数理解析研究所講究録 1388 巻 2004 年 74-80

The initial value problem for Schrödinger equations on the torus

HIROYUKI CHIHARA* 千原浩之 (東北大理)
Mathematical Institute
Tohoku University
Sendai 980-8578, Japan
e-mail chihara@math.tohoku.ac.jp

This note is a summary of a paper [2]. We are concerned with the initial value problems for linear Schrödinger-type equations of the form

$$Lu \equiv \partial_t u - i\Delta u + \vec{b}(x)\cdot\nabla u + c(x)u = f(t,x) \quad \text{in} \quad \mathbb{R}\times\mathbb{T}^n, \tag{1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{T}^n, \tag{2}$$

and for semilinear Schrödinger equations of the form

$$\partial_t u - i\Delta u = F(u,\nabla u,\bar{u},\nabla\bar{u}) \quad \text{in} \quad \mathbb{R}\times\mathbb{T}^n, \tag{3}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{T}^n, \tag{4}$$

where $u(t,x)$ is a complex valued unknown function of $(t,x) = (t,x_1,\dots,x_n) \in \mathbb{R}\times\mathbb{T}^n$, $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, $i=\sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j=1,\dots,n)$, $\nabla = (\partial_1,\cdots,\partial_n)$, $\Delta = \nabla\cdot\nabla$, and $\vec{b}(x) = (b_1(x),\dots,b_n(x))$, $c(x)$, $f(t,x)$ and $u_0(x)$ are given functions. Suppose that $b_1(x),\dots,b_n(x)$ and $c(x)$ are smooth functions on $\mathbb{T}^n$, and that $F(u,v,\bar{u},\bar{v})$ is a smooth function on $\mathbb{R}^{2+2n}$, and

$$F(u,v,\bar{u},\bar{v}) = O(|u|^2+|v|^2) \quad \text{near} \quad (u,v)=0.$$

In [7], Mizohata proved that, when $x\in\mathbb{R}^n$, if the initial value problem (1)-(2) is L^2-well-posed, then it follows that

$$\sup_{(t,x,\omega)\in\mathbb{R}^{1+n}\times S^{n-1}} \left|\int_0^t \operatorname{Im}\vec{b}(x-\omega s)\cdot\omega ds\right| < +\infty, \tag{5}$$

where $\vec{b}\cdot\xi = b_1\xi_1+\cdots+b_n\xi_n$. Moreover, he gave sufficient condition for L^2-well-posedness which is slightly stronger than (5). In particular, (5) is also sufficient condition for L^2-well-posedness when $n=1$. Roughly speaking, (5) gives an upper bound of the strength of the real vector field $(\operatorname{Im}\vec{b}(x))\cdot\nabla$. In other words, if $(\operatorname{Im}\vec{b}(x))\cdot\nabla$ can be dominated by so-called local smoothing effect of $e^{it\Delta}$, then (5) must holds. After his results, many authors investigated the necessary and sufficient condition, and some weaker sufficient conditions were discovered. Unfortunately, however, the characterization of L^2-well-posedness for (1)-(2) remains open except for one-dimensional case. Such linear theories were applied to solving (1)-(2) in case $x\in\mathbb{R}^n$. See, e.g., [3] for linear equations, [1], for nonlinear equations, and references therein.

*Supported by JSPS Grant-in-Aid for Scientific Research ♯14740095.

On the other hand, the periodic case is completely different from the Euclidean case. The local smoothing effect of $e^{it\Delta}$ fails because the hamiltonian flow generated by the hamiltonian vector field $2\xi \cdot \nabla$ is completely trapped. See [4] for the relationship between the global behavior of the hamiltonian flow and the local smoothing effect.

The purpose of this note is to present the necessary and sufficient condition of L^2-well-posedness of (1)-(2), and apply this condition to (3)-(4). To state a definition and our results, we here introduce notation. Let $s \in \mathbb{R}$. $H^s(\mathbb{T}^n)$ denotes the set of all distributions on $\mathbb{T}^n$ satisfying

$$\|u\|_s^2 = \int_{\mathbb{T}^n} |(1-\Delta)^{s/2}u(x)|^2 dx < +\infty.$$

Set $L^2(\mathbb{T}^n) = H^0(\mathbb{T}^n)$, and $\|\cdot\| = \|\cdot\|_0$ for short. Let I be an interval in $\mathbb{R}$. $C(I; H^s(\mathbb{T}^n))$ denotes the set of all $H^s(\mathbb{T}^n)$-valued continuous function on I. Similarly $L^1(I; H^s(\mathbb{T}^n))$ is the set of $H^s(\mathbb{T}^n)$-valued integrable functions on I.

$$\frac{\partial}{\partial u} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} u} - i\frac{\partial}{\partial \operatorname{Im} u}\right), \quad \frac{\partial}{\partial \bar{u}} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} u} + i\frac{\partial}{\partial \operatorname{Im} u}\right),$$
$$\frac{\partial}{\partial v_j} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} v_j} - i\frac{\partial}{\partial \operatorname{Im} v_j}\right), \quad \frac{\partial}{\partial \bar{v}_j} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} v_j} + i\frac{\partial}{\partial \operatorname{Im} v_j}\right).$$

We here give the definition of L^2-well-posedness.

Definition 1. The initial-boundary value problem (1)-(2) is said to be L^2-well-posed if for any $u_0 \in L^2(\mathbb{T}^n)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{T}^n))$, (1)-(2) has a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^n))$.

It follows from Banach's closed graph theorem that the condition required in Definition 1 is equivalent to a seemingly stronger condition, that is, for any $u_0 \in L^2(\mathbb{T}^n)$ and for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{T}^n))$, (1)-(2) has a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^n))$, and for any $T > 0$ there exists $C_T > 0$ such that

$$\|u(t)\| \leqslant C_T \left(\|u_0\| + \left| \int_0^t \|f(s)\| ds \right| \right), \quad t \in [-T, T]. \tag{6}$$

Firstly, we present L^2-well-posedness results for linear equations.

Theorem 2. *The following conditions are mutually equivalent:*

1. *(1)-(2) is L^2-well-posed.*
2. *For $x \in \mathbb{T}^n$ and $\alpha \in \mathbb{Z}^n$*

$$\int_0^{2\pi} \operatorname{Im} \vec{b}(x - \alpha s) \cdot \alpha ds = 0. \tag{7}$$

3. *There exists a scalar function $\phi(x) \in C^\infty(\mathbb{T}^n)$ such that $\nabla \phi(x) = \operatorname{Im} \vec{b}(x)$.*

When $n = 1$, set $b(x) = b_1(x)$. The condition (7) is reduced to

$$\int_0^{2\pi} \operatorname{Im} b(x)dx = 0. \tag{8}$$

The condition (7) is the natural torus version of (5). More precisely, (7) is a special case of Ichinose's necessary condition of L^2-well-posedness discovered in [5]. On the other hand, the condition 3 corresponds to Ichinose's sufficient condition of L^2-well-posedness discovered in [6]. Theorem 2 makes us expect analogous results for nonlinear equations. In fact, we have local existence and local ill-posedness results as follows.

Theorem 3. *Let $s > n/2 + 2$. Suppose that there exists a smooth real-valued function $\Phi(u, \bar{u})$ on $\mathbb{R}^2$ such that for any $u \in C^1(\mathbb{T}^n)$*

$$\nabla\Phi(u, \bar{u}) = \operatorname{Im} \nabla_v F(u, \nabla u, \bar{u}, \nabla\bar{u}). \tag{9}$$

Then for any $u_0 \in H^s(\mathbb{T}^n)$, there exists $T > 0$ depending on $\|u_0\|_s$ such that (3)-(4) possesses a unique solution $u \in C([-T, T]; H^s(\mathbb{T}^n))$. Furthermore, Let $\{u_{0,k}\}$ be a sequence of initial data belonging to $H^s(\mathbb{T}^n)$, and let $\{u_k\}$ be a sequence of corresponding solutions. If

$$u_{0,k} \longrightarrow u_0 \quad in \quad H^s(\mathbb{T}^n) \quad as \quad k \to \infty,$$

then for any $m < s$

$$u_k \longrightarrow u \quad in \quad C([0, T]; H^m(\mathbb{T}^n)) \quad as \quad k \to \infty. \tag{10}$$

Theorem 4. *Suppose that there exists a holomorphic n-vector function*

$$\vec{G}(u) = (G_1(u), \cdots, G_n(u)), \quad u \in \mathbb{C}$$

such that $G(u) \not\equiv 0$, and

$$F(u, \nabla u, \bar{u}, \nabla\bar{u}) = \nabla \cdot \vec{G}(u) \tag{11}$$

for any $u \in C^1(\mathbb{T}^n)$. Then (3)-(4) is not *locally well-posed in the sense of Theorem 3.*

It seems to be hard to show the continuous dependence of the solution on the initial data because the gain of derivative of $e^{it\Delta}$ fails when $x \in \mathbb{T}^n$. To prove Theorem 4, we construct a sequence of solutions which are real-analytic in x by using the idea of the abstract Cauchy-Kowalewski theorem. Hence it is essential that $G(u)$ is holomorphic.

In what follows we give the sketch the proofs of Theorems 2 and 4. We omit the sketch of the proof of Theorem 3.

Proof of Theorem 2. To prove 1⇒2, we suppose that the condition 2 fails, and construct a sequence of approximate solutions $\{u_l(t, x)\}$ which break an energy inequality (6). Suppose that there exist $x_0 \in \mathbb{T}^n$ and $\alpha \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\int_0^{2\pi} \operatorname{Im} \vec{b}(x_0 - \alpha s) \cdot \alpha ds \equiv 4\pi b_0 \neq 0.$$

Without loss of generality, we can assume that $b_0 > 0$. It follows that there exists a small positive constant δ such that

$$\int_0^{2\pi} \operatorname{Im} \vec{b}(x - s\alpha) \cdot \alpha ds \geqslant 2\pi b_0 \tag{12}$$

for any $x \in D$, which is defined by

$$D = \bigcup_{\substack{\beta \in \mathbf{Z}^n \\ \alpha \in \mathbf{R}}} \{x \in \mathbb{R}^n \mid |x - x_0 - 2\pi\beta - \alpha a| \leqslant 2\delta\}.$$

Fix an arbitrary $T > 0$. We construct a sequence $\{u_l\}_{l=1,2,3,\ldots}$ by

$$u_l(t,x) = \exp(i\phi_l(t,x))\psi(x),$$
$$\phi_l(t,x) = -l^2 t\alpha \cdot \alpha + l\alpha \cdot x - \frac{1}{2}\int_0^{2l(t-T)} \vec{b}(x - \alpha s) \cdot \alpha ds,$$

where the amplitude function ψ is a smooth function on $\mathbb{T}^n$ and supported on $D/2\pi\mathbb{Z}^n$. It is easy to see that

$$\|u_l(T)\| = 1, \quad \|u_l(0)\| = O(\exp(-lb_0T)), \quad \|Lu_l(t)\| = O(\exp(-lb_0(T-t)/2)),$$

which means that the energy inequality fails for $\{u_l\}$.

Next we give the sketch of the proof 2⇒3 in case $n \geqslant 2$. Suppose (7). Since $\operatorname{Im}\vec{b} \in (C(\mathbb{T}^n))^n$, $\operatorname{Im}\vec{b}(x)$ is represented by a Fourier series

$$\operatorname{Im}\vec{b}(x) = \sum_{\beta \in \mathbf{Z}^n} \vec{b}_{\mathrm{I},\beta} e^{i\beta\cdot x}, \quad \vec{b}_{\mathrm{I},\beta} \in \mathbb{C}^n. \tag{13}$$

The substitution of (13) into (7) gives

$$0 = \sum_{\beta \in \mathbf{Z}^n} \vec{b}_{\mathrm{I},\beta} \cdot \alpha e^{i\beta\cdot x} \int_0^{2\pi} e^{-i\alpha\cdot\beta s} ds = 2\pi \sum_{\beta\cdot\alpha=0} \vec{b}_{\mathrm{I},\beta} \cdot \alpha e^{i\beta\cdot x}. \tag{14}$$

Then it follows that $\vec{b}_{\mathrm{I},\beta} \cdot \alpha = 0$ for any $\alpha \in \mathbb{Z}^n$. Since the orthogonal complement of $\beta \neq 0$ is spanned by some $\alpha^1, \ldots, \alpha^{n-1} \in \mathbb{Z}^{\kappa}$, there exists $a_\beta \in \mathbb{C}$ such that $\vec{b}_{\mathrm{I},\beta} = a_\beta \beta$ for $\beta \neq 0$. On the other hand, (14) implies $\vec{b}_{\mathrm{I},0} = 0$ since $V_0 = \mathbb{R}^n$ is spanned by $e_1, \ldots, e_n \in \mathbb{Z}^n$. Then we have

$$\operatorname{Im}\vec{b}(x) = \sum_{\beta \neq 0} a_\beta \beta e^{i\beta\cdot x}.$$

If we set

$$\phi(x) = -i\sum_{\beta \neq 0} a_\beta e^{i\beta\cdot x},$$

then $\nabla\phi(x) = \operatorname{Im}\vec{b}(x)$.

It is easy to prove 3⇒1. Since $\exp(\pm\phi(x)/2)$ is a smooth function on $\mathbb{T}^n$, a mapping $u \mapsto v = \exp(-\phi(x)/2)u$ is automorphic on $L^2(\mathbb{T}^n)$. Multiplying $Lu = f$ by $\exp(\phi(x)/2)$, we have

$$(\partial_t - i\Delta + \operatorname{Re}\vec{b}(x) \cdot \nabla + \tilde{c}(x))v = g(t,x), \tag{15}$$

where $\tilde{c}(x) \in C^\infty(\mathbb{T}^n)$ and $g(t,x) = \exp(-\phi(x)/2)f(t,x)$. It is easy to obtain forward and backward energy inequalities in t. The duality arguments proves that (1)-(2) is L^2-well-posed. □

Proof of Theorem 4. We will construct a sequence which fails to satisfy (19). It suffices to do it for one dimensional case since a one dimensional counter example is also an any dimensional counter example. Suppose that there exists a nonconstant holomorphic function $G(u)$ in $\mathbb{C}$ such that for $u \in C^1(\mathbb{T})$

$$F(u, \nabla u, \bar{u}, \nabla\bar{u}) = \frac{\partial}{\partial x}G(u) = G'(u)u_x.$$

Set $g = G'$ for short. If u is a smooth solution to (3), then

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}} u(t,x)dx &= \int_{\mathbb{T}} \partial_t u(t,x)dx \\ &= \int_{\mathbb{T}} \frac{\partial}{\partial x}\{u_x(t,x) + G(u(t,x))\}dx \\ &= 0.\end{aligned} \tag{16}$$

We here express u by a Fourier series

$$u(t,x) = \sum_{l\in\mathbb{Z}} u_l(t)e^{ilx}.$$

Then (16) implies $u_0(t) \equiv u_0(0)$. Set $u_0(0) = z_0$ and $v(t,x) = u(t,x) - z_0$ for short. Since $g(0) = 0$ and $u_x = v_x$, there exists an appropriate complex constant z_0 such that

$$g(u)u_x = -(\mu + i\lambda)v_x + h(v)v_x,$$

where $\mu \in \mathbb{R}$, $\lambda > 0$, and h is holomorphic in $\mathbb{C}$. Then, v solves

$$v_t - iv_{xx} + (\mu + i\lambda)v_x = h(v)v_x.$$

In what follows, fix z_0. Note that $u(t,x) \equiv z_0$ is a solution to (3)-(4).

Suppose that the conclusion of Theorem 3 holds. Consider the initial value problem of the form $v^{(m)}$ solves the initial value problem of the form

$$v_t^{(m)} - iv_{xx}^{(m)} + (\mu + i\lambda)v_x^{(m)} = h(v^{(m)})v_x^{(m)} \quad \text{in} \quad (0,T) \times \mathbb{T}, \tag{17}$$

$$v^{(m)}(0,x) = \frac{e^{imx}}{(1+m)^s} \quad \text{in} \quad \mathbb{T}, \tag{18}$$

where $s > 5/2$, $m = 1, 2, 3, \ldots$. Since $\{v^{(m)}(0,x)\}$ is bounded in $H^s(\mathbb{T})$ and

$$v^{(m)}(0,x) \longrightarrow 0 \quad \text{in} \quad H^\sigma(\mathbb{T}) \quad \text{as} \quad m \to \infty$$

for any $\sigma < s$, it follows from the hypothesis that

$$v^{(m)} \longrightarrow 0 \quad \text{in} \quad C([0,T]; H^\sigma(\mathbb{T})) \quad \text{as} \quad m \to \infty \tag{19}$$

for any $\sigma < s$. We investigate a formal Fourier series solution to (17)-(18) of the form

$$w^{(m)}(t,x) = \sum_{l=1}^{\infty} w_l^{(m)}(t) e^{ilmx}. \tag{20}$$

The substitution of (20) into (17)-(18) gives

$$\begin{aligned} &\frac{d}{dt} w_l^{(m)}(t) + (il^2m^2 + i\mu lm - \lambda lm) w_l^{(m)}(t) \\ &= \sum_{p=1}^{\infty} h_p \sum_{\substack{l_0+\cdots+l_p=l \\ l_0,\ldots,l_p \geqslant 1}} il_0 m \prod_{j=0}^{p} w_{l_j}^{(m)}(t), \end{aligned} \tag{21}$$

$$w_l^{(m)}(0) = \begin{cases} (1+m)^{-s} & \text{if} \quad l = 1 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

For $l = 1$, (21)-(22) is concretely solved by

$$w_1^{(m)}(t) = (1+m)^{-s} \exp(-i(m^2 + \mu m)t + \lambda m t). \tag{23}$$

For $l \geqslant 2$, we apply the idea of the abstract Cauchy-Kowalewski theorem to (21)-(22). We can show that there exists $T_m \in (0,T)$ such that the formal series (20) converges in $C([0,T_m]; H^s(\mathbb{T}))$. Then it follows from the hypothesis that

$$v^{(m)} = w^{(m)} \quad \text{in} \quad C([0,T_m]; H^s(\mathbb{T})).$$

Finally we can find $\delta > 0$, $\alpha \in (0,1)$ and $t_m \in (0,T_m)$ such that

$$\begin{aligned} \sup_{t\in[0,T]} \|v^{(m)}(t)\|_{(1-\alpha)s} &\geqslant \|v^{(m)}(t_m)\|_{(1-\alpha)s} \\ &= \|w^{(m)}(t_m)\|_{(1-\alpha)s} \\ &= \left(\sum_{l=1}^{\infty} (1+lm)^{2(1-\alpha)s} |w_l^{(m)}(t_m)|^2 \right)^{1/2} \\ &\geqslant (1+m)^{(1-\alpha)s} |w_1^{(m)}(t_m)| \\ &= (1+m)^{-s\alpha} \exp(\lambda m t_m) \\ &= \delta, \end{aligned}$$

which contradicts (19). Here we omit the detail. □

References

[1] H. Chihara, *Gain of regularity for semilinear Schrödinger equations*, Math. Ann. **315** (1999), 529–567.

[2] H. Chihara, *The initial value problem for Schrödinger equations on the torus*, Int. Math. Res. Not. **2002:15** (2002), 789–820.

[3] S. Doi, *Remarks on the Cauchy problem for Schrödinger-type equations*, Comm. in Partial Differential Equations **21** (1996), 163–178.

[4] S. Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math. J. **82** (1996), 679–706.

[5] W. Ichinose, *On L^2 well posedness of the Cauchy problem for Schrödinger type equations on the Riemannian manifold and Maslov theory*, Duke Math. J. **56** (1988), 549–588.

[6] W. Ichinose, *A note on the Cauchy problem for Schrödinger type equations on the Riemannian manifold*, Math. Japon. **35** (1990), 205–213.

[7] S. Mizohata, "On the Cauchy Problem" Academic Press, 1985.