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Kyoto University
Small global solutions for the nonlinear Dirac equation

Shuji Machihara* Makoto Nakamura† and Tohru Ozawa‡

1 Introduction

In this note we study the Cauchy problem for the nonlinear Dirac equation (NLD) in space-time \( \mathbb{R}^{1+n} \):

\[
\partial_t \psi = (iA_0 + \sum_{j=1}^{n} A_j \partial_j) \psi + \lambda |(A_0 \psi | \psi)|^{(p-1)/2} \psi,
\]

\( \psi(0) = \phi \),

where \( \psi : \mathbb{R}^{1+n} \to \mathbb{C}^{N(n)} \) is a function of \((t, x) \in \mathbb{R} \times \mathbb{R}^n\), \( \phi : \mathbb{R}^n \to \mathbb{C}^{N(n)} \) is a given Cauchy data, \( \lambda \in \mathbb{C} \) and \( p > 1 \) are constants, \( \partial_t = \partial/\partial t, \partial_j = \partial/\partial x_j \) with space variable \( x = (x_1, \ldots, x_n) \), \((\cdot | \cdot)\) denotes the inner product in \( \mathbb{C}^{N(n)} \). \( A_0, \ldots, A_n \) denote the \( N(n) \times N(n) \) matrices satisfying \( A_i A_j + A_j A_i = 2 \delta_{ij} I \), where \( \delta_{ij} \) is Kronecker's delta and \( I \) is the unit matrix. \( N(n) \) is an integer depending on the space dimension \( n \).

There are several ways to construct the set of matrices satisfying the anticommutation relation above. The set of \( (A_0^{(n)}, \ldots, A_n^{(n)}) \) for \( n \) dimensional case can be derived from \( (A_0^{(n-1)}, \ldots, A_{n-1}^{(n-1)}) \) for \( n-1 \) dimensional case inductively.

Example 1 For \( n = 1 \), \( A_0^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( A_1^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

For \( n \geq 2 \), \( A_j^{(n)} = \begin{pmatrix} 0 & A_j^{(n-1)} \\ A_j^{(n-1)} & 0 \end{pmatrix} \), \( j = 0, \ldots, n-1 \), \( A_n^{(n)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \).

Then \( N(n) = 2^n \).

Example 2 For \( n = 1 \), \( A_0^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( A_1^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Let \( m \) be an integer.

For \( n = 4m + 2 \), \( A_j^{(n)} = A_j^{(n-1)} \), \( j = 0, \ldots, n-1 \), \( A_n^{(n)} = iA_0^{(n-1)} \cdots A_{n-1}^{(n-1)} \).

For \( n = 4m + 1, 4m + 3 \), \( A_j^{(n)} = \begin{pmatrix} 0 & A_j^{(n-1)} \\ A_j^{(n-1)} & 0 \end{pmatrix} \), \( j = 0, \ldots, n-1 \), \( A_n^{(n)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \).

*Department of Mathematics, Shimane University, Matsue 690-0815, Japan
†Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan
‡Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
For \( n = 4m + 4, \ A_{j}^{(n)} = A_{j}^{(n-1)}, \ j = 0, \ldots, n - 1, \ A_{n}^{(n)} = A_{0}^{(n-1)} \cdots A_{n-1}^{(n-1)}. \)

Then \( N(n) = 2^{[(n+1)/2]}, \) where \([a]\) denotes the largest integer which is less than or equal to \( a.\)

We consider the global existence of solution with small data for \( NLD. \) The case of \( n = 3\) have been already studied in [3], [7]. Our basic tool for the proof is Strichartz estimate for Klein–Gordon equation which works for the space-time norm \( L_{t}^{q}B_{x}, \ q \geq 2, \) where \( B_{x} \) denotes suitable Besov spaces on \( \mathbb{R}^{n}. \) These estimates have been studied for \( q > 2, \) though the estimates for \( q = 2 \) i.e. \( L_{t}^{2}B_{x} \) have been excluded until lately and play an important role for our results. First study on the estimate for \( q = 2 \) was given by Lindblad and Sogge [6] and Ginibre and Velo [4] independently in 1995 for the wave equation. Keel and Tao [5] proved the end point estimate in 1998 for wave and Schrödinger equations. For Klein–Gordon equation, estimate on \( q = 2 \) can be found in [7]. In this note we give estimates applicable to more general norm in space variables.

Before stating our results, we shall give a scaling approach in this problem. For instance let us consider the massless case of \( NLD: \)

\[
\partial_{t}\psi = \sum_{j=1}^{n} A_{j}\partial_{j}\psi + \lambda |(A_{0}\psi|\psi)|^{(p-1)/2}\psi. \tag{1.2}
\]

We scale the function \( \psi \) in the form

\[
\psi_{\gamma}(t, x) = \gamma^{1/p-1}\psi(\gamma t, \gamma x), \quad \gamma > 0. \tag{1.3}
\]

Then we see that \( \psi_{\gamma} \) is a solution of (1.2) if and only if \( \psi \) is a solution of (1.2). We take the initial data belonging to the homogeneous Sobolev space \( \dot{H}^{s}, \)

\[
||\psi_{\gamma}(0)||_{\dot{H}^{s}} = \gamma^{s-n/2+1/(p-1)}||\psi(0)||_{\dot{H}^{s}}. \tag{1.4}
\]

Therefore we may think \( s(p) := n/2 - 1/(p - 1) \) as a critical exponent for \( NLD. \)

Now we give our results.

**Theorem 3** Let \( n, p, \phi \) satisfy the following conditions:

1. \( n = 1, \ p \geq 5, \ |\phi||_{B_{2,1}^{1}} \ll 1, \ s = 1/2 + 1/(p - 1), \)
2. \( n = 2, \ 3 < p \leq 5, \ |\phi||_{B_{2,1}^{1}} \ll 1, \ s = 1/2 + 1/(p - 1), \)
3. \( n = 2, \ p > 5, \ |\phi||_{\dot{H}^{s(p)}} \ll 1, \)
4. \( n = 3, \ p = 3, \ |\phi||_{\dot{H}^{s}} \ll 1, \ s > 1, \)
5. \( n = 3, \ p > 3, \ |\phi||_{\dot{H}^{s(p)}} \ll 1, \)
6. \( n \geq 4, \ p = 3, \ |\phi||_{B_{2,1}^{3}} \ll 1, \)
7. \( n \geq 4, \ p > 3, \ |\phi||_{\dot{H}^{s(p)}} \ll 1, \ (s(p) < (p - 1)/2 \text{ if } p \neq \text{odd}). \)

Then \( NLD \) has a solution \( \psi \in C(\mathbb{R}; X), \) where \( X \) denotes the space of data indicated above.
Remark The cases (4), (5) were proved in [7], [3] respectively.

We close this section by introducing some notation. For any $r$ with $1 \leq r \leq \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any $r$ with $1 < r < \infty$, $H^s_r$ [resp. $\dot{H}^s_r$] denotes the inhomogeneous [resp. homogeneous] Sobolev space. For any $s \in \mathbb{R}$ and any $r$, $m$ with $1 \leq r, m \leq \infty$, $B^s_{r,m}$ [resp. $\dot{B}^s_{r,m}$] denotes the inhomogeneous [resp. homogeneous] Besov space. We make abbreviations such as $H^s = H^s_2$, $\dot{H}^s = \dot{H}^s_2$, and $L^r \sim B^s_{r,m} = L^r(\mathbb{R}; B)$. Occasionally we use $\lesssim$ to mean $\leq C$, where $C$ is a positive constant.

We give some properties for Sobolev and Besov spaces which seem to be important for following argument (see [1]).

\[ H^\alpha_p \hookrightarrow H^\beta_q, \quad B^\alpha_{p,m} \hookrightarrow B^\beta_{q,m} \]  
(1.5)  
with $\alpha - n/p = \beta - n/q$, $\alpha \geq \beta$.

\[ H^\alpha_p \hookrightarrow B^\alpha_{p,2} \quad [\text{resp. } B^\alpha_{p,2} \hookrightarrow H^\alpha_p] \]  
(1.6)  
with $p \leq 2$ [resp. $p \geq 2$].

\[ B^\alpha_{2,1} \hookrightarrow B^\alpha_{2,2} = H^\alpha \hookrightarrow B^\beta_{2,1} \]  
(1.7)  
with $\alpha > \beta$. We often use the embedding

\[ B^0_{\infty,1} \hookrightarrow L^\infty. \]  
(1.8)

2 Proof

We employ a contraction argument to obtain the solution. For this purpose, we prepare two lemmas, Strichartz estimate and interpolation estimate for $L^\infty$ norm.

For simplicity we set $f(\psi) = \lambda |(A_0 \psi | \psi)|^{(p-1)/2} \psi$, and $\omega = (1 - \Delta)^{1/2}$. The solutions for NLD satisfy the following integral equation (see [7]):

\[ \psi(t) = U(t) \phi + \int_0^t U(t - t') f(\psi(t')) dt' \]  
(2.1)  
with $U(t) := \cos t\omega + (iA_0 + \sum A_j \partial_j)\omega^{-1} \sin t\omega$. We investigate the operator $U(t)$. We give the following lemma which is often called Strichartz estimate.

Lemma 4 Let $k = 1, 2$. The following estimate holds.

\[ \|U(t)u\|_{L^q(\mathbb{R}; B^{\sigma}_{r,k})} \lesssim \|u\|_{B^{\sigma}_{r,k}}, \]  
(2.2)

where $0 \leq 1/q \leq 1/2, 0 \leq 1/r \leq 1/2 - 2/(n - 1 + \theta)q$, $(n + \theta, q) \neq (3, 2)$ and

\[ \frac{n}{2} - \frac{n - 1 - \theta}{n - 1 + \theta} \frac{1}{q} - \frac{\sigma - n}{r} = 0, \]  
(2.3)  
for $0 \leq \theta \leq 1$. 

\[ \frac{1}{q} \leq 1/2 \]
From (2.2), we have the homogeneous estimate as follows,
\[
\left\| \int_0^t U(t-t') f(t') dt' \right\|_{L^q(\mathbb{R}; B^{-\sigma}_{r,k})} \leq \int_{-\infty}^\infty \left\| U(t) U(-t') f(t') \right\|_{L^q(\mathbb{R}; B^{-\sigma}_{r,k})} dt'.
\]
(2.4)

**Remark**
(i) The estimate for \( k = 2 \) was proved in [7].
(ii) We may replace \( K(t) = e^{\pm it\omega} \) for \( U(t) \).

(\dot{i}i) From the condition (2.3), if we take \( \theta = 0 \) and substitute the inhomogeneous norms by the homogeneous ones, i.e. \( B^{-\sigma}_{r,k} \rightarrow \dot{B}^{-\sigma}_{r,k} \), \( B^0_{2,k} \rightarrow \dot{B}^0_{2,k} \), then the estimates (2.2) and (2.4) satisfy scaling invariance.

**Proof of Lemma 4**
We concentrate on the case \( k = 1 \). From duality argument, it is sufficient to prove that
\[
\left\| \int_{-\infty}^\infty U(-t') F(t') dt' \right\|_{\dot{B}^\sigma_{r,\infty}} \lesssim \left\| F \right\|_{L^q_t B^\sigma_{r,\infty}},
\]
(2.5)
where \( q' \) and \( r' \) denote the H"older conjugate of \( q \) and \( r \) respectively. In fact in [7] we find
\[
\left\| \int_{-\infty}^\infty U(-t') \varphi_k \ast F(t') dt' \right\|_{L^2} \lesssim 2^{k\sigma} \left\| \varphi_k \ast F \right\|_{L^q_t L^{r'}},
\]
(2.6)
where \( \{\varphi_k\}_{0}^\infty \) is the Littlewood–Paley dyadic decomposition on \( \mathbb{R}^n \) and \( q, r, \sigma \) are as in Lemma 4. We take supremum of \( k \) on both sides to obtain (2.5).

We use the following Gagliardo–Nirenberg type interpolation inequality ([3] or see [8] for more general cases).

**Lemma 5** The following estimate holds.
\[
\left\| f \right\|_{L^\infty} \lesssim \left\| f \right\|_{H^\alpha_p}^{\delta} \left\| f \right\|_{H^\beta_q}^{1-\delta},
\]
(2.7)
where \( 1 \leq p, q \leq \infty \), \( 0 < \alpha, \beta < \infty \), \( 0 < \delta < 1 \), \( \alpha > n/p \), \( \beta < n/q \), \( \delta(\alpha - n/p) + (1 - \delta)(\beta - n/q) = 0 \).

**Proof of Theorem 3**
We define the complete metric space \( \Phi = \Phi(p, s, k, M) \) for \( NLD \) as
\[
\Phi = \{ \psi \in L^\infty(\mathbb{R}; B^s_{2,k}) \cap L^{p-1}(\mathbb{R}; L^\infty); \left\| \psi \right\|_{L^p_t B^s_{2,k}} + \left\| \psi \right\|_{L^p_t H^s_{2,k}} \leq M \}. \tag{2.8}
\]
We find a unique solution of \( NLD \) in \( \Phi \) for sufficiently small data \( \psi \) and \( M \). For any \( s \in \mathbb{R} \), \( k = 1, 2 \) and \( q, r, \sigma \) satisfying the condition in Lemma 4, we have from (2.1), (2.2) and (2.4),
\[
\left\| \psi \right\|_{L^q_t B^{-\sigma}_{r,k}} \lesssim \left\| \phi \right\|_{B^s_{2,k}} + \left\| f(\psi) \right\|_{L^1_t B^s_{2,k}} \lesssim \left\| \phi \right\|_{B^s_{2,k}} + \left\| \psi \right\|_{L^p_t L^\infty} \left\| \psi \right\|_{L^\infty_t B^s_{2,k}},
\]
(2.9)
So we concentrate on the $L_t^pL^\infty$ norm. We apply Lemma 4 to estimate it in $L_t^qB^{s-\sigma}_{r,k}$.

(1) $n = 1$. For any $p \geq 5$, we take $k = 1$ and

$$(s, q, r, \sigma, \theta) = \left( \frac{1}{2} + \frac{1}{(p-1)}, p-1, \infty, \frac{1}{2} + \frac{1}{(p-1)}, \frac{4}{(p-1)} \right)$$

and use $B_{\infty,1}^0 \hookrightarrow L^\infty$ to obtain the theorem.

(2) $n = 2$. For any $3 < p \leq 5$, we take $k = 1$ and

$$(s, q, r, \sigma, \theta) = \left( \frac{1}{2} + \frac{1}{(p-1)}, p-1, \infty, \frac{1}{2} + \frac{1}{(p-1)}, (5-p)/(p-1) \right)$$

and use $B_{\infty,1}^0 \hookrightarrow L^\infty$ to obtain the theorem.

(3) $n = 2$. For any $p > 5$, we take $k = 2$ and $0 < \delta < 1$ satisfying $(1-\delta)(p-1) \geq 4$. From (2.7), we estimate

$$||\psi||_{L_t^{p-1}L^\infty} \lesssim ||\psi||_{L_t^qH^s} ||\psi||_{L_t^{(1-\delta)(p-1)}H^{\delta\epsilon-\sigma}}^{1-\delta}. \quad (2.12)$$

We take

$$(s, q, r, \sigma, \theta) = \left( 1 - \frac{1}{(p-1)}, 1 - \delta(p-1), 1/2 - \frac{2}{(1-\delta)(p-1)}, 3/(1-\delta)(p-1), 0 \right)$$

(2.13)

to obtain the theorem.

(6) $n \geq 4$, $p = 3$. We take $k = 1$ and

$$(s, q, r, \sigma, \theta) = \left( \frac{n-1}{2}, 2, 2(n-1)/(n-3), (n+1)/2(n-1), 0 \right)$$

(2.14)

and use $B_{r,1}^{s-\sigma} \hookrightarrow L^\infty$ to obtain the theorem.

(7) $n \geq 4$. For any $p > 3$, we take $k = 2$ and $0 < \delta < 1$ satisfying $(1-\delta)(p-1) \geq 2$.

From (2.7), we estimate (2.12) and take

$$(s, q, r, \sigma, \theta) = \left( n/2 - 1/(p-1), (1-\delta)(p-1), (1/2-2/(1-\delta)(p-1)(n-1))^{-1}, (n+1)/(1-\delta)(p-1)(n-1), 0 \right)$$

to obtain the theorem.

### 3 Application for nonlinear Klein–Gordon equations

We apply the previous argument for the Klein–Gordon equation with derivative coupling ($NLKG$):

$$\partial_t^2 u - \Delta u + m^2 u = \lambda f(u), \quad u(0) = u_0, \partial_t u(0) = u_1, \quad (3.1)$$

where $u : \mathbb{R}^{1+n} \rightarrow \mathbb{C}$ is unknown, $u_0, u_1 : \mathbb{R}^n \rightarrow \mathbb{C}$ are given Cauchy data, $m > 0$ and $\lambda \in \mathbb{C}$ are constants. We consider the nonlinear term $f(u)$ of following types:

$$f(u) = \partial_j(u^p), \partial_k(u^p), \prod_{j=1}^n (\partial_j u)^{p_j}, (\partial_t u)^{p_0}, \prod_{j=1}^n (\partial_j u)^{p_j}, \quad (3.2)$$
where $1 \leq j \leq n$.

We give the results only. For simplicity we set $\varphi = (u_0, u_1)$ and $\|\varphi\|_{B^{s}_{2,k}} := \|u_0\|_{B^{s}_{2,k}} + \|u_1\|_{B^{s-1}_{2,k}}$.

**Theorem 6** Let $n, p, \phi$ satisfy the following conditions:

1. $n = 1$, $p \geq 5$, $\|\varphi\|_{B^{s}_{2,1}} \ll 1$, $s = 1/2 + 1/(p - 1)$,
2. $n = 2$, $3 < p \leq 5$, $\|\varphi\|_{B^{s}_{2,1}} \ll 1$, $s = 1/2 + 1/(p - 1)$,
3. $n = 2$, $p > 5$, $\|\varphi\|_{H^s(p)} \ll 1$,
4. $n = 3$, $p = 3$, $\|\varphi\|_{H^s(p)} \ll 1$,
5. $n = 3$, $p > 3$, $\|\varphi\|_{H^s(p)} \ll 1$,
6. $n \geq 4$, $p = 3$, $\|\varphi\|_{B^{s}_{2,1}} \ll 1$,
7. $n \geq 4$, $p > 3$, $\|\varphi\|_{H^s(p)} \ll 1$, $(s(p) < p$ if $p$ is not integer).

(1) - (7) for $f = \partial_j (u^p)$, (4) - (7) for $f = \partial_t (u^p)$.

Then NLKG has a solution $\psi \in C(\mathbb{R}; X)$, where $X$ denotes the space of data $\varphi$ indicated above.

**Remark** In this case $s(p)$ is scaling critical exponent for massless NLKG ($m = 0$).

**Theorem 7** Let $n, p, \phi$ satisfy the following conditions:

1. $n = 1$, $p \geq 5$, $\|\varphi\|_{B^{s+1}_{2,1}} \ll 1$, $s = 1/2 + 1/(p - 1)$,
2. $n = 2$, $3 < p \leq 5$, $\|\varphi\|_{B^{s+1}_{2,1}} \ll 1$, $s = 1/2 + 1/(p - 1)$,
3. $n = 2$, $p > 5$, $\|\varphi\|_{H^{s(p)+1}} \ll 1$, $s > 1$,
4. $n = 3$, $p = 3$, $\|\varphi\|_{H^{s+1}} \ll 1$, $s > 1$,
5. $n = 3$, $p > 3$, $\|\varphi\|_{H^{s(p)+1}} \ll 1$,
6. $n \geq 4$, $p = 3$, $\|\varphi\|_{B^{s+1}_{2,1}} \ll 1$,
7. $n \geq 4$, $p > 3$, $\|\varphi\|_{H^{s(p)+1}} \ll 1$,

(1) - (7) for $f = \prod_{j=1}^{n} (\partial_j u)^{p_j}$, $p_1 + \cdots + p_n = p$, $p_j \in \mathbb{Z}^+ \cup \{0\}$ or $p_j > \max\{1, s\}$,

(4) - (7) for $f = (\partial_t u)^{p_0} \prod_{j=1}^{n} (\partial_j u)^{p_j}$, $p_0 + \cdots + p_n = p$, $p_j \in \mathbb{Z}^+ \cup \{0\}$ or $p_j > \max\{1, s\}$.

Then NLKG has a solution $\psi \in C(\mathbb{R}; X)$, where $X$ denotes the space of data $\varphi$ indicated above.

**Remark** In this case $s(p) + 1$ is scaling critical exponent for massless NLKG ($m = 0$).

**References**


