The equivariant determinant of elliptic operators and the group action (Perspectives of Hyperbolic Spaces II)

Author(s)
Tsuboi, Kenji

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The equivariant determinant of elliptic operators and the group action.

(Kenji Tsuboi, Tokyo University of Marine Science and Technology)

1. Equivariant determinant of elliptic operators

Let $M = M^{2m}$ be a $2m$-dimensional closed connected oriented Riemannian manifold and $G$ a finite group acting on $M$. The $G$-action is assumed to be orientation-preserving, isometric and effective. Let $D : \Gamma(E) \to \Gamma(F)$ be a $G$-equivariant elliptic operator where $E, F$ are complex $G$-vector bundles. Then $\ker D$ and $\text{coker} D$ are finite dimensional $G$-modules.

Equivariant determinant of $D$ is defined by

$$G \ni g \mapsto \det(D, g) = \frac{\det(g|\ker D)}{\det(g|\text{coker} D)} \in S^1 \subset \mathbb{C}^*$$

and $\det_D := \det(D, \cdot) : G \to S^1$ is a group homomorphism.

Then an additive group homomorphism $I_D : G \to \mathbb{R}/\mathbb{Z}$ is defined by

$$I_D(g) := \frac{1}{2\pi \sqrt{-1}} \log \det(D, g) \pmod{\mathbb{Z}}.$$

This additive group homomorphism has the following properties:

$$I_D(gh) = I_D(hg) = I_D(g) + I_D(h), \quad I_D([G, G]) = 0, \quad I_D(1) = 0.$$
Next proposition is proved by using the linear algebra (see [2]).

**Proposition** If $g^p = 1$ ($p \geq 2$), we have

$$I_D(g) \equiv \frac{p-1}{2p} \text{Ind}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \xi_p^{-k}} \text{Ind}(D, g^k) \quad (\text{mod } \mathbb{Z})$$

where $\xi_p = e^{2\pi \sqrt{-1}/p}$ is the primitive $p$-th root of unity,

$$\text{Ind}(D, g^k) = \text{Tr}(g^k \mid \ker D) - \text{Tr}(g^k \mid \cok D) \in \mathbb{C}$$

is the equivariant index of $D$ evaluated at $g^k$ and

$$\text{Ind}(D) = \text{Ind}(D, 1) = \dim \ker D - \dim \cok D \in \mathbb{Z}$$

is the numerical index of $D$.

2. **Cyclic action on Riemann surfaces and its rotation angles**

   Let $\Sigma^\sigma$ be the compact Riemann surface of genus $\sigma$ ($\sigma \geq 2$). Assume that a finite group $G$ acts on $\Sigma^\sigma$ as a biholomorphic automorphism with respect to some complex structure of $\Sigma^\sigma$.

   Let $g \in G$ be any element of order $p$ and set $\mathbb{Z}_p = \langle g \rangle$. Then $\pi : \Sigma^\sigma \rightarrow \Sigma^\sigma / \mathbb{Z}_p$ is a branched covering with $b$ branch points $y_1, \cdots, y_b \in \Sigma^\sigma / \mathbb{Z}_p$ of order $(n_1, \cdots, n_b)$, where $\pi^{-1}(y_i) = \{q_i, g \cdot q_i, \cdots, g^{r_i-1} \cdot q_i\}$ consists of $r_i := p/n_i$ points.

   For $1 \leq i \leq b$, assume that $g^{r_i} \mid T_{\pi^{-1}(y_i)}\Sigma^\sigma = \xi^{t_i}_{n_i} = \xi^r_{p}$ where $1 \leq t_i \leq n_i - 1$ and $t_i$ is prime to $n_i$.

   **Problem 1** Can we determine the rotation angles $r_1 t_1, \cdots, r_b t_b$ by using the equivariant determinant?

   Let $D_\ell$ be the $\otimes^\ell T\Sigma^\sigma$-valued Dirac operator on $\Sigma^\sigma$ defined by the complex structure of $\Sigma^\sigma$. Then using the Atiyah-Singer index formula, we can show the next formula (see [2]).

   **Formula** Set

   $$\Phi_i := zt_i(n_i - 1)(7n_i - 11) + 6 \sum_{j=\left\lfloor \frac{t_i + 1}{n_i} \right\rfloor + 1}^{\left\lfloor \frac{t_i + n_i + 1}{zt_i} \right\rfloor} f_{n_i} \left( \left\lfloor \frac{tn_i - 1}{zt_i} \right\rfloor - \ell - 1 \right)$$
where \( f_{n_{i}}(x) = x^{2} - (n_{i} - 2)x - (n_{i} - 1)^{2} \) and \([\ ]\) is the Gauss's symbol. Then for any integers \( \ell, z \), \( 12p \, I_{D_{\ell}}(g^{z}) \) is an integer and we have

\[
12p \, I_{D_{\ell}}(g^{z}) \equiv 6(p - 1)(1 - \sigma)(2\ell + 1) + \sum_{i=1}^{b} r_{i} \Phi_{i} \pmod{12p}.
\]

**Remark** Assume that \( \mu \nu \) is prime to \( p \). Then since \( p \, I_{D_{\ell}}(g) = 0 \),

\[
\mu \, I_{D_{\ell}}(g^{\nu}) = \mu \nu \, I_{D_{\ell}}(g) = 0 \iff I_{D_{\ell}}(g) = 0.
\]

**Example 1** (Dihedral group) Assume that \( p \) is odd. Let

\[
G = D(p) = \langle g, h \mid g^{p} = h^{2} = 1, g^{-1}h^{-1}gh = g^{-2} \rangle
\]

be the dihedral group. Then since \( g^{-2} \in [G, G] \), it follows that

\[
I_{D_{\ell}}(g^{-2}) = -2I_{D_{\ell}}(g) = 0 \quad (\forall \ell \in \mathbb{N}) \iff I_{D_{\ell}}(g) = 0 \quad (\forall \ell \in \mathbb{N}).
\]

**Example 2** (Symmetric group) Assume that \( p \) is odd. Let \( \tau_{1} = (1, 2), \tau_{2} = (1, 3), \ldots, \tau_{p-1} = (1, p) \) be the transpositions of \( p \) letters and \( S(p) \) the symmetric group of the \( p \) letters. Let \( g \in S(p) \) be an element of order \( p \) defined by \( g = \tau_{1}\tau_{2}\cdots\tau_{p-1} = (p, p - 1, \cdots, 2, 1) \).

Then we have

\[
0 = I_{D_{\ell}}(1) = I_{D_{\ell}}((g\tau_{p-1}\cdots\tau_{2}\tau_{1})^{2})
\]

\[
= I_{D_{\ell}}(g^{2}) + I_{D_{\ell}}(\tau_{p-1}^{2}) + \cdots + I_{D_{\ell}}(\tau_{1}^{2}) = 2I_{D_{\ell}}(g)
\]

\[
\iff I_{D_{\ell}}(g) = 0 \quad (\forall \ell \in \mathbb{N}).
\]

**Problem 2** Can we determine the rotation angles \( r_{1}t_{1}, \ldots, r_{b}t_{b} \) of \( g \) under the condition that \( I_{D_{\ell}}(g) = 0 \) for any integers \( \ell \)?

Assume that the order \( p \) of \( g \) is an odd prime number hereafter.

(Hence we have \( n_{i} = p, \; r_{i} = 1 \) for \( 1 \leq i \leq b \).

Then the we have the following formula.

**Formula** (Riemann-Hurwitz equation)

\[
\sigma = p(\tau - 1) + \frac{b(p - 1)}{2} + 1 \iff \tau = \frac{1}{p} \left( \sigma - \frac{b(p - 1)}{2} - 1 \right) + 1
\]

where \( \tau \) is the genus of \( \Sigma^{\sigma}/\mathbb{Z}_{p} \).
Let $F := \{q_1, \ldots, q_b\} \subset \Sigma^\sigma$ be the fixed point set of the $\mathbb{Z}_p$-action and $\pi : \Sigma^\sigma \longrightarrow \Sigma^\tau = \Sigma^\sigma / \mathbb{Z}_p$ the branched covering with branch points $\pi(q_1), \ldots, \pi(q_b)$ of order $(p, \ldots, p)$.

Assume that $g|T_{q_i}\Sigma^\sigma = \xi_p^{t_i} \ (1 \leq t_i \leq p - 1) \text{ for } 1 \leq i \leq b$.

Set $\Sigma_0^\sigma := \Sigma^\sigma - F$ and $\Sigma_0^\tau := \Sigma_0^\sigma / \mathbb{Z}_p$, then we have the next exact sequence:

$$\pi_1(\Sigma_0^\sigma) \longrightarrow$$
$$\pi_1(\Sigma_0^\tau) = \langle a_1, b_1, \ldots, a_\tau, b_\tau, x_1, \ldots, x_b \mid \prod_{k=1}^\tau [a_k, b_k]x_1 \cdots x_b = 1 \rangle$$
$$\partial \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

where $x_i$ is represented by a loop around the branch point $\pi(q_i)$.

Let $\overline{t}_i$ denotes the mod.$p$ inverse of $t_i \ (1 \leq i \leq b)$. Then since

$$\prod_{k=1}^\tau [a_k, b_k]x_1 \cdots x_b = 1, \ \partial([a_k, b_k]) = 0, \ \partial(x_i) = \overline{t}_i \in \mathbb{Z}_p,$$

it follows that

$$\sum_i \partial(x_i) = 0 \in \mathbb{Z}_p \iff \sum_{i=1}^b \overline{t}_i \equiv 0 \pmod{p} \cdots (1)$$

Conversely, if $b \geq 2$, $\sigma$, $\tau$ satisfy the Riemann-Hurwitz equation and the equality (1) holds, then there exists a $\mathbb{Z}_p$-action on $\Sigma^\sigma$ with $b$-fixed points such that the genus of $\Sigma^\sigma / \mathbb{Z}_p$ is $\tau$ and that the rotation angles are $t_1, \ldots, t_b$ (see [1]).

By definition, rotation angles $(t_1, \ldots, t_b)$ are equivalent to the rotation angles $(t'_1, \ldots, t'_b)$ if there exists an integer $s$ such that $t'_i = st_i \ (\forall i)$ or $(t'_1, \ldots, t'_b)$ is a permutation of $(t_1, \ldots, t_b)$.

In the following tables, the equivalence class of rotation angles of $g$ is simply called "rotation angles", the rotation angles of $g$ such that $\sum_{i=1}^b t_i \equiv 0 \pmod{p}$ is called "possible rotation angles" and the rotation angles of $g$ such that $I_D_\ell(g) = 0 \ (\forall \ell \in \mathbb{N})$ is called "admissible rotation angles".
\( p = 5 \)

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( b )</th>
<th>Possible rotation angles</th>
<th>Admissible rotation angles</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>((1,1,2))</td>
<td>none</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>((1,1,1,3)) ((1,1,4,4)) ((1,2,3,4))</td>
<td>((1,1,4,4)) ((1,2,3,4))</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>((1,4))</td>
<td>((1,4))</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>((1,1,1,1,1)) ((1,1,1,2,4)) ((1,1,2,2,3))</td>
<td>((1,1,1,1,1))</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>((1,1,2))</td>
<td>none</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>((1,1,1,1,2,2)) ((1,1,1,1,3,4)) ((1,1,1,2,3,3)) ((1,1,1,4,4,4)) ((1,1,2,3,4,4))</td>
<td>((1,1,1,4,4,4)) ((1,1,2,3,4,4))</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>((1,1,1,3)) ((1,1,4,4)) ((1,2,3,4))</td>
<td>((1,1,4,4)) ((1,2,3,4))</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>((1,1,1,1,1,1,4)) ((1,1,1,1,1,2,3)) ((1,1,1,1,2,4,4)) ((1,1,1,1,3,3,3)) ((1,1,1,2,2,3,4)) ((1,1,1,3,3,4,4))</td>
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</tr>
<tr>
<td>11</td>
<td>5</td>
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$p = 7$

<table>
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<tr>
<th>$\sigma$</th>
<th>$b$</th>
<th>Possible rotation angles</th>
<th>Admissible rotation angles</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>none</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(1, 1, 3) ( (1, 2, 4) )</td>
<td>(1, 2, 4)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>5</td>
<td>none</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>(1, 1, 1, 2) ( (1, 1, 4, 5) ) ( (1, 1, 6, 6) ) ( (1, 2, 5, 6) )</td>
<td>(1, 1, 6, 6) ( (1, 2, 5, 6) )</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>(1, 6)</td>
<td>(1, 6)</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>(free action)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>(1, 1, 1, 1, 5) ( (1, 1, 1, 3, 6) ) ( (1, 1, 1, 4, 4) ) ( (1, 1, 2, 3, 5) ) ( (1, 1, 2, 4, 6) ) ( (1, 1, 3, 3, 4) )</td>
<td>(1, 1, 2, 4, 6)</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>(1, 1, 3) ( (1, 2, 4) )</td>
<td>(1, 2, 4)</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>none</td>
<td>none</td>
</tr>
</tbody>
</table>
3. Higher dimensional case

Let $M$ be a $2m$-dimensional almost complex manifold with $\mathbb{Z}_p$-action and $q_i$ ($1 \leq i \leq b$) the fixed points of the generator $g$ of $\mathbb{Z}_p$. Then for $1 \leq i \leq b$, the tangent space $T_{q_i}M$ is decomposed into

$$T_{q_i}M = \bigoplus_{j=1}^{m} V_{i,j} \quad (\dim_{\mathbb{C}} V_{i,j} = 1, \quad g|V_{i,j} = \xi_{p}^{t_{i,j}}).$$

We call $\{t_{1,1}, \cdots, t_{1,m}\}, \cdots, \{t_{b,1}, \cdots, t_{b,m}\}$ the rotation angles of $g$.

**Example 3** Let $D(5) = \langle g, h | g^5 = h^2 = 1, g^{-1}h^{-1}gh = g^{-2} \rangle$ be the dihedral group. Then since $\Sigma^5$ can be embedded symmetrically
into $\mathbb{R}^3$ with respect to the $\pi$-rotation around $x$-axis and $2\pi/5$-rotation around $z$-axis, $D(5)$ can act on $\Sigma^5$ and $g$ acts on $\Sigma^5$ with 2-fixed points of the rotation angles $(1,4), (2,3)$. Hence the diagonal action of $D(5)$ on $\Sigma^5 \times \Sigma^5$ gives an action of $g$ on $\Sigma^5 \times \Sigma^5$ with 4-fixed points of the rotation angles

$$(1,4) \times (1,4) = (\{1,1\}, \{1,4\}, \{1,4\}, \{4,4\}) ,$$

$$(1,4) \times (2,3) = (\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\})$$

and we have $I_D(g) = 0 \in \mathbb{Z}_5$ for any $D(5)$-equivariant elliptic operator $D$ because $-2I_D(g) = I_D(g^{-2}) = I_D(g^{-1}h^{-1}gh) = 0$.

Now assume that $\mathbb{Z}_5 = \langle g \rangle$ acts on $\Sigma^5 \times \Sigma^5$ and that the action preserves some almost complex structure of $\Sigma^5 \times \Sigma^5$. Let $L$ be the complex $\mathbb{Z}_5$-line bundle defined by

$$L = (\bigwedge^2 \mathcal{T}^\mathbb{C} (\Sigma^5 \times \Sigma^5))^\ell$$

and $D_\ell$ the $L$-valued Dirac operator on $\Sigma^5 \times \Sigma^5$.

**Problem 3** Can we determine the rotation angles of $g$ under the condition that $g$ has 4-fixed points and $I_{D_\ell}(g) = 0 \in \mathbb{Z}_5$ for any integers $\ell$?

Let $(\{s_1, t_1\}, \{s_2, t_2\}, \{s_3, t_3\}, \{s_4, t_4\})$ be the rotation angles of $g$. Then using the Atiyah-Singer index formula, we can prove the next equality.

$$I_{D_\ell}(g) = \frac{32}{5} (2\ell + 1)^2 - \frac{1}{5} \sum_{i=1}^{4} \sum_{k=1}^{4} \xi_5^{-k(s_i + t_i)} \frac{\xi_5^{k\ell(s_i + t_i)}}{(1 - \xi_5^{-k})(1 - \xi_5^{-ks_i})(1 - \xi_5^{-kt_i})}$$

Equivalence of rotation angles is defined as follows:

$$(\{1,2\}, \{1,2\}, \{2,3\}, \{3,4\}) \equiv (\{3,4\}, \{2,1\}, \{3,2\}, \{1,2\})$$

$$\equiv (\{2,4\}, \{2,4\}, \{4,1\}, \{1,3\}) \equiv (\{3,1\}, \{3,1\}, \{1,4\}, \{4,2\}) \equiv \cdots$$

Then we can obtain the following result.
Result. The (equivalence class of) rotation angles do not satisfy the condition $I_{D_{l}}(g) = 0$ ($\forall \ell$) unless
\[
(\{1, 1\}, \{1, 4\}, \{1, 4\}, \{4, 4\}) , \quad (\{1, 1\}, \{2, 3\}, \{1, 4\}, \{4, 4\}) ,
(\{1, 1\}, \{2, 3\}, \{2, 3\}, \{4, 4\}) , \quad (\{1, 2\}, \{1, 2\}, \{3, 4\}, \{3, 4\}) ,
(\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\})
\]
(see Example 3).

Remark. Let $N$ be the number of the equivalence classes of rotation angles. Then we have
\[
N \geq \frac{4^8}{2^4 \times 4! \times 4} = \frac{128}{3} \implies N \geq 43
\]

References