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NOTE ON REALIZATION OF CUSP CROSS-SECTIONS OF COMPLEX HYPERBOLIC ORBIFOLDS

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INTRODUCTION

Long and Reid [3] has shown that every flat manifold of dimension $n \geq 3$ arises as some cusp cross-section of a finite volume cusped (real) hyperbolic orbifold. Mcreynolds has proved that every 3-dimensional infranil manifold is a cusp of a complex hyperbolic 2-orbifold. Long and Reid [4] has also proved that some compact flat 3-manifold cannot be a cusp cross-section of a 1-cusped finite volume hyperbolic manifold. Inthis note we give a negative answer similarly to the flat case.

Theorem 1. *There exists a 3-dimensional closed Heisenberg infranil-nilmanifold which cannot be a cusp cross-section of a 1-cusped finite volume complex hyperbolic 2-manifold.*

1. HEISENBERG LIE GROUPS

Let $\mathbf{K}$ be the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ respectively. Denote $c = 1, 2$ accroding to $\mathbb{R}, \mathbb{C}$. We define the bilinear form Q on the $\mathbf{K}$-vector space $\mathbf{K}^{n+2}$:

$$Q(z, w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}.$$

Let $P : \mathbf{K}^{n+2} - \{0\} \longrightarrow \mathbf{K}\mathbb{P}^{n+1}$ be the projection onto the $c(n+1)$-dimensional $\mathbf{K}$-projective space respectively.

When $\mathrm{GL}(n+2, \mathbf{K})$ is the group of all invertible $(n+2) \times (n+2)$-matrices with entries in $\mathbf{K}$, the $\mathbf{K}$-Lorentz group $\mathbf{O}(n+1, 1; \mathbf{K})$ is defined to be the subgroup $\{A \in \mathrm{GL}(n+2, \mathbf{K}) \, |Q(Az, Aw) = Q(z, w) \ \forall \ z, w \in \mathbf{K}^{n+1}\}$. The kernel of this action is the center $C(\mathbf{K})$ isomorphic to $\{\pm 1\}$ if $\mathbf{K} = \mathbb{R}$ or the circle S^1 if $\mathbf{K} = \mathbb{C}$. Let $\mathrm{PO}(n+1, 1; \mathbf{K})$ be the quotient group $\mathbf{O}(n+1, 1; \mathbf{K})/C(\mathbf{K})$, It is customary to write $\mathrm{PO}(n+1, 1; \mathbf{K})$ as $\mathrm{PO}(n+1, 1), \mathrm{PU}(n+1, 1)$ respectively. If we choose the quadratic space $V_0^{c(n+2)-1} = \{z \in \mathbf{K}^{n+2} - \{0\} | \ Q(z, z) = 0\}$,

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then $P(V_0^{c(n+2)-1})$ is the $(c(n+1)-1)$-dimensional sphere $S^{c(n+1)-1}$ in $\mathbf{K}\mathbb{P}^{n+1}$. As the group $\mathbf{PO}(n+1,1;\mathbf{K})$ leaves $S^{c(n+1)-1}$ invariant and transitive. This gives the geometry $(\mathbf{PO}(n+1,1;\mathbf{K}), S^{c(n+1)-1})$. According to whether $\mathbf{K} = \mathbb{R}, \mathbb{C}$, we get the conformally flat geometry, $(\mathrm{PO}(n+1,1), S^n)$, the spherical CR geometry $(\mathrm{PU}(n+1,1), S^{2n+1})$. The imaginary part of $\mathbb{C}$, $\mathrm{Im}\mathbb{C}$ is the real vector space $\mathbb{R}$. (In addition, $\mathrm{Im}\mathbb{R} = 0$.)

The $\mathbf{K}$-Heisenberg nilpotent Lie group $\mathcal{N}_{\mathbf{K}}$ is the the product $\mathrm{Im}\mathbf{K} \times \mathbf{K}^n$ with group law:

$$(a,z)\cdot(b,w) = (a+b-\mathrm{Im}\langle z,w\rangle,\ z+w), \tag{1.1}$$

where the Hermitian inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{K}^n$ is defined as

$$\langle z,w\rangle = \bar{z}_1\cdot w_1 + \bar{z}_2\cdot w_2 + \cdots + \bar{z}_n\cdot w_n.$$

Here $\bar{z}$ is the complex conjugate of z. It is easy to see that $\mathcal{N}_{\mathbf{K}}$ is 2-step nilpotent, i.e. $[\mathcal{N}_{\mathbf{K}}, \mathcal{N}_{\mathbf{K}}] = (\mathrm{Im}\mathbf{K}, 0)$. Identified $(\mathrm{Im}\mathbf{K}, 0)$ with $\mathrm{Im}\mathbf{K}$, it is the central subgroup $\mathcal{C}(\mathcal{N}_{\mathbf{K}})$ of $\mathcal{N}_{\mathbf{K}}$. This induces a canonical central group extension:

$$1 \to \mathcal{C}(\mathcal{N}_{\mathbf{K}}) \to \mathcal{N}_{\mathbf{K}} \xrightarrow{P} \mathbf{K}^n \to 1. \tag{1.2}$$

Acoording to whether $\mathbf{K} = \mathbb{R}, \mathbb{C}$, $\mathcal{N}_{\mathbf{K}}$ is described as the vector space $\mathbb{R}^n$, and the Heisenberg nilpotent Lie group $\mathcal{N}$. Each space $\mathbb{R}^n$, $\mathcal{N}$ has the conformally flat structurte, spherical CR structure respectively. Let $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}})$ be the subgroup of the automorphism group $\mathrm{Aut}(\mathcal{N}_{\mathbf{K}})$ whose elements preserve the geometric structure on $\mathcal{N}_{\mathbf{K}}$ respectively. Then $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}})$ is isomorphic to the semidirect product $\mathbb{R}^n \rtimes (\mathrm{O}(n) \times \mathbb{R}^+)$, $\mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ respectively. Note that $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}})$ for $\mathbf{K} = \mathbb{C}$ are called generalized similarity transformations generated by translations, rotations and similarities in the sense of each geometry. Obviously as $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}})$ acts transitively on $\mathcal{N}_{\mathbf{K}}$, we arrive at the $\mathbf{K}$-Heisenberg geometry: $(\mathrm{Sim}(\mathbb{R}^n), \mathbb{R}^n)$, $(\mathrm{Sim}(\mathcal{N}), \mathcal{N})$. Note that the group $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}})$ acts on $\mathcal{N}_{\mathbf{K}}$ as follows: $(x \in \mathbb{R}^n, (b,v) \in \mathcal{N} = \mathbb{R} \times \mathbb{C}^n)$:

$$\begin{aligned} &\text{If } (z,(A,t)) \in \mathbb{R}^n \rtimes (\mathrm{O}(n) \times \mathbb{R}^+),\\ &\quad ((z,(A,t))\cdot x = z + t\cdot Ax.\\ &\text{If } ((a,z),(A,t)) \in \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+),\\ &\quad ((a,z),A,t))\cdot(b,v) = (a + t^2\cdot b, t\cdot Av). \end{aligned} \tag{1.3}$$

Letting $\mathrm{O}(n,\mathbf{K}) = \mathrm{O}(n), \mathrm{U}(n)$ respectively, we write the above group $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}}) = \mathcal{N}_{\mathbf{K}} \rtimes (\mathrm{O}(n,\mathbf{K}) \times \mathbb{R}^+)$.

On the other hand, we observe that $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}})$ is realized as the maximal amenable Lie subgroup of $\mathbf{PO}(n+1,1;\mathbf{K}), S^{c(n+1)-1})$. Choose the

standard basis $\{e_1, \dots, e_{n+2}\}$ of $\mathbf{K}^{n+2}$ with respect to the Hermitian form Q for which $Q(e_1, e_1) = -1, Q(e_i, e_j) = \delta_{ij}$ $(i, j = 2, \dots, n+2)$, $Q(e_1, e_j) = 0$ $(j = 2, \dots, n+2)$.
Let $P : (\mathbf{O}(n+1, 1; \mathbf{K}), V_0^{c(n+2)-1}) \to (\mathbf{PO}(n+1, 1; \mathbf{K}), S^{c(n+1)-1})$ be the equivariant projection as before.
If we put $f_1 = (e_1 + e_{n+2})/\sqrt{2}$, $f_{n+2} = (e_1 - e_{n+2})/\sqrt{2}$ respectively, then the vectors f_1, f_{n+2} lie in the cone $V_0^{c(n+2)-1}$ of $\mathbf{K}^{n+2}$. We call $P(f_1) = \infty$ the point at infinity (north pole) in $S^{c(n+1)-1}$. (Similarly, $P(f_{n+2}) = 0$ the origin (south pole) of $S^{c(n+1)-1}$.) The stabilizer at $\{\infty\}$ of the isometry group $\mathrm{Iso}(\mathbb{H}_{\mathbf{K}}^{n+1})$ is isomorphic to $\mathbf{PO}(n+1, 1; \mathbf{K})_\infty \rtimes \langle \tau \rangle$, where τ is the identity, $\mathbf{K} = \mathbb{R}$ or the involution, $\mathbf{K} = \mathbb{C}$. The geometry $(\mathbf{PO}(n+1, 1; \mathbf{K}), S^{c(n+1)-1})$ restricts the geometry $(\mathbf{PO}(n+1, 1; \mathbf{K})_\infty, S^{c(n+1)-1} - \{\infty\})$ which is isomorphic to the $\mathbf{K}$-Heisenberg geometry $(\mathrm{Sim}(\mathcal{N}_{\mathbf{K}}), \mathcal{N}_{\mathbf{K}})$. Moreover we observe how $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}})$ is realized as the stabilizer of $\mathrm{PO}(n+1, 1; \mathbf{K})$ at ∞ under the identification $\mathcal{N}_{\mathbf{K}} = S^{c(n+1)-1} - \{\infty\}$. First note that if G is a subgroup of $\mathrm{PO}(n+1, 1; \mathbf{K})$ which leaves f_1 invariant, then PG is isomorphic to $\mathrm{PO}(n+1, 1; \mathbf{K})_\infty$. Now each element g of G has the following form with respect to the basis $\{f_1, e_2, \dots, e_{n+1}, f_{n+2}\}$:

$$g = \begin{pmatrix} \lambda & \lambda^t\bar{y}B & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix}$$

satisfying that

(1) $\lambda, \mu \in \mathbf{K}^*$ with $\bar{\lambda}\mu = 1$.
(2) B is a matrix contained in $\mathrm{O}(n), \mathrm{U}(n)$ respectively.
(3) y is an n-th column vector, and $z \in \mathbf{K}$ with $\bar{z}\mu + \bar{\mu}z = |y|^2$.

Then $\mathbf{K}$-Heisenberg Lie group $\mathcal{N}_{\mathbf{K}}$ is the subgroup consisting of the following matrices for $\mathbf{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively;

$$\begin{pmatrix} 1 & {}^t\bar{y} & \frac{|y|^2}{2} \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & {}^t\bar{y} & \frac{|y|^2}{2} - \mathrm{i}a \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be checked that the correspondence

$$\begin{pmatrix} \lambda & \lambda^t\bar{y}B & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix} \mapsto ((-\mathrm{Im}(z\bar{\lambda}), y\bar{\lambda}), (B, \lambda)) \tag{1.4}$$

is an isomorphim of G onto $\mathbb{R}^n \rtimes (\mathrm{O}(n) \times \mathbb{R}^*)$ (respectively $\mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{C}^*)$.) As the center $\mathcal{C}(\mathbf{K}) = \{\pm 1\}, S^1, \{\pm 1\}$ respectively, this induces an isomorphism from $PG = \mathrm{PO}(n+1, 1; \mathbf{K})_\infty$ onto $\mathrm{Sim}(\mathcal{N}_{\mathbf{K}}) = \mathcal{N}_{\mathbf{K}} \rtimes (\mathrm{O}(n, \mathbf{K}) \times \mathbb{R}^+)$ respectively. Denote the group of rigid motions of $\mathcal{N}_{\mathbf{K}}$

by $\mathrm{E}(\mathcal{N}_{\mathbf{K}}) = \mathcal{N}_{\mathbf{K}} \rtimes \mathrm{O}(n, \mathbf{K})$. Form the group $\mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}}) = \mathrm{E}(\mathcal{N}_{\mathbf{K}}) \rtimes \langle \tau \rangle$ which is a subgroup of $\mathrm{Iso}(\mathbb{H}^{n+1}_{\mathbf{K}})_{\infty}$.

Definition 1.1. *We call* $\mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}})$ *the* **K**-*Heisenberg euclidean group. A generalized* **K**-*Heisenberg infranilmanifold (orbifold) is a compact manifold (orbifold)* $\mathcal{N}_{\mathbf{K}}/\Gamma$ *such that* Γ *is a (torsion free) discrete co-compact subgroup of* $\mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}})$. *In addition, if* Γ *belongs to* $\mathrm{E}(\mathcal{N}_{\mathbf{K}})$, *then* $\mathcal{N}_{\mathbf{K}}/\Gamma$ *is called a* **K**-*Heisenberg infranilmanifold.*

Given a noncompact finite volume hyperbolic manifold $\mathbb{H}^{n+1}_{\mathbf{K}}/G$, the form of a cusp-cross section is described as a generalized **K**-Heisenberg infranilmanifold:

$$\mathcal{N}_{\mathbf{K}}/\Gamma \quad \text{where } G_{\infty} = \Gamma \subset \mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}}). \tag{1.5}$$

An automorphism h of the **K**-Heisenberg euclidean group $\mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}})$ is defined by $h = (h_0, \hat{h}) : \mathcal{N}_{\mathbf{K}} \to \mathcal{N}_{\mathbf{K}}$, more precisely $h \in \mathrm{O}(n)$, $h = (1, \hat{h}) \in \mathrm{U}(n)$. The group $\mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}})$ acts on $\mathcal{N}_{\mathbf{K}}$ as follows (see (1.3)): if $(b, w) \in \mathcal{N}_{\mathbf{K}}$,

$$\begin{aligned}((a,z),h)\cdot(b,w) &= (a,z)\cdot h(b,w) = (a,z)\cdot(h_0(b),\hat{h}(w))\\ &= ((a + h_0(b) - \mathrm{Im}\langle z, \hat{h}(w)\rangle), z + \hat{h}(w))\end{aligned}$$

We can define a map $\Psi_{\theta} : \mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}}) \longrightarrow \mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}})$ for each real nonzero number θ:

$$\Psi_{\theta}((a,z),h) = ((\theta^2 \cdot a, \theta \cdot z), h) \tag{1.6}$$

for $(a, z) \in \mathcal{N}_{\mathbf{K}}$, $h \in \mathrm{O}(n, \mathbf{K}) \rtimes \langle \tau \rangle$.

As $((a, z), h)((b, w), g) = ((a + h(b) - \mathrm{Im}\langle z, h(w)\rangle), h \circ g)$, it is easy to see that Ψ_{θ} is an isomorphism of $\mathrm{E}^{\tau}(\mathcal{N}_{\mathbf{K}})$ onto istself.

2. Geometric boundary

We shall consider whether every Heisenberg infranilmanifold can be arised, up to diffeomorphism, as a cusp cross-section of a complete finite volume 1- cusped complex hyperbolic manifold. In [1], Burns and Epstein has obtained the CR-invariant $\mu(M)$ on the 3-dimensional strictly pseudoconvex CR-manifolds M provided that the holomorphic line bundle is trivial. Let N be a compact strictly pseudoconvex complex 2-dimensional manifold with smooth boundary M. Then they have shown the following equality in [2]:

$$\int_N c_2 - \frac{1}{3}c_1^2 = \chi(N) - \frac{1}{3}\int_N \bar{c}_1^2 + \mu(M). \tag{2.1}$$

Here $\bar{c}_1$ is a lift of c_1 by the inclusion $j^* : H^2(N, M : \mathbb{R}) \to H^2(N : \mathbb{R})$.

Let $\mathrm{E}^{\tau}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(1)$ be the 3-dimensional $\mathbb{C}$-Heisenberg euclideanm group (cf. 1.1). Let $L : \mathrm{E}^{\tau}(\mathcal{N}) \to \mathrm{U}(1)$ be the holonomy homomorphism.

Theorem 2.1. *There exists a 3-dimensional infarnilmanifold $\mathcal{N}/\Gamma$ which does not bound a complete complex hyperbolic 2-manifold (no cusp cross-section of one cusped complex hyperbolic manifold).*

Proof. There exists a 3-dimensional Heisenberg infranilmanifold $M = \mathcal{N}/\Gamma$ but not a homogeneous space and the holonomy group $L(\Gamma)$ is odd cyclic (see [5] for the classification.) Suppose that M is realized as a cusp-cross section of a complete finite volume one-cusped complex hyperbolic manifold $W = \mathbb{H}^2_{\mathbb{C}}/\pi$. Then we view M as a boundary of $\bar{W}$ where $\bar{W}\backslash\partial W$ supports a complete complex hyperbolic structure. The spherical CR-structure on M is induced from the complex hyperbolic structure on W. Let $p : \tilde{\bar{W}} \to \bar{W}$ be the finite cover, say of order ℓ, whose induced covering $\tilde{M}$ of M is now a nilmanifold (possiblly consists of a finite number of such manifolds). We may assume ℓ is odd prime (see [5]). Since W admits a complete Einstein-Kähler metric, we know that $c_2 - \frac{1}{3}c_1^2 = 0$. Moreover, since $\tilde{M}$ is a spherical CR manifold with trivial holomorphic line bundle, it follows that $\mu(\tilde{M}) = 0$. Applying the above equality to $\tilde{\bar{W}}$, we have $\chi(\tilde{W}) = \frac{1}{3}\int_{\tilde{W}} \bar{c}_1^2$. As $p^*(\bar{c}_1(W)) = \bar{c}_1(\tilde{W})$ by naturality and $p_*[\tilde{W}] = \ell[W]$,

$$\int_{\tilde{W}} \bar{c}_1^2 = \langle \bar{c}_1^2(\tilde{W}), [\tilde{W}] \rangle = \langle \bar{c}_1^2(W), \ell[W] \rangle. \tag{2.2}$$

Since $\chi(\tilde{W}) = \ell\chi(W)$, it follows that

$$3\chi(W) = \langle \bar{c}_1^2(W), [W] \rangle. \tag{2.3}$$

As a consequence, $\bar{c}_1(W)$ could be an integer, i.e. $\bar{c}_1(W) \in H^2(W, N : \mathbb{Z})$ so that $j^*\bar{c}_1(W) = c_1(W) \in H^2(W : \mathbb{Z})$.

On the other hand, given a CR structure on M, there is the canonical splitting $TM \otimes \mathbb{C} = B^{1,0} \oplus B^{0,1}$ where $B^{1,0}$ is the holomorphic line bundle. Since M is an infranilmanifold but not homogeneous, $B^{1,0}$ is nontrivial, i.e. $c_1(B^{1,0}) \neq 0$. (In fact, it is a torsion element in $H^2(N : \mathbb{Z})$, because the ℓ-fold covering $\tilde{M}$ has the trivial holomorphic bundle.) The spherical CR manifold M has a characteristic CR vector field (Reeb field) ξ. If ϵ^1 is the vector field on M pointing outward to W, then the vector fields $\langle \epsilon^1, \xi \rangle$ generates a trivial holomorphic line bundle $T\mathbb{C}$ on M for which $TW \otimes \mathbb{C}|M = B^{1,0} + T\mathbb{C}^{1,0} \oplus B^{0,1} + T\mathbb{C}^{0,1}$. In particular,

$$0 = i^*j^*(\bar{c}_1(W)) = i^*c_1(W) = c_1(B^{1,0} + T\mathbb{C}^{1,0}) = c_1(B^{1,0}),$$

which is a contradiction.

□

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