数理解析研究所講究録 1388 巻 2004 年 81-87

Small global solutions for the nonlinear Dirac equation

Shuji Machihara[*], Makoto Nakamura[†] and Tohru Ozawa[‡]

町原秀二

島根大学総合理工学部

1 Introduction

In this note we study the Cauchy problem for the nonlinear Dirac equation (NLD) in space-time $\mathbb{R}^{1+n}$:

$$\partial_t\psi = (iA_0 + \sum_{j=1}^{n} A_j\partial_j)\psi + \lambda|(A_0\psi|\psi)|^{(p-1)/2}\psi, \tag{1.1}$$
$$\psi(0) = \phi,$$

where $\psi : \mathbb{R}^{1+n} \to \mathbb{C}^{N(n)}$ is a function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, $\phi : \mathbb{R}^n \to \mathbb{C}^{N(n)}$ is a given Cauchy data, $\lambda \in \mathbb{C}$ and $p > 1$ are constants, $\partial_t = \partial/\partial t, \partial_j = \partial/\partial x_j$ with space variable $x = (x_1, \ldots, x_n)$, $(\cdot|\cdot)$ denotes the inner product in $\mathbb{C}^{N(n)}$. $A_0, \ldots, A_n$ denote the $N(n) \times N(n)$ matrices satisfying $A_iA_j + A_jA_i = 2\delta_{ij}I$, where δ_{ij} is Kronecker's delta and I is the unit matrix. $N(n)$ is an integer depending on the space dimension n.

There are several ways to construct the set of matrices satisfying the anticommutation relation above. The set of $(A_0^{(n)}, \ldots, A_n^{(n)})$ for n dimensional case can be derived from $(A_0^{(n-1)}, \ldots, A_{n-1}^{(n-1)})$ for $n-1$ dimensional case inductively.

Example 1 *For* $n = 1$, $A_0^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_1^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

For $n \geq 2$, $A_j^{(n)} = \begin{pmatrix} 0 & A_j^{(n-1)} \\ A_j^{(n-1)} & 0 \end{pmatrix}$, $j = 0, \ldots, n-1$, $\quad A_n^{(n)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

Then $N(n) = 2^n$.

Example 2 *For* $n = 1$, $A_0^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_1^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Let* m *be an integer.*

For $n = 4m+2$, $A_j^{(n)} = A_j^{(n-1)}$, $j = 0, \ldots, n-1$, $\quad A_n^{(n)} = iA_0^{(n-1)} \cdots A_{n-1}^{(n-1)}$.

For $n = 4m+1, 4m+3$, $A_j^{(n)} = \begin{pmatrix} 0 & A_j^{(n-1)} \\ A_j^{(n-1)} & 0 \end{pmatrix}$, $j = 0, \ldots, n-1$, $\quad A_n^{(n)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

[*]Department of Mathematics, Shimane University, Matsue 690-0815, Japan

[†]Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan

[‡]Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

For $n = 4m+4$, $A_j^{(n)} = A_j^{(n-1)}$, $j = 0, \ldots, n-1$, $A_n^{(n)} = A_0^{(n-1)} \cdots A_{n-1}^{(n-1)}$.

Then $N(n) = 2^{[(n+1)/2]}$, *where* $[a]$ *denotes the largest integer which is less than or equal to* a.

We consider the global existence of solution with small data for NLD. The case of $n = 3$ have been already studied in [3], [7]. Our basic tool for the proof is Strichartz estimate for Klein–Gordon equation which works for the space–time norm $L_t^q B_x$, $q \geq 2$, where B_x denotes suitable Besov spaces on $\mathbb{R}^n$. These estimates have been studied for $q > 2$, though the estimates for $q = 2$ i.e. $L_t^2 B_x$ have been excluded until lately and play an important role for our results. First study on the estimate for $q = 2$ was given by Lindblad and Sogge [6] and Ginibre and Velo [4] independently in 1995 for the wave equation. Keel and Tao [5] proved the end point estimate in 1998 for wave and Schrödinger equations. For Klein–Gordon equation, estimate on $q = 2$ can be found in [7]. In this note we give estimates applicable to more general norm in space variables.

Before stating our results, we shall give a scaling approach in this problem. For instance let us consider the massless case of NLD:

$$\partial_t \psi = \sum_{j=1}^{n} A_j \partial_j \psi + \lambda |(A_0 \psi | \psi)|^{(p-1)/2} \psi. \tag{1.2}$$

We scale the function ψ in the form

$$\psi_\gamma(t,x) = \gamma^{\frac{1}{p-1}} \psi(\gamma t, \gamma x), \quad \gamma > 0. \tag{1.3}$$

Then we see that ψ_γ is a solution of (1.2) if and only if ψ is a solution of (1.2). We take the initial data belonging to the homogeneous Sobolev space $\dot{H}^s$,

$$\|\psi_\gamma(0)\|_{\dot{H}^s} = \gamma^{s-n/2+1/(p-1)} \|\psi(0)\|_{\dot{H}^s}. \tag{1.4}$$

Therefore we may think $s(p) := n/2 - 1/(p-1)$ as a critical exponent for NLD.

Now we give our results.

Theorem 3 *Let* n, p, ϕ *satisfy the following conditions:*

(1) $n = 1$, $p \geq 5$, $\|\phi\|_{B^s_{2,1}} \ll 1$, $s = 1/2 + 1/(p-1)$,
(2) $n = 2$, $3 < p \leq 5$, $\|\phi\|_{B^s_{2,1}} \ll 1$, $s = 1/2 + 1/(p-1)$,
(3) $n = 2$, $p > 5$, $\|\phi\|_{H^{s(p)}} \ll 1$,
(4) $n = 3$, $p = 3$, $\|\phi\|_{H^s} \ll 1$, $s > 1$,
(5) $n = 3$, $p > 3$, $\|\phi\|_{H^{s(p)}} \ll 1$,
(6) $n \geq 4$, $p = 3$, $\|\phi\|_{B^{s(3)}_{2,1}} \ll 1$,
(7) $n \geq 4$, $p > 3$, $\|\phi\|_{H^{s(p)}} \ll 1$, $(s(p) < (p-1)/2$ *if* $p \neq$ *odd*).

Then NLD *has a solution* $\psi \in C(\mathbb{R}; X)$, *where* X *denotes the space of data indicated above.*

Remark The cases (4), (5) were proved in [7], [3] respectively.

We close this section by introducing some notation. For any r with $1 \le r \le \infty, L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty, H_r^s$ [resp. $\dot{H}_r^s$] denotes the inhomogeneous [resp. homogeneous] Sobolev space. For any $s \in \mathbb{R}$ and any r, m with $1 \le r, m \le \infty, B_{r,m}^s$ [resp. $\dot{B}_{r,m}^s$] denotes the inhomogeneous [resp. homogeneous] Besov space. We make abbreviations such as $H^s = H_2^s, \dot{H}^s = \dot{H}_2^s$, and $L_t^q B = L^q(\mathbb{R}; B)$. Occasionally we use $\lesssim$ to mean $\le C$, where C is a positive constant.

We give some properties for Sobolev and Besov spaces which seem to be important for following argument (see [1]).

$$H_p^\alpha \hookrightarrow H_q^\beta, \qquad B_{p,m}^\alpha \hookrightarrow B_{q,m}^\beta \tag{1.5}$$

with $\alpha - n/p = \beta - n/q, \ \alpha \ge \beta$.

$$H_p^\alpha \hookrightarrow B_{p,2}^\alpha \quad [\text{resp. } B_{p,2}^\alpha \hookrightarrow H_p^\alpha] \tag{1.6}$$

with $p \le 2$ [resp. $p \ge 2$].

$$B_{2,1}^\alpha \hookrightarrow B_{2,2}^\alpha = H^\alpha \hookrightarrow B_{2,1}^\beta \tag{1.7}$$

with $\alpha > \beta$. We often use the embedding

$$B_{\infty,1}^0 \hookrightarrow L^\infty. \tag{1.8}$$

2 Proof

We employ a contraction argument to obtain the solution. For this purpose, we prepare two lemmas, Strichartz estimate and interpolation estimate for L^∞ norm.

For simplicity we set $f(\psi) = \lambda|(A_0\psi|\psi)|^{(p-1)/2}\psi$, and $\omega = (1-\Delta)^{1/2}$. The solutions for NLD satisfy the following integral equation (see [7]):

$$\psi(t) = U(t)\phi + \int_0^t U(t-t')f(\psi(t'))dt' \tag{2.1}$$

with $U(t) := \cos t\omega + (iA_0 + \sum A_j\partial_j)\omega^{-1}\sin t\omega$. We investigate the operator $U(t)$. We give the following lemma which is often called Strichartz estimate.

Lemma 4 *Let $k = 1, 2$. The following estimate holds.*

$$\|U(t)u\|_{L^q(\mathbb{R}; B_{r,k}^{-\sigma})} \lesssim \|u\|_{B_{2,k}^0}, \tag{2.2}$$

where $0 \le 1/q \le 1/2, \ 0 \le 1/r \le 1/2 - 2/(n-1+\theta)q, \ (n+\theta, q) \ne (3, 2)$ *and*

$$\frac{n}{2} - \frac{n-1-\theta}{n-1+\theta}\frac{1}{q} - \sigma - \frac{n}{r} = 0, \tag{2.3}$$

for $0 \le \theta \le 1$.

From (2.2), we have the homogeneous estimate as follows,

$$\left\| \int_0^t U(t-t')f(t')dt' \right\|_{L^q(\mathbb{R};B^{-\sigma}_{r,k})} \leq \int_{-\infty}^{\infty} \|U(t)U(-t')f(t')\|_{L^q(\mathbb{R};B^{-\sigma}_{r,k})} dt'$$
$$\lesssim \|f\|_{L^1(\mathbb{R};B^0_{2,k})}. \tag{2.4}$$

Remark (i) The estimate for $k=2$ was proved in [7]. (ii) We may replace $K(t) = e^{\pm it\omega}$ for $U(t)$. (iii) From the condition (2.3), if we take $\theta = 0$ and substitute the inhomogeneous norms by the homogeneous ones, i.e. $B^{-\sigma}_{r,k} \to \dot{B}^{-\sigma}_{r,k}, B^0_{2,k} \to \dot{B}^0_{2,k}$, then the estimates (2.2) and (2.4) satisfy scaling invariance.

Proof of Lemma 4

We concentrate on the case $k=1$. From duality argument, it is sufficient to prove that

$$\left\| \int_{-\infty}^{\infty} U(-t')F(t')dt' \right\|_{B^0_{2,\infty}} \lesssim \|F\|_{L^{q'}_t B^{\sigma}_{r',\infty}}, \tag{2.5}$$

where q' and r' denote the Hölder conjugate of q and r respectively. In fact in [7] we find

$$\left\| \int_{-\infty}^{\infty} U(-t')\varphi_k * F(t')dt' \right\|_{L^2} \lesssim 2^{k\sigma} \|\varphi_k * F\|_{L^{q'}_t L^{r'}}, \tag{2.6}$$

where $\{\varphi_k\}_0^\infty$ is the Littlewood–Paley dyadic decomposition on $\mathbb{R}^n$ and q, r, σ are as in Lemma 4. We take supremum of k on both sides to obtain (2.5).

We use the following Gagliardo–Nirenberg type interpolation inequality ([3] or see [8] for more general cases).

Lemma 5 *The following estimate holds.*

$$\|f\|_{L^\infty} \lesssim \|f\|^{\delta}_{H^\alpha_p} \|f\|^{1-\delta}_{H^\beta_q}, \tag{2.7}$$

where $1 \leq p, q \leq \infty,\ 0 < \alpha, \beta < \infty,\ 0 < \delta < 1,\ \alpha > n/p,\ \beta < n/q,\ \delta(\alpha - n/p) + (1-\delta)(\beta - n/q) = 0.$

Proof of Theorem 3

We define the complete metric space $\Phi = \Phi(p, s, k, M)$ for NLD as

$$\Phi = \{\psi \in L^\infty(\mathbb{R}; B^s_{2,k}) \cap L^{p-1}(\mathbb{R}; L^\infty);\ \|\psi\|_{L^\infty_t B^s_{2,k}} + \|\psi\|_{L^{p-1}_t L^\infty} \leq M\}. \tag{2.8}$$

We find a unique solution of NLD in Φ for sufficiently small data ψ and M. For any $s \in \mathbb{R},\ k = 1, 2$ and q, r, σ satisfying the condition in Lemma 4, we have from (2.1), (2.2) and (2.4),

$$\|\psi\|_{L^q_t B^{s-\sigma}_{r,k}} \lesssim \|\phi\|_{B^s_{2,k}} + \|f(\psi)\|_{L^1_t B^s_{2,k}}$$
$$\lesssim \|\phi\|_{B^s_{2,k}} + \|\psi\|^{p-1}_{L^{p-1}_t L^\infty} \|\psi\|_{L^\infty_t B^s_{2,k}}. \tag{2.9}$$

So we concentrate on the $L_t^{p-1}L^\infty$ norm. We apply Lemma 4 to estimate it in $L_t^q B_{r,k}^{s-\sigma}$.

(1) $n=1$. For any $p \geq 5$, we take $k=1$ and

$$(s,q,r,\sigma,\theta) = \big(1/2+1/(p-1), p-1, \infty, 1/2+1/(p-1), 4/(p-1)\big) \tag{2.10}$$

and use $B_{\infty,1}^0 \hookrightarrow L^\infty$ to obtain the theorem.

(2) $n=2$. For any $3 < p \leq 5$, we take $k=1$ and

$$(s,q,r,\sigma,\theta) = \big(1/2+1/(p-1), p-1, \infty, 1/2+1/(p-1), (5-p)/(p-1)\big) \tag{2.11}$$

and use $B_{\infty,1}^0 \hookrightarrow L^\infty$ to obtain the theorem.

(3) $n=2$. For any $p>5$, we take $k=2$ and $0<\delta<1$ satisfying $(1-\delta)(p-1)\geq 4$. From (2.7), we estimate

$$\|\psi\|_{L_t^{p-1}L^\infty} \lesssim \|\psi\|_{L_t^\infty H^s}^{\delta} \|\psi\|_{L_t^{(1-\delta)(p-1)} H_r^{s-\sigma}}^{1-\delta}. \tag{2.12}$$

We take

$$(s,q,r,\sigma,\theta) = \big(1-1/(p-1), (1-\delta)(p-1), 1/2-2/(1-\delta)(p-1), 3/(1-\delta)(p-1), 0\big) \tag{2.13}$$

to obtain the theorem.

(6) $n \geq 4$, $p=3$. We take $k=1$ and

$$(s,q,r,\sigma,\theta) = \big((n-1)/2, 2, 2(n-1)/(n-3), (n+1)/2(n-1), 0\big) \tag{2.14}$$

and use $B_{r,1}^{s-\sigma} \hookrightarrow L^\infty$ to obtain the theorem.

(7) $n \geq 4$. For any $p>3$, we take $k=2$ and $0<\delta<1$ satisfying $(1-\delta)(p-1)\geq 2$. From (2.7), we estimate (2.12) and take

$$\begin{aligned}(s,q,r,\sigma,\theta) = \big(n/2-1/(p-1), (1-\delta)(p-1), (1/2-2/(1-\delta)(p-1)(n-1))^{-1},\\ (n+1)/(1-\delta)(p-1)(n-1), 0\big)\end{aligned}$$

to obtain the theorem.

3 Application for nonlinear Klein–Gordon equations

We apply the previous argument for the Klein–Gordon equation with derivative coupling ($NLKG$):

$$\partial_t^2 u - \Delta u + m^2 u = \lambda f(u), \quad u(0)=u_0,\ \partial_t u(0) = u_1, \tag{3.1}$$

where $u : \mathbb{R}^{1+n} \to \mathbb{C}$ is unknown, $u_0, u_1 : \mathbb{R}^n \to \mathbb{C}$ are given Cauchy data, $m>0$ and $\lambda \in \mathbb{C}$ are constants. We consider the nonlinear term $f(u)$ of following types:

$$f(u) = \partial_j(u^p), \partial_t(u^p), \prod_{j=1}^n (\partial_j u)^{p_j}, (\partial_t u)^{p_0} \prod_{j=1}^n (\partial_j u)^{p_j}, \tag{3.2}$$

where $1 \leq j \leq n$.

We give the results only. For simplicity we set $\varphi = (u_0, u_1)$ and $\|\varphi\|_{B^s_{2,k}} := \|u_0\|_{B^s_{2,k}} + \|u_1\|_{B^{s-1}_{2,k}}$.

Theorem 6 *Let n, p, ϕ satisfy the following conditions:*

$$
\begin{array}{lllll}
(1) & n=1, & p \geq 5, & \|\varphi\|_{B^s_{2,1}} \ll 1, & s = 1/2 + 1/(p-1), \\
(2) & n=2, & 3 < p \leq 5, & \|\varphi\|_{B^s_{2,1}} \ll 1, & s = 1/2 + 1/(p-1), \\
(3) & n=2, & p > 5, & \|\varphi\|_{H^{s(p)}} \ll 1, & \\
(4) & n=3, & p = 3, & \|\varphi\|_{H^s} \ll 1, & s > 1, \\
(5) & n=3, & p > 3, & \|\varphi\|_{H^{s(p)}} \ll 1, & \\
(6) & n \geq 4, & p = 3, & \|\varphi\|_{B^{s(3)}_{2,1}} \ll 1, & \\
(7) & n \geq 4, & p > 3, & \|\varphi\|_{H^{s(p)}} \ll 1, & (s(p) < p \quad \textit{if } p \neq \textit{integer}),
\end{array}
$$

$(1)-(7)$ for $f = \partial_j(u^p)$, $(4)-(7)$ for $f = \partial_t(u^p)$.

Then $NLKG$ has a solution $\psi \in C(\mathbb{R}; X)$, where X denotes the space of data φ indicated above.

Remark In this case $s(p)$ is scaling critical exponent for massless $NLKG$ $(m = 0)$.

Theorem 7 *Let n, p, ϕ satisfy the following conditions:*

$$
\begin{array}{lllll}
(1) & n=1, & p \geq 5, & \|\varphi\|_{B^{s+1}_{2,1}} \ll 1, & s = 1/2 + 1/(p-1), \\
(2) & n=2, & 3 < p \leq 5, & \|\varphi\|_{B^{s+1}_{2,1}} \ll 1, & s = 1/2 + 1/(p-1), \\
(3) & n=2, & p > 5, & \|\varphi\|_{H^{s(p)+1}} \ll 1, & \\
(4) & n=3, & p = 3, & \|\varphi\|_{H^{s+1}} \ll 1, & s > 1, \\
(5) & n=3, & p > 3, & \|\varphi\|_{H^{s(p)+1}} \ll 1, & \\
(6) & n \geq 4, & p = 3, & \|\varphi\|_{B^{s(3)+1}_{2,1}} \ll 1, & \\
(7) & n \geq 4, & p > 3, & \|\varphi\|_{H^{s(p)+1}} \ll 1, &
\end{array}
$$

$(1)-(7)$ for $f = \prod_{j=1}^n (\partial_j u)^{p_j}$, $p_1 + \cdots + p_n = p$, $p_j \in \mathbb{Z}^+ \cup \{0\}$ or $p_j > \max\{1, s\}$,
$(4)-(7)$ for $f = (\partial_t u)^{p_0} \prod_{j=1}^n (\partial_j u)^{p_j}$, $p_0 + \cdots + p_n = p$, $p_j \in \mathbb{Z}^+ \cup \{0\}$ or $p_j > \max\{1, s\}$.

Then $NLKG$ has a solution $\psi \in C(\mathbb{R}; X)$, where X denotes the space of data φ indicated above.

Remark In this case $s(p)+1$ is scaling critical exponent for massless $NLKG$ $(m = 0)$.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin / Heiderberg / New York (1976).

[2] Y. CHOQUET-BRUHAT, C. DE WITT-MORETTE AND M. DILLARD-BLEICK, *Analysis, Manifolds and Physics*, North Holland (1977).

[3] M. ESCOBEDO AND L. VEGA, *A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s > 1$*, SIAM J. Math. Anal., **2** (1997), 338–362.

[4] J. GINIBRE, G. VELO, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. **133** (1995), 50–68.

[5] M. KEEL AND T. TAO, *Endpoint Strichartz estimates*, Amer. J. Math., **120** (1998), 955–980.

[6] H. LINDBLAD AND C. D. SOGGE, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. **130** (1995), 357–426.

[7] S. MACHIHARA, K. NAKANISHI, AND T. OZAWA, *Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation*, to appear in Rev. Mat. Iberoamericana.

[8] S. MACHIHARA AND T. OZAWA, *Interpolation Inequalities in Besov spaces*, to appear in Proc. in AMS.