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<td>Trace fields of genus 3 surfaces with regular fundamental polygons (Perspectives of Hyperbolic Spaces II)</td>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1387: 170-178</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25798">http://hdl.handle.net/2433/25798</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Trace fields of genus 3 surfaces with regular fundamental polygons

1. Introduction

Let $\Gamma \subset \text{SL}(2, \mathbb{R})$ be a Fuchsian group. The trace field $\text{tr}(\Gamma)$ of $\Gamma$ is the field generated over $\mathbb{Q}$ by the traces of elements in $\Gamma$. In [5] M. Naätänen and T. Kuusalo determined the trace fields of all Fuchsian groups of signature $(2; 0)$ with a regular polygon as a fundamental polygon. In the present paper we shall consider the trace fields for the case of signature $(3, 0)$ analogously.

2. Regular fundamental polygons and trace fields

By Euler's formula we see that there are 4 regular polygons to be a compact surface of genus three.

1. 30-gon with each angle $2\pi/3$,
2. 20-gon with each angle $\pi/2$,
3. 14-gon with each angle $2\pi/7$,
4. 12-gon with each angle $\pi/6$.

By using a computer we can show the side-pairing patterns for each polygon.

Theorem 1. There exist 927 side-pairing patterns for 30-gon, 297 for 20-gon, 112 for 14-gon and 82 for 12-gon up to mirror images.

The following is mentioned for the case of $(2, 0)$ in [5].

Lemma 2. Let $\Gamma$ be a Fuchsian group of signature $(3; 0)$ with a regular $2n$-gon as a fundamental polygon $(n = 6, 7, 10, 15)$. Then $\Gamma$ is a subgroup of the triangle group $\Lambda_n$ of type $(2, 2n/(n-5), 2n)$.

Proposition 3.(cf. Hilden, Lozano and Montesinos-Amilibia [3]) Let $\Lambda_n^2$ be the subgroup of $\Lambda_n$ generated by the squares of the elements of $\Lambda_n$. Then it follows that

$$\text{tr}(\Lambda_n^2) \subset \text{tr}(\Gamma) \subset \text{tr}(\Lambda_n).$$

Proposition 4.(cf. Hilden, Lozano and Montesinos-Amilibia [3])

$$\text{tr}(\Lambda_n) = \mathbb{Q} \left( \cos \frac{\pi}{2n}, \cos \frac{(n-5)\pi}{2n}, \cos \frac{\pi}{2} \right) = \mathbb{Q} \left( \cos \frac{\pi}{2n} \right),$$

$$\text{tr}(\Lambda_n^2) = \mathbb{Q} \left( \cos \frac{\pi}{n}, \cos \frac{(n-5)\pi}{n}, \cos \frac{\pi}{2n} \cos \frac{(n-3)\pi}{2n} \cos \frac{\pi}{2} \right) = \mathbb{Q} \left( \cos \frac{\pi}{n} \right).$$

We denote by $C_k$ the $k$-th side of the regular $2n$-gon. Suppose that the polygon is centered at the origin such that the middle points of $C_n$ and $C_{2n}$ are real.

Lemma 5. Let $F_n$ be a hyperbolic translation of the regular $2n$-gon identifying a pair of opposite sides $C_n$ and $C_{2n}$. Then the diagonal entries of $F_n$ are equal to $1 + 4 \cos^2(\pi/n)$.

A proof of this lemma is analogous to that of Lemma 2.1 in [5].

Definition 6. A side-pairing $T$ of the regular $2n$-gon is the composite $T = R_n^k F_n R_n^{-1}$ of $F_n$ and the

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Supported by Grant-in-Aid for Scientific Research (No.15740096), The Ministry of Education, Culture, Sports, Science and Technology, Japan.
rotation $R_n$ around the origin by angle $\pi/n$. Then $T$ is said to be odd or even if $k - l$ is odd or even, respectively.

**Theorem 7.** Let $\Gamma$ be a Fuchsian group of signature $(3;0)$ with a regular $2n$-gon as a fundamental polygon. Then $\text{tr}(\Gamma) = \mathbb{Q}(\cos(\pi/n))$ if all side-pairings are even, and $\text{tr}(\Gamma) = \mathbb{Q}(\cos(\pi/(2n)))$ if some side-pairing is odd.

See Theorem 2.2 in [5] for a proof.

By considering the side-pairings for each polygons we have the following:

**Theorem 8.** The polygons only with even side-pairings are listed as follows:

<table>
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<tr>
<th>2n</th>
<th>Side-pairings</th>
<th>Trace field</th>
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<tbody>
<tr>
<td>20</td>
<td>Side-pairings in Figure 1</td>
<td>$\mathbb{Q}(\cos(\pi/10))$</td>
</tr>
<tr>
<td>14</td>
<td>Side-pairings in Figure 2</td>
<td>$\mathbb{Q}(\cos(\pi/7))$</td>
</tr>
<tr>
<td>12</td>
<td>Side-pairings in Figure 3</td>
<td>$\mathbb{Q}(\cos(\pi/6))$</td>
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Here, $P_j$ denotes the 30-gon endowed with $j$-th side-pairing pattern in [6].

![Figure 1: 20-gons only with even side-pairings](image-url)
An extremal surface of genus $g$ in the sense of C. Bavard has the regular $(12g-6)$-gon as a fundamental polygon. We see that every extremal surface of genus 3 admitting two extremal disks has the trace field $Q(\cos(\pi/30))$ (see Figure 9).

References


Figure 4: 12-gons with odd side-pairings
Figure 5: 14-gons with odd side-pairings
Figure 6: 20-gons with odd side-pairings (1)
Figure 7: 20-gons with odd side-pairings (2)
Figure 8: 20-gons with odd side-pairings (3)
Figure 9: Side-pairing patterns which induce extremal surfaces admitting two extremal disks