) = \sum_{\ell=0}^{n-1} e^{2\pi i(\ell^2 m + \ell j)/n}.$$

Problem 3: Find the deep and subtle relation between $H(t,x) = e^{t\Delta}\delta(x)$ and $S(t,x)$ above.

3.3. **Problem for system of PDE.** We regard Feynman's problem as calling a **new methodology of solving systems of PDE**. By the way, a system of PDE has **two non-commutativities**:
(i) One from $[\partial_q, q] = 1$ (Heisenberg relation),
(ii) the other from $[A, B] \neq 0$ (A, B are coefficient matrices of a system of PDE).

Non-commutativity from Heisenberg relation is nicely treated by using Fourier transformations (the theory of Ψ.D.Op).

Here, we want to give a **new method of treating non-commutativity** $[A, B] \neq 0$, using Fourier transformations on functions on superspace $\mathfrak{R}^{m|n}$.

Dogmatic opinion. For a given system of PDE, if we may reduce that system to scalar PDEs by diagonalization, then we must doubt whether it is truely necessary to use matrix representation. Therefore, if we need to represent some equations using matrices, we should try to treat system of PDE as it is, **without diagonalization.**

Remark. We may consider the method employed here, as a trial to **extend the "method of characteristics" to PDE with matrix-valued coefficients.**

3.4. **Efetov's representation formula.** Let $\mathfrak{U}_N$ be a set of Hermitian $N \times N$ matrices, which is identified with $\mathbb{R}^{N^2}$ as a topological space. In this set, we introduce a probability measure $d\mu_N(H)$ on $\mathfrak{U}_N$ by

$$d\mu_N(H) = \prod_{k=1}^{N} d(\Re H_{kk}) \prod_{j<k}^{N} d(\Re H_{jk}) d(\Im H_{jk}) P_{N,J}(H), \tag{3.3}$$
$$P_{N,J}(H) = Z_{N,J}^{-1} \exp\Big[-\frac{N}{2J^2} \operatorname{tr} H^* H \Big]$$

where $H = (H_{jk})$, $H^* = (H^*_{jk}) = (\overline{H}_{kj}) = {}^t\overline{H}$, $\prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk})$ being the Lebesgue measure on $\mathbb{R}^{N^2}$, and $Z_{N,J}^{-1}$ is the normalizing constant given by $Z_{N,J} = 2^{N/2}(J^2\pi/N)^{3N/2}$.

Let $E_\alpha = E_\alpha(H)$ $(\alpha = 1, \cdots, N)$ be real eigenvalues of $H \in \mathfrak{U}_N$.

We put

$$\rho_N(\lambda) = \rho_N(\lambda; H) = N^{-1} \sum_{\alpha=1}^{N} \delta(\lambda - E_\alpha(H)), \tag{3.4}$$

where δ is the Dirac's delta. Denoting

$$\langle f \rangle_N = \langle f(\cdot) \rangle_N = \int_{\mathfrak{U}_N} d\mu_N(H)\, f(H),$$

for a function f on $\mathfrak{U}_N$, we get

Theorem 3.3 (Wigner's semi-circle law).

$$\lim_{N\to\infty} \langle \rho_N(\lambda) \rangle_N = w_{sc}(\lambda) = \begin{cases} (2\pi J^2)^{-1}\sqrt{4J^2 - \lambda^2} & \textit{for} \quad |\lambda| < 2J, \\ 0 & \textit{for} \quad |\lambda| > 2J. \end{cases} \tag{3.5}$$

Remark. By definition, the limit in (3.5) is interpreted as

$$\lim_{N\to\infty} \langle \phi, \int_{\mathfrak{U}_N} d\mu_N(H)\, N^{-1} \sum_{\alpha=1}^{N} \delta(\cdot - E_\alpha(H)) \rangle = \langle \phi, w_{sc} \rangle = \int_{\mathbb{R}} d\lambda\, \phi(\lambda) w_{sc}(\lambda)$$

for any $\phi \in C_0^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R})$. $\langle \cdot, \cdot \rangle$ stands for the duality between $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$. We need more interpretation to give the meaning to $\int_{\mathfrak{U}_N} d\mu_N(H)\, N^{-1} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H))$.

Seemingly, there exist several methods to prove this fact. Here, we want to explain a new derivation of this fact using odd variables obtained by Efetov. That is, the key expression obtained by introducing new auxiliary variables, is

$$\langle \rho_N(\lambda) \rangle_N = \pi^{-1} \Im \int_{\Omega} dQ \left(\{(\lambda - i0)I_2 - Q\}^{-1} \right)_{bb} \exp\left[-N\mathcal{L}(Q) \right] \tag{3.6}$$

where I_n stands for $n \times n$-identity matrix and

(3.7)

$$\mathcal{L}(Q) = \operatorname{str}\left[(2J^2)^{-1} Q^2 + \log((\lambda - i0)I_2 - Q) \right],$$
$$\Omega = \Big\{ Q = \begin{pmatrix} x_1 & \rho_1 \\ \rho_2 & ix_2 \end{pmatrix} \,\Big|\, x_1, x_2 \in \mathfrak{R}_{\mathrm{ev}},\ \rho_1, \rho_2 \in \mathfrak{R}_{\mathrm{od}} \Big\} \cong \mathfrak{R}^{2|2}, \quad dQ = \frac{dx_1 dx_2}{2\pi} d\rho_1 d\rho_2,$$
$$\left(((\lambda - i0)I_2 - Q)^{-1} \right)_{bb} = \frac{(\lambda - i0 - x_1)(\lambda - i0 - ix_2) + \rho_1\rho_2}{(\lambda - i0 - x_1)^2(\lambda - i0 - ix_2)}.$$

Here in (3.6), **the parameter N appears only in one place**. This formula is formidably charming but **not yet directly justified**, like Feynman's expression of certain quantum objects using his measure.

3.5. **Deformation quantization.** We follow the presentation in Hirshfeld and Henselder [27].

Deformation quantization achieves the passage from classical mechanics to quantum mechanics by the replacement of the pointwise multiplication of functions on phase space by the star product.

> Let M be an open subset of $\mathbb{R}^d$ with a Poisson structure
> $$\{f,g\} = \alpha^{ij}(x)\partial_i f(x)\partial_j g(x) \quad \text{with} \quad \alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0$$
> Find an associative product $\bigstar$ on $C^\infty(M)[[\hbar]]$ such that for $f, g \in C^\infty(M)$
> $$f\bigstar g(x) = f(x)g(x) + \frac{i\hbar}{2}\{f,g\}(x) + O(\hbar^2).$$
> Kontsevich [36] gave a solution of this problem, therefore the Field medal was awarded.

A prototype is the construction of quantum electrodynamics (QED) by quantization of Maxwell's classical electrodynamics. Quantization proceeds according to Dirac's prescription of replacing the classical Poisson brackets of phase space variables by the commutators of quantum mechanical operators.

On the other hand, because of the Pauli principle, ferminionic systems do not have a classical limit in the sense that bosonic systems do.

Problem 4: How do we extend Kontsevich's result containing fermions, not only quantum mechanics but also quantum field theory?

See, for example, Cattaneo and Felder [13]. Here, they write down symbolically

$$f\bigstar g(x) = \int_{\gamma(\pm\infty)=x} f(\gamma(1))g(\gamma(0)) \exp\,\big[i\hbar^{-1}\int_\gamma d^{-1}\omega\big]d\gamma, \tag{3.8}$$

where (M,ω) is the symplectic manifold and the integral over trajectories $\gamma : \mathbb{R} \to M$ is to be understood as an expansion around the classical solution $\gamma(t) = x$, which is a constant function of time since the Hamiltonian vanishes. In that paper, they extend also the expression (3.8), to the bosonic quantum field theory.

By the way, photon and electron should be treated on the same footing, so claimed in Berezin and Marinov [8] and they introduced a prototype of superanalysis.

Problem 5: Some years before, the constructive quantum field theory is in fashion, for example Glim and Jaffe [25]. Can we use this theory to give a transparent and simple way of getting Kontsevich's result.

4. Other Problems

Problem 6: Recently, I found an article entitled *The Cauchy-Riemann equations in infinite dimensions* by Lempert [38]. Relate this to functional derivative equations (FDE)!

Problem 7: Use triangulation of manifolds and wavelets to treat eigen-values, eigen-functions of Laplacian. See, Dodziuk [18], Cheeger [14], Müller [43].

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