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Biholomorphic maps between asymptotic Teichmüller spaces

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1 Introduction

Let $R$ be a hyperbolic Riemann surface. The asymptotic Teichmüller space $AT(R)$ of $R$ is a quotient space of the Teichmüller space $T(R)$, which was introduced by Gardiner and Sullivan [7] when $R$ is the upper half-plane and by Earle, Gardiner and Lakic [1], [2], [6, Chap. 14] when $R$ is an arbitrary hyperbolic Riemann surface.

In this note, we investigate basic properties of asymptotic Teichmüller spaces. In particular, we prove that if $R$ is of analytically finite type, then $AT(R)$ consists of just one point. Furthermore, we prove that for a Riemann surface $R$ and a Riemann surface $R'$ from which finitely many points are removed, their asymptotic Teichmüller spaces are biholomorphically equivalent.

An element of the Teichmüller modular group $Mod(R)$ induces an isometric automorphism of $T(R)$. Similarly, an element of $Mod(R)$ also induces an isomorphism of $AT(R)$. Such an isomorphism is called geometric and the set of all geometric isomorphisms of $AT(R)$ is denoted by $G(R)$. We give a sufficient condition for $G(R)$ to act on $AT(R)$ non-trivially. This condition is crucial for further observations of the action of geometric isomorphisms.

2 Preliminaries

2.1 Teichmüller space and Teichmüller modular group

Throughout this note, we assume that a Riemann surface $R$ is hyperbolic. Namely, it is represented by a quotient space $\mathbb{H}/\Gamma$ of the upper half-plane $\mathbb{H}$ by a torsion free Fuchsian group $\Gamma$. We say that $R$ is of the analytically finite type if it is compact except for finitely many punctures. Furthermore we say that $R$ is of the topologically finite type if it is compact except for finitely many punctures and holes.
First we recall the definition of Teichmüller spaces and Teichmüller modular groups (see [12]). Fix a Riemann surface $R$. We say that two quasiconformal maps $f_1$ and $f_2$ on $R$ are equivalent if there exists a conformal map $h$ of $f_1(R)$ onto $f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every points of the ideal boundary fixed throughout. The Teichmüller space $T(R)$ with the base Riemann surface $R$ is the set of all equivalence classes $[f]$ of quasiconformal maps $f$. A distance between two points $[f_1]$ and $[f_2]$ in $T_0(R)$ is defined by $d_T([f_1],[f_2]) = \log K(f)$, where $f$ is an extremal quasiconformal map in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then $d_T$ is a complete metric on $T(R)$, which is called the Teichmüller distance.

We say that two quasiconformal automorphisms $g_1$ and $g_2$ of $R$ are equivalent if $g_2 \circ g_1^{-1}$ is homotopic to the identity by a homotopy that keeps every points of the ideal boundary fixed throughout. The Teichmüller modular group $\text{Mod}(R)$ is the set of all equivalence classes $[g]$ of quasiconformal automorphisms $g$ of $R$. Every element $\chi = [g] \in \text{Mod}(R)$ induces an automorphism $\chi_\ast$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is an isometry with respect to $d_T$. Let $\text{Isom}(T(R))$ be the group of all orientation preserving isometric automorphisms of $T(R)$, which coincides with the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism $\iota_T : \text{Mod}(R) \to \text{Isom}(T(R))$ by $\chi \mapsto \chi_\ast$. With a few exceptional surfaces, $\iota_T$ is faithful. This was first proved in [2]. Other proofs were given by Epstein [4] and Matsuzaki [10]. Furthermore, it was proved by Markovic [9] that $\iota_T$ is surjective. Hence we can identify $\text{Mod}(R)$ with $\text{Isom}(T(R))$.

### 2.2 Asymptotic Teichmüller space

We say that a quasiconformal map $f$ on $R$ is asymptotically conformal if for every $\epsilon > 0$, there exists a compact subset $E$ of $R$ such that the maximal dilatation $f$ is less than $1 + \epsilon$ on $R - E$. A Teichmüller equivalence class $[f] \in T(R)$ is called asymptotically conformal if it is represented by an asymptotically conformal map. The set of all asymptotically conformal classes in $T(R)$ is denoted by $T_0(R)$. It was proved in [2] that $T_0(R)$ is a closed and connected complex submanifold of $T(R)$.

We define the asymptotic Teichmüller space of $R$. We say that two quasiconformal maps $f_1$ and $f_2$ on $R$ are asymptotically equivalent if there exists an asymptotically conformal map $h$ of $f_1(R)$ onto $f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every points of the ideal boundary fixed throughout. The asymptotic Teichmüller space $AT(R)$ with the base Riemann surface $R$ is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal maps $f$ on $R$. Since a conformal map is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. Note that for two equivalence classes $[f_1]$ and $[f_2]$ in $T(R)$, $\pi([f_1]) = \pi([f_2])$ if and only if $[f_2 \circ f_1^{-1}] \in T_0(f_1(R))$. It was proved in [2] that the asymptotic Teichmüller space $AT(R)$ has a complex manifold structure.
such that \( \tau \) is holomorphic, and it was proved by Earle, Markovic and Saric [3] that \( T_0(R) \) and \( AT(R) \) are contractible.

### 2.3 Boundary dilatation

For a quasiconformal map \( f \) of \( R \), the boundary dilatation of \( f \) is defined by
\[
H^*(f) = \inf \{ K(f'|_{R-e}) \mid E \subset R : \text{compact} \}.
\]
Furthermore, for a point \( \tau = [f] \in T'(R) \), the boundary dilatation of \( \tau \) is defined by
\[
H(\tau) = \inf \{ H^*(g) \mid g \in [f] \}.
\]
Set \( K_0(\tau) = \inf \{ K(g) \mid g \in [f] \} \). Then clearly, \( H(\tau) \leq K_0(\tau) \). A point \( \tau \in T(R) \) is said to be a Strebel point if \( H(\tau) < K_0(\tau) \). It was proved by Lakic [8] that the set of all Strebel points are open and dense in \( T(R) \).

A distance between two points \( \tau_1 = [[f_1]] \) and \( \tau_2 = [[f_2]] \) in \( AT(R) \) is defined by
\[
d_{AT}(\tau_1, \tau_2) = \log H([f_2 \circ f_1^{-1}]).
\]
Then \( d_{AT} \) is a complete metric on \( AT(R) \), which is called the asymptotic Teichmüller distance. It was proved in [6, Chap. 15] that for any point \( [[f]] \in AT(R) \), there exists an element \( f_0 \in [[f]] \) such that \( H([[f]]) = H^*(f_0) \). We call such \( f_0 \) asymptotically extremal.

### 3 Results

#### 3.1 Biholomorphic maps

First we observe a modification of a quasiconformal map around a point.

**Lemma 1** Let \( R \) be a Riemann surface and \( p \) a point of \( R \). For a quasiconformal map \( f \) of \( R \), suppose that the Teichmüller equivalence class \( [f] \) belongs to \( T_0(R) \). Then the Teichmüller equivalence class \( [f|_{R-\{p\}}] \) belongs to \( T_0(R-\{p\}) \).

**Proof.** We take a sufficiently small constant \( \epsilon > 0 \) so that \( U_\epsilon = \{ q \in R \mid d(p, q) < \epsilon \} \) is simply connected. Since \( [f] \in T_0(R) \), we may assume that \( f \) is an asymptotically conformal map. For the Beltrami coefficient \( \mu \) of \( f \) and for \( t \in [0, 1] \), we set \( \mu_t = (1-t)\mu \) on \( U_\epsilon \) and \( \mu_t = \mu \) on \( R-U_\epsilon \). Let \( f_t \) be a quasiconformal map on \( R \) whose Beltrami coefficient is \( \mu_t \). Then \( f_t(0 \leq t \leq 1) \) is a homotopy connecting \( f_0 = f \) and \( f_1 \). We take a quasiconformal map \( h_t : f_t(R) \to f(R) \) so that \( h_t = f \circ f_t^{-1} \) on \( f_t(R)-f_t(U_\epsilon) \) and \( h_t \) is conformal on \( f_t(U_{\epsilon/2}) \) and it satisfies \( h_t \circ f_t(p) = f(p) \). Furthermore we take the \( h_t \) so that it is continuous on \( t \) and \( h_0 \) is the identity. Set \( g_t := h_t \circ f_t : R \to f(R) \), which is a homotopy connecting \( g_0 = f \) and \( g_1 \). Since \( g_t(p) = f(p) \), we have \( [g_t|_{R-\{p\}}] = [f|_{R-\{p\}}] \in T(R-\{p\}) \). Since \( g_1 \) is conformal on \( U_{\epsilon/2} \) and \( g_1 = f \) on \( R-U_\epsilon \), we see that \( g_1|_{R-\{p\}} \) is asymptotically conformal. Thus \( [f|_{R-\{p\}}] = [g_1|_{R-\{p\}}] \in T_0(R-\{p\}) \).

**Corollary 2** Let \( R \) be a Riemann surface of analytically finite type. Then \( AT(R) \) is singleton.

**Proof.** By definition, \( R \) is a compact Riemann surface \( \overline{R} \) from which at most finitely many points \( \{p_i\}_{i=1}^n \) are removed. We take an arbitrary Teichmüller
equivalent class \([f] \in T(R)\). The quasiconformal map \(f\) of \(R\) extends to a quasiconformal map \(\overline{f}\) of \(\overline{R}\) and we have \([\overline{f}] \in T(\overline{R}) = T_0(\overline{R})\). Then by Lemma \(1\), we have \([\overline{f}]_{R-\{p_1\}} \in T_0(\overline{R} - \{p_1\})\). Again by Lemma \(1\), we see that \([\overline{f}]_{R-\{p_1,p_2\}} \in T_0(R - \{p_1,p_2\})\). By repeating this process, we conclude that \([f] \in T_0(R)\), which implies the assertion. ■

On a biholomorphic equivalence between asymptotic Teichmüller spaces, we have the following.

**Theorem 3** Let \(R\) be a Riemann surface and \(p\) a point of \(R\). Then the asymptotic Teichmüller spaces \(AT(R)\) and \(AT(R - \{p\})\) are biholomorphically equivalent.

**Proof.** Every quasiconformal map \(f\) of \(R - \{p\}\) extends to a quasiconformal map \(\overline{f}\) of \(\overline{R}\). Since the map of \(T(R - \{p\})\) onto \(T(R)\) defined by \([f] \mapsto [\overline{f}]\) is holomorphic (see [12, §5.3]) and the projection \(\pi : T(R) \to AT(R)\) is holomorphic, the map \(\psi : AT(R - \{p\}) \to AT(R)\) defined by \([\overline{f}] \mapsto [\overline{f}]\) is holomorphic. We will prove that \(\psi\) is injective. Suppose that \([\overline{f}] = [\text{id}]\) in \(AT(R)\). Then \([\overline{f}] \in T_0(R)\). By Lemma \(1\), we have \([f] \in T_0(R - \{p\})\). Thus \([\overline{f}] = [\text{id}]\) in \(AT(R - \{p\})\), which means that \(\psi\) is injective. ■

For a Riemann surface \(R\) of topologically finite type with \(n\) boundary components, the asymptotic Teichmüller space \(AT(R)\) is biholomorphically equivalent to the product space \(AT(D)^n\) of the asymptotic Teichmüller of the unit disk \(D\) in \(C\). This was proved by Miyachi [11].

### 3.2 Geometric isomorphisms on \(AT(R)\)

Similar to the action of the Teichmüller modular group \(\text{Mod}(R)\) on \(T(R)\), every element \(\chi = [g] \in \text{Mod}(R)\) induces an automorphism \(\chi_*\) of \(AT(R)\) by \([f] \mapsto [f \circ g^{-1}]\), which is an isometry with respect to \(d_{AT}\). Let \(\text{Isom}(AT(R))\) be the group of all orientation preserving isometric automorphisms of \(AT(R)\). Then we have a homomorphism \(\iota_{AT} : \text{Mod}(R) \to \text{Isom}(AT(R))\) by \(\chi \mapsto \chi_*\). It is different from the case of \(\iota_T\) that the homomorphism \(\iota_{AT}\) is not faithful for any hyperbolic Riemann surface \(R\). Indeed, let \([g_0] \in \text{Mod}(R)\) be a Dehn twist along a simple closed geodesic \(c\) on \(R\). Since \([g_0]\) has a representative that is the identity outside of the collar of \(c\), we see that \([g_0] \in \ker \iota_{AT}\), whereas \([g_0] \neq [\text{id}]\) as an element of \(\text{Mod}(R)\). Hence \(\iota_{AT}\) is not faithful. Thus we define the geometric isomorphism group by

\[
G(R) = \text{Mod}(R)/\ker \iota_{AT}.
\]

We call an element of \(G(R)\) geometric isomorphism and denote the equivalence class of \([g] \in \text{Mod}(R)\) in \(G(R)\) by \([g]\).

We give a sufficient condition for \([g] \notin \ker \iota_{AT}\), namely \([g]\) acts non-trivially on \(AT(R)\). For a non-trivial simple closed curve \(c\), let \(\ell(c)\) be the hyperbolic length of the geodesic that is homotopic to \(c\), and \(d\) the hyperbolic distance on \(R\).
\textbf{Theorem 4} Let $g$ be a quasiconformal automorphism of $R$. Suppose that there exist a sequence $\{c_n\}_{n=1}^{\infty}$ of simple closed geodesics on $R$ and a positive constant $\delta$ independent of $n$ such that $d(p, c_n) \to \infty$ for a point $p \in R$ and

$$\left| \frac{\ell(g(c_n))}{\ell(c_n)} - 1 \right| \geq \delta$$

for all $n$. Then the class $[g] \in \text{Mod}(R)$ is not asymptotically conformal. Namely, the action of $[[g]] \in G(R)$ on $AT(R)$ is not trivial.

A proof of Theorem 4 is given in the author's forthcoming paper [5].

\textbf{References}


