Blow-up solutions for quasilinear degenerate elliptic equation
(Partial Differential Equations and Time-Frequency Analysis)

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Blow-up solutions for quasilinear degenerate elliptic equation

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Abstract

We treat the equations with a positive nonlinearity in the right hand side. Namely

$$\left\{ \begin{array}{ll}
L_p(u) = \lambda f(u), & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right. \quad (0.1)$$

where

$$L_p(u) = -\text{div}(|\nabla u|^{p-2}\nabla u) \quad (0.2)$$

Here $\lambda \geq 0$, and the nonlinearity $f$ is, roughly speaking, positive, increasing and strictly convex on $[0, +\infty)$. In connection with combustion theory and other applications, we are interested in the study of positive minimal solutions. This is a résumé of the preprint [9].

1 Introduction.

In connection with combustion theory and other applications, we are interested in the study of positive solutions of the following:

$$\left\{ \begin{array}{ll}
L_p(u) = \lambda f(u), & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right. \quad (1.1)$$

where

$$L_p(u) = -\text{div}(|\nabla u|^{p-2}\nabla u) \quad (1.2)$$

Here $\lambda \geq 0$, and the nonlinearity $f$ is, roughly speaking, positive, increasing and strictly convex on $[0, +\infty)$. 
When $p = 2$, it is known that there is a finite number $\lambda^*$ such that (1.2) has a classical positive solution $u \in C^2(\Omega)$ if $0 < \lambda < \lambda^*$. On the other hand no solution exists, even in the weak sense, for $\lambda > \lambda^*$. This value $\lambda^*$ is often called the extremal value and solutions for this extremal value are called extremal solutions. It has been a very interesting problem to study the properties of these extremal solutions.

As for a nonlinearity $f(t)$ we adopt the following.

Definition 1.1 $f(t) \in C^1([0, +\infty))$, increasing, strictly convex and

$$f(0) > 0, \quad \lim_{t \to \infty} \frac{f'(t)t}{f(t)} > p - 1.$$  

Definition 1.2 (Weak solution)

A function $u \in W_0^{1,p}(\Omega)$ is called a weak solution if $f(u)$ satisfy

$$\text{dist}(x, \partial\Omega) \cdot f(u) \in L^1(\Omega)$$

and $u$ satisfies

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi - \lambda f(u)\varphi) \, dx = 0$$

for all $\varphi \in C^1_0(\Omega)$.

Lemma 1.1 Let $u \in W_0^{1,p}(\Omega) \cap L^\infty$ be a weak solution. Then $\exists C > 0$ and $\exists \sigma \in (0,1)$ such that

$$\begin{cases} 
|\nabla u| \leq C, \\
|\nabla u(x) - \nabla u(y)| \leq C|x - y|^{\sigma}.
\end{cases}$$  

Then we have

Lemma 1.2

$\exists u$; a classical solution for a sufficiently small $\lambda > 0$.

2 Minimal solution and extremal solution

Definition 2.1 (Minimal solution)

The minimal solution $u_\lambda \in C^1(\overline{\Omega})$ is the smallest solution among all possible solutions.
Then we have

**Lemma 2.1** $\exists_{1} u_{\lambda} \in C^{1}(\overline{\Omega})$; the minimal solution for a sufficiently small $\lambda \geq 0$.

**Lemma 2.2** $u_{\lambda}$ satisfies:

1. $u_{\lambda} \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$,
2. For $\lambda > 0$, $u_{\lambda} > 0$ in $\Omega$ and $u_{\lambda} = 0$ on $\partial\Omega$.
3. monotone increasing and left-continuous on $\lambda$.

**Definition 2.2 (Extremal value $\lambda^*$)**

The extremal value $\lambda^*$ is the supremum of $\mu$ such that:

(a) For $\forall \lambda \in (0,\mu]$, $\exists u_{\lambda}$ (minimal solution).

(b) The following Hardy type inequality is valid:

$$
\int_{\Omega} |\nabla u_{\lambda}|^{p-2} \left( |\nabla \varphi|^{2} + (p-2) \frac{(\nabla u_{\lambda}, \nabla \varphi)^{2}}{|\nabla u_{\lambda}|^{2}} \right) dx 
\geq \lambda \int_{\Omega} f'(u_{\lambda}) \varphi^{2} dx
$$

for any $\varphi \in V_{\lambda,p}(\Omega)$.

$$V_{\lambda,p}(\Omega) = \{ \varphi : ||\varphi||_{V_{\lambda,p}} < +\infty, \varphi = 0 \text{ on } \partial\Omega \},$$

$$||\varphi||_{V_{\lambda,p}} = \left( \int_{\Omega} |\nabla u_{\lambda}(x)|^{p-2} |\nabla \varphi|^{2} dx \right)^{\frac{1}{2}}.$$ 

Under these preparations, we see

**Proposition 2.1**

$$u_{\lambda^*}(x) = \lim_{\lambda \rightarrow \lambda^*} u_{\lambda}(x) \text{ a.e..}$$

Moreover $u_{\lambda^*} \in W_{0}^{1,p}(\Omega)$ is a weak solution.

**Proof:** From the definition of $V_{\lambda,p}(\Omega)$, we see $u_{\lambda} \in V_{\lambda,p}(\Omega)$. By the assumption we have

$$(p - 1) \int_{\Omega} |\nabla u_{\lambda}|^{p} dx \geq \lambda \int_{\Omega} f'(u_{\lambda}) u_{\lambda}^{2} dx$$
Since $u_{\lambda}$ is a solution of (2.3), we have
\[ \int_{\Omega} |\nabla u_{\lambda}|^p \, dx = \int_{\Omega} f(u_{\lambda})u_{\lambda} \, dx. \]
Then for any $\varepsilon > 0$ there is a positive number $C_{\varepsilon} > 0$ such that
\[ (p - 1 + \varepsilon)f(t)t \leq f'(t)t^2 + C_{\varepsilon}. \]
Hence
\[ \int_{\Omega} f'(u_{\lambda})u_{\lambda}^2 \, dx \leq \frac{p - 1}{p - 1 + \varepsilon} \int_{\Omega} f'(u_{\lambda})u_{\lambda}^2 \, dx + C'_{\varepsilon}. \]
Here $C'_{\varepsilon}$ is a positive number independent of each $\lambda < \lambda^*$. Then, for some positive number $C$
\[ \int_{\Omega} |\nabla u_{\lambda}|^p \, dx = \lambda \int_{\Omega} f(u_{\lambda})u_{\lambda} \, dx \leq C \]
\[ \int_{\Omega} f'(u_{\lambda})u_{\lambda}^2 \, dx \leq C, \]
and so $u_{\lambda}$ is uniformly bounded in $W^{1,p}_0(\Omega)$ for $\lambda < \lambda^*$. Therefore $\{u_{\lambda}\}$ contains a weakly convergent subsequence in $W^{1,p}_0(\Omega)$. Since $u_{\lambda}$ is increasing in $\lambda$, the limit $u^* = \lim_{\lambda \to \lambda^*} u_{\lambda}$ uniquely exists a.e. and clearly $u^* \in W^{1,p}_0(\Omega)$ becomes a weak solution.

**Definition 2.3 (Singular solution)**
A unbounded solution is called singular.

3 The linearized operator of $L_p(u)$ at $u_{\lambda}$

Recall the linearized operator and $V_{\lambda,p}$:
\[ L'_p(u)\varphi = -\text{div} \left( |\nabla u|^{p-2}(\nabla \varphi + (p - 2)\frac{\nabla u \cdot \nabla \varphi}{|\nabla u|^2}\nabla u) \right). \]
When $p \geq 2$, $L_p(u)$ is Frechet differentiable in $W^{1,p}_0(\Omega)$. But if $1 < p < 2$, it is not differentiable. Therefore we have to prepare proper space for the linearized operator $L'_p(u_{\lambda})$ with $u_{\lambda}$ being the minimal solution.
Definition 3.1 Let us set
\[ \| \varphi \|_{V_{\lambda,p}} = \left( \int_{\Omega} |\nabla u_{\lambda}(x)|^{p-2} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}}, \]
\[ V_{\lambda,p}(\Omega) = \{ \varphi : \| \varphi \|_{V_{\lambda,p}} < +\infty, \varphi = 0 \text{ on } \partial \Omega \}. \]

Lemma 3.1 (Coercivity) For \( \forall \varphi \in V_{\lambda,p}(\Omega), \)
\[ \varphi \in V_{\lambda,p}(\Omega) \implies L_p'(u_{\lambda}) \varphi \in [V_{\lambda,p}(\Omega)]' \]
\[ |\langle L_p'(u_{\lambda}) \varphi, \psi \rangle_{V_{\lambda,p}' \times V_{\lambda,p}}| \geq C \| \nabla \varphi \|_{V_{\lambda,p}}^2. \]

We need more notations.

Definition 3.2
\[ F_{\lambda,p} = \{ x \in \Omega : |\nabla u_{\lambda}(x)| = 0 \}. \]

Definition 3.3
\[ \tilde{V}_{\lambda,p}(\Omega) = \left\{ \psi \in C_0^\infty(\Omega) : |\nabla \psi| \equiv 0 \text{ on some nbd of } F_{\lambda,p} \right\}. \]

Lemma 3.2 Assume that \( 0 < \lambda < \lambda^* \).
If \( p \geq 2 \), then
\[ \tilde{V}_{\lambda,p}(\Omega) \subset W^{1,p}_0(\Omega) \subset V_{\lambda,p}(\Omega), \]
If \( 1 < p < 2 \), then
\[ \tilde{V}_{\lambda,p}(\Omega) \subset V_{\lambda,p}(\Omega) \subset W^{1,p}_0(\Omega) \]

Definition 3.4 (Differentiability in \( V_{\lambda,p}(\Omega) \))
\( L_p(\cdot) \) is said to be differentiable at \( u_{\lambda} \) in the direction to \( \varphi \) in \( V_{\lambda,p}(\Omega) \), if
\[ \frac{1}{t} (L_p(u_{\lambda} + t \varphi) - L_p(u_{\lambda}) - L_p'(u_{\lambda}) \varphi) = o(1), \quad \text{in } [V_{\lambda,p}(\Omega)]', \]
In addition if \( S \) is dense in \( V_{\lambda,p}(\Omega) \), then \( L_p(\cdot) \) is said to be differentiable at \( u_{\lambda} \) in \( V_{\lambda,p}(\Omega) \) a.e. respectively.
Then we see

**Proposition 3.1** Let $u_\lambda$ be the minimal solution. Then, $L_p(\cdot)$ is differentiable at $u_\lambda$ in the direction to $\forall \varphi \in \tilde{V}_{\lambda,p}(\Omega)$.

**Definition 3.5** Let us set for $\forall$ compact set $F \subset \Omega$

$$
\text{Cap}(F, |\nabla u_\lambda|^{p-2}) = \inf \left[ \int_\Omega |\nabla u_\lambda|^{p-2} |\nabla \varphi|^2 \, dx : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ on } F \right]
$$

Then we see

**Proposition 3.2** If $\text{Cap}(F_{\lambda,p}, |\nabla u_\lambda|^{p-2}) = 0$, then $\tilde{V}_{\lambda,p}(\Omega) = V_{\lambda,p}(\Omega)$.

**Corollary 3.1** If $\text{Cap}(F_{\lambda,p}, |\nabla u_\lambda|^{p-2}) = 0$, then $L_p(\cdot)$ is differentiable at $u_\lambda$ in $V_{\lambda,p}(\Omega)$ a.e.

**Remark 3.1** The denseness of $\tilde{V}_{\lambda,p}$ in $V_{\lambda,p}$ is not completely essential in this talk. In most cases it is sufficient that a first eigenfunction can be approximated by elements in $\tilde{V}_{\lambda,p}$.

**Remark 3.2** In the case that $p \geq 2$, we have $W^{1,p}_0(\Omega) \subset V_{\lambda,p}(\Omega)$. But we can not take $W^{1,p}_0(\Omega)$ as $S$ in the definition. Because $L_p(u_\lambda + t\varphi)$ with $\varphi \in W^{1,p}_0(\Omega)$ does not belong to $[V_{\lambda,p}(\Omega)]'$ but to $[W^{1,p}_0(\Omega)]'$ in general.

But $L'_p(u_\lambda)$ is continuous from $W^{1,p}_0(\Omega)$ to its dual $[W^{1,p}_0(\Omega)]'$, hence we can give an alternative definition of differentiability of $L_p(\cdot)$ in $[W^{1,p}_0(\Omega)]'$.

**Definition 3.6 (Differentiability in $W^{1,p}_0(\Omega)$)**

Let $p \in [2, +\infty)$ and let $u_\lambda$ be the minimal solution for $\lambda \in (0, \lambda^*)$. $L_p(\cdot)$ is said to be differentiable at $u_\lambda$ in $W^{1,p}_0(\Omega)$, if for any $\varphi \in W^{1,p}_0(\Omega)$ it holds that as $t \to 0$

$$
\frac{1}{t} (L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - L'_p(u_\lambda)\varphi) = o(1), \quad \text{in } [W^{1,p}_0(\Omega)]'.
$$

**Proposition 3.3** Let $u_\lambda$ be the minimal solution for $\lambda \in (0, \lambda^*)$. If $p \in [2, +\infty)$, then $L_p(\cdot)$ is differentiable at $u_\lambda$ in direction to $W^{1,p}_0(\Omega)$. 
The linearized operator $L'_p(u_\lambda)$

Let $u_\lambda \in C^{1,\sigma}(\Omega)$ be the minimal solution.

\[
\begin{cases}
-\text{div}(|\nabla u_\lambda|^{p-2}\nabla u_\lambda) = \lambda f(u_\lambda) & \text{in } \Omega \\
u_\lambda = 0 & \text{on } \partial\Omega,
\end{cases}
\]

Lemma 4.1 For $\forall \lambda \in (0, \lambda^*)$, we have for $\forall \varphi \in C_0^1(\Omega)$:

\[
\int_\Omega |\nabla u_\lambda|^{p-1}|\nabla \varphi| \, dx \geq C \int_\Omega |\varphi| \, dx \tag{4.1}
\]

\[
\int_\Omega |\nabla u_\lambda|^{2(p-1)}|\nabla \varphi|^2 \, dx \geq C \int_\Omega \varphi^2 \, dx \tag{4.2}
\]

\[
\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla \varphi|^2 \geq C \int_\Omega \varphi^2 \, dx \tag{4.3}
\]

Here $C$ is a positive number independent of each $\varphi$.

Let us recall $F_{\lambda,p} = \{x \in \Omega : |\nabla u_\lambda| = 0\}$.

Corollary 4.1

1. $F_{\lambda,p}$ is discrete in $\Omega$.
2. $L'_p(u_\lambda): V_{\lambda,p} \rightarrow [V_{\lambda,p}]'$ is invertible.
3. $L'_p(u_\lambda)$ is extended to a self-adjoint operator on $L^2(\Omega)$.

Definition 4.1 By $I$ we denote the imbedding operator from $V_{\lambda,p}(\Omega)$ into $L^2(\Omega)$ defined by

\[
I: \varphi \in V_{\lambda,p}(\Omega) \longrightarrow \varphi \in L^2(\Omega)
\]

Then we can show

Proposition 4.1 The imbedding operator

\[
I: \varphi \in V_{\lambda,p}(\Omega) \longrightarrow \varphi \in L^2(\Omega)
\]

is compact.

Corollary 4.2 The operator

\[
M_{\lambda,p} \equiv I_{V \rightarrow L^2} \circ (L'_p(u_\lambda))^{-1}|_{L^2}
\]

is compact from $L^2(\Omega)$ into $L^2(\Omega)$. 
5 Differentiability of $u_{\lambda}$ w.r.t. $\lambda$ ($p \geq 2$)

**Theorem 5.1** Assume $2 \leq p < \infty$ and the operator $L'_{p}(u_{\lambda}) - \lambda f'(u_{\lambda})$ on $L^{2}(\Omega)$ has a positive first eigenvalue for $\forall \lambda \in (0, \lambda^{*})$.

Then $u_{\lambda}$ is left differentiable with respect to $\forall \lambda \in (0, \lambda^{*})$, and $v_{\lambda} \equiv \left( \frac{du_{\lambda}}{d\lambda} \right)_{-} \in V_{\lambda,p}(\Omega)$ satisfies

\[
\begin{align*}
L'_{p}(u_{\lambda})v_{\lambda} - \lambda f'(u_{\lambda})v_{\lambda} &= f(u_{\lambda}), & \text{in } \Omega \\
v_{\lambda} &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

**Remark 5.1 1.**

\[
\frac{1}{p-1}u_{\lambda} \leq \lambda v_{\lambda}, \quad \text{if } v_{\lambda} \text{ exists.}
\]

6 Behaviors of $u_{\lambda}$ and $\frac{du_{\lambda}}{d\lambda}$ near $\lambda = 0$

Let $\varphi_{0} \geq 0$ be the unique solution of

\[
L_{p}(\varphi_{0}) = 1 \quad \text{in } \Omega; \quad \varphi_{0} = 0 \quad \text{on } \partial \Omega.
\]

**Lemma 6.1** For $\forall \varepsilon_{0} \in (0, \lambda^{*})$, $\exists C > 0$ such that for $\forall \lambda \in [0, \varepsilon_{0}]$:

1. $\int_{\Omega} |\nabla u_{\lambda}|^{q} d\lambda \leq C \lambda^{\frac{q}{p-1}}$ for $\forall q \geq 0$.
2. $|\nabla u_{\lambda}| \leq C \lambda^{\frac{1}{p-1}}$ a.e.
3. $\lambda^{\frac{1}{p-1}} \varphi_{0} \leq u_{\lambda} \leq C \lambda^{\frac{1}{p-1}}$

**Lemma 6.2** For $\forall \varepsilon_{0} \in (0, \lambda^{*})$, $\exists C > 0$ such that we have:

If $p \geq 2$, then for $\forall \lambda \in [0, \varepsilon_{0}]$

1. $\int_{\Omega} u_{\lambda} d\lambda \geq C \lambda^{-\frac{p-2}{p-1}}$
2. $\int_{\Omega} |\nabla u_{\lambda}| d\lambda \geq C \lambda^{-\frac{p-2}{p-1}}$

If $1 < p < 2$, then for $\forall \lambda \in [0, \varepsilon_{0}]$

3. $\int_{\Omega} u_{\lambda} d\lambda \leq C \lambda^{\frac{2}{p-2}}$
4. $\int_{\Omega} |\nabla u_{\lambda}|^{2} d\lambda \leq C \lambda^{\frac{2}{p-2}}$. 
7  Positivity of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ for a small $\lambda$

Theorem 7.1 $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ has a positive first eigenvalue if $\lambda$ is sufficiently small.

In other words, $\exists \mu > 0$ such that

$$\langle (L_p'(u_\lambda) - \lambda f'(u_\lambda))\varphi, \varphi \rangle_{V_{\lambda,p} \times V_{\lambda,p}} \geq \mu \int_{\Omega} \varphi^2 dx,$$

for any $\varphi \in V_{\lambda,p}(\Omega)$.

**Proof:** A scaling argument;

$$u_\lambda = \lambda^{\frac{1}{p-1}}w_\lambda$$

Then as $\lambda \to 0$

$$w_\lambda \to w_0 :$$

$$\left\{ \begin{array}{l} L_p(w_0) = 1 \quad \text{in } \Omega \\ w_0 = 0 \quad \text{on } \partial\Omega \end{array} \right.$$  

The linearized operator at $w_0$ has a positive first eigen value! From this fact we can show the assertion.

8  Nonnegativity of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$

Definition 8.1 Let $\hat{\varphi}_\lambda \in V_{\lambda,p}(\Omega)$ be the first eigenfunction of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$

Definition 8.2 (Accessibility Condition) The first eigenfunction $\hat{\varphi}_\lambda$ is said to satisfy $(AC)$ if for $\forall \epsilon > 0$ there exists a nonnegative $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ such that

$$L_p'(u_\lambda)(\varphi - \hat{\varphi}_\lambda) + |\varphi - \hat{\varphi}_\lambda| \leq \epsilon \max(\hat{\varphi}_\lambda, \text{dist}(x, \partial\Omega)) \quad \text{in } \Omega.$$

Theorem 8.1 Assume $(AC)$. Then the 1st eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ is nonnegative.
Remark 8.1 (1) In case that $\Omega$ is radially symmetric, the minimal solution is also radial. Hence this condition is easily verified.
(2) Since $L_p$ is not Frechet differentiable in general, we need Lemma which combines $L_p$ with its linearized operator $L'(u_\lambda)$.

A Sketch of proof of Theorem:

Assume that $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ has a negative first eigenvalue $\mu$

$$L_p'(u_\lambda)\varphi - \lambda f'(u_\lambda)\varphi = \mu\varphi, \quad (\mu < 0, \varphi \in \tilde{V}_{\lambda,p}(\Omega)).$$

\[\downarrow\]

Lemma 8.1 (Key Lemma) Assume $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$. Then $\exists \psi_t \in C^0([0, T], V_{\lambda,p}(\Omega))$ s.t.

$$\begin{cases}
L_p(u_\lambda - t\psi_t(x)) = L_p(u_\lambda) - tL_p'(u_\lambda)\varphi & \text{in } \Omega, \\
\psi_t = 0 & \text{on } \partial\Omega,
\end{cases}$$

Moreover for a small $\rho > 0$ and $\Omega_\rho = \{a \in \Omega : \text{dist}(x, \partial\Omega) < \rho\}$

$$\lim_{t \to 0} ||\psi_t - \varphi||_{C^1(\overline{\Omega_\rho})} = 0.$$

\[\downarrow\]

For small $\forall t > 0, \exists x_t \in \Omega$ and $\exists r_t > 0$ s.t.

$$L_p(u_\lambda) - tL_p'(u_\lambda)\varphi \leq \lambda f(u_\lambda - t\psi_t) \quad \text{in } B_{r_t}(x_t).$$

\[\downarrow\]

$$0 \leq \lambda f'(u_\lambda)(\varphi - \psi_t) + \mu\varphi + o(1)|\psi_t| \quad \text{in } B_{r_t}(x_t).$$

Or,

$$0 \leq \lambda f'(u_\lambda)\left(1 - \frac{\psi_t}{\varphi}\right) + \mu + o(1)\frac{|\psi_t|}{\varphi} \quad \text{in } B_{r_t}(x_t).$$

\[\downarrow\]

Since $\Omega$ is bounded, we can assume $\lim_{t \to +0} x_t = \exists x^0 \in \overline{\Omega}$.

\[\downarrow\]

$$0 \leq \mu$$

Contradiction!!
9 Proof of Key lemma

**Lemma 9.1 (Key Lemma)** Assume \( \varphi \in \tilde{V}_{\lambda,p}(\Omega) \). Then \( \exists! \psi_{t} \in C^{0}([0, T], V_{\lambda,p}(\Omega)) \) s.t.

\[
\begin{align*}
L_{p}(u_{\lambda} - t\psi_{t}(x)) = L_{p}(u_{\lambda}) - tL'_{p}(u_{\lambda})\varphi & \quad \text{in } \Omega, \\
\eta_{t} = 0 & \quad \text{on } \partial\Omega.
\end{align*}
\]

Moreover for a small number \( \rho > 0 \)

\[
\lim_{t \to 0} ||\psi_{t} - \varphi||_{C^{1}(\overline{\Omega_{\rho}})} = 0.
\]

**Extremely rough sketch of Proof:**

The former part follows from the invertibility of \( L'(u_{\lambda}) \) and monotonicity of \( L_{p} \).

The latter part follows from the energy inequalities

\[
||W_{t}||_{W^{n,2}(\Omega_{\rho'})} \leq C(n, \rho, \rho')||W_{t}||_{V_{\lambda,p}(\Omega)} + t \quad \text{as } t \to +0.
\]

involving \( W_{t} = \psi_{t} - \varphi \). After all, from Sobolev imbedding theorem the assertion follows.

10 The extremal solution

**Theorem 10.1** Let \( u_{\lambda}^{*} \) be the singular extremal solution. Moreover, assume that \( f(t) \) satisfies

\[
\frac{f'(t)}{f(t)^{\frac{p-2}{p-1}}} \quad \text{is nondecreasing on } [0, \infty).
\]

Then if \( \lambda > \lambda^{*} \), there is no solution even in the weak sense.

**Lemma 10.1** Let \( u \in W_{0}^{1,p}(\Omega) \) be a solution. Let \( \Psi \in C^{2}(\mathbb{R}) \) be concave, with \( \Psi' \) bounded and \( \Psi(0) = 0 \). Then \( v = \Psi(u) \) satisfies

\[
L_{p}(v) \geq \lambda|\Psi'(u)|^{p-2}\Psi'(u)f(u).
\]

For a given \( \epsilon \in (0, 1) \) we set

\[
\tilde{f} = (1 - \epsilon)f.
\]
\[ h(u) = \int_0^u \frac{ds}{f(s)^{\frac{1}{p-1}}} \quad \text{and} \quad \tilde{h}(u) = \int_0^u \frac{ds}{\tilde{f}(s)^{\frac{1}{p-1}}} \]

**Lemma 10.2** Assuming (10.1), we set \( \Psi(u) = \tilde{h}^{-1}(h(u)) \).

then

1. \( \Psi(0) = 0 \) and \( 0 \leq \Psi(u) \leq u \) for all \( u \geq 0 \).
2. If \( h(+\infty) < +\infty \) and \( \tilde{f} \neq f \), then \( \Psi(+\infty) < +\infty \).
3. \( \Psi \) is increasing, concave, and \( \Psi' \leq 1 \) for all \( u \geq 0 \).

**Proof of Theorem:** Assume that \( \exists u; \text{solution for some } \lambda > \lambda^* \). Set \( v = \Psi(u) = \tilde{h}^{-1}(h(u)) \). Then \( v \) satisfies

\[
\begin{cases}
L_p(v) \geq \lambda(1-\varepsilon)f(v) & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Hence \( v \) is a supersolution.

**Proposition 10.1** Assume that \( p \geq 2 \). For any \( \varphi \in V_{\lambda, p}(\Omega) \)

\[
\langle (L_p'(u_{\lambda}^*) - \lambda^* f'(u_{\lambda}^*))\varphi, \varphi \rangle_{V_{\lambda, p}' \times V_{\lambda, p}} \geq 0.
\]

A weaker result holds for \( 1 < p < 2 \).

**Proposition 10.2** Assume \( 1 < p \leq 2 \). Let \( u \in W_{0}^{1,p}(\Omega) \) be a singular solution such that for any \( \varphi \in V_{\lambda, p}(\Omega) \)

\[
\langle (L_p'(u_{\lambda}) - \lambda f'(u_{\lambda}))\varphi, \varphi \rangle_{V_{\lambda, p}' \times V_{\lambda, p}} \geq 0.
\]

Moreover we assume that

\[ |\nabla u| \geq |\nabla u_{\lambda}| \quad \text{in } \Omega \quad (p \neq 2). \]

Then we have \( \lambda = \lambda^* \) and \( u = u_{\lambda}^* \).

A weaker result holds for \( p > 2 \).
11 Weighted Hardy’s inequality in a ball

Theorem 11.1 Suppose that a positive integer $N$ and a real number $\alpha$ satisfy $N + \alpha > 2$. Then it holds that for any $u \in W_0^1(\Omega)$

$$\int_{\Omega} |\nabla u|^2 |x|^\alpha \, dx \geq H(N, \nabla, \alpha) \int_{\Omega} |u|^2 |x|^{\alpha-2} \, dx + \lambda_1 \left( \frac{\omega_N}{|\Omega|} \right) \int_{\Omega} |u|^2 |x|^\alpha \, dx.$$  

Here

$$H(N, \nabla, \alpha) = \left( \frac{n-2+\alpha}{2} \right)^2,$$

$\omega_N$ is a volume of $N$-dimensional unit ball, and $\lambda_1$ is the first eigenvalue of the Dirichlet problem given by:

$$\lambda_1 = \inf \left[ \int_{B_1^2} |\nabla v|^2 \, dx : v \in W^{1,2}_0(B_1^2), \int_{B_1^2} v^2 \, dx = 1 \right],$$

where by $B_1^2$ and $\nabla_2$ we denote the two dimensional unit ball and the gradient.

Remark 11.1 When $\alpha = 0$, this result was initially established in [3] by H. Brezis and J.L. Vázquez. They also investigated in [3] fundamental properties of blow-up solutions of some nonlinear elliptic problems.

For the sake of the self-containedness, we give a proof of Theorem in the case $\alpha = 0$. By the spherically symmetric decreasing rearrangement, it suffices to show the inequality in the case that $\Omega = B$; a unit ball in $\mathbb{R}^N$ and $u \in C_0^1(B)$ is radially symmetric. Set $u = r^{-\beta}v$ for $u \in C_0^1(B)$ and $\beta = \frac{N-2}{2}$.

$$\int_{B} |\nabla u|^2 \, dx = H(N, \nabla, 0) \int_{B} \frac{u^2}{|x|^2} \, dx$$  

$$= N\omega_N \left( \int_{0}^{1} |u'|^2 r^{N-1} \, dr - H(N, \nabla, 0) \int_{0}^{1} u^2 r^{N-3} \, dr \right)$$  

$$= N\omega_N \left( \int_{0}^{1} |v'|^2 r \, dr \right) \geq \lambda_1 N\omega_N \int_{0}^{1} v^2 r \, dr$$  

$$= \lambda_1 \int_{B} u^2 \, dx$$

This proves the assertion.
Example

\[
\begin{align*}
 f_q(u) &= (1 + u)^q, \quad (q > p - 1) \\
 f_e(u) &= e^u. \\
 \lambda_N(p, q) &= \left(\frac{p}{q-p+1}\right)^{p-1} (N - \frac{pq}{q-p+1}), \\
 \lambda_N(p) &= p^{p-1} (N - p). \\
 U_{p,q}(r) &= r^{-Q} - 1, \quad Q = \frac{p}{q-p+1} \\
 U_p(r) &= -p \log r.
\end{align*}
\]

Lemma 12.1 \( U_p \in W_0^{1,p}(B) \) if \( N > p \) and \( U_{p,q} \in W_0^{1,p}(B) \) if \( N > p + pQ \).

Moreover:

\[
\begin{align*}
 L_p(U_{p,q}) &= \lambda_N(p, q)(U_{p,q} + 1)^q \quad \text{in} B \\
 U_{p,q} &= 0 \quad \text{on} \ \partial B, \\
 L_p(U_p) &= \lambda_N(p)e^{U_p} \quad \text{in} B \\
 U_p &= 0 \quad \text{on} \ \partial B.
\end{align*}
\]

As \( q \to +\infty \), for any \( r \in (0, 1) \)

\[
( f_q(U_{p,q}(r)), q\lambda_N(p, q), qU_{p,q}(r) ) \to ( f_e(U_p(r)), \lambda_N(p), U_p(r) )
\]

Proposition 12.1 (Exponential case) Assume that \( 1 < p \leq 2 \). Then \( U_p \) is the singular extremal, iff \( N \geq \frac{p}{p-1} \).

Proposition 12.2 (Exponential case) Assume \( p > 2 \). Then \( U_p \) is the singular extremal, if \( N > 5p \).

Proposition 12.3 (Polynomial case) Assume \( 1 < p \leq 2 \). Then \( U_{p,q} \) is the singular extremal, iff

\[
N \geq \frac{p(1 + qQ) + 2\sqrt{pqQ}}{p-1}.
\]
Proposition 12.4 (Polynomial case) Assume $p > 2$. Then $U_{p,q}$ is the singular extremal with $f = f_p$, if

$$N \geq Q(3q - 1 + 2\sqrt{q(q - 1)}).$$

Remark 12.1 (1) When $p > 2$, it is unknown if $U_p; 5p > N \geq p_{p_{-;}}^{+3}$ ($U_{p,q}$; $Q(3q - 1 + 2\sqrt{q(q - 1)}) > N \geq$ becomes the extremal.

(2) $1 < p \leq 2$. If $N > p_{p-;}^{+3}$, then

$$L'(U_p) - \lambda_N(p)e^{U_p}$$

has a positive first eigenvalue $\mu(\lambda_N(p))$.

If $N = p_{p-;}^{+3}$, then this does not have a 1st eigenfunction in $W_0^{1,p}(B)$. However, the weighted Hardy inequality gives a positive value for $\mu(\lambda_N(p))$ defined as

$$\mu(\lambda_N(p)) = \lim_{\lambda \rightarrow \lambda_N(p)} \mu(\lambda) = \lambda_1 p^{p-2}(p - 1).$$

References


