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On Hölder Type Inequality in Besov Spaces with Applications to the Navier–Stokes Equations

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1 Introduction.

(Equations). We consider the nonstationary Navier–Stokes equations in $\mathbb{R}^n$ $(n \geq 2)$:

$$\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0, \qquad \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = u_0, \qquad \operatorname{div} u_0 = 0 \qquad \text{in } \mathbb{R}^n. \end{cases}$$

Here, $u = u(x,t) = (u^1(x,t), u^2(x,t), \ldots, u^n(x,t))$ and $p = p(x,t)$ stand for the unknown velocity and unknown scalar function, respectively; u_0 is a given initial velocity. Throughout this paper we do not distinguish the space of vector–valued from scalar functions.

The existence of the locally–in–time solution to (NS) is well known when the initial data in L^p, see [16] or [11]. It should be noted that L^∞ solution is also constructed by [8] and [12].

(Function Spaces). Our purpose in this paper is to construct the locally–in–time solution to (NS) with nondecaying initial data. The spaces which we treat are larger than L^∞. Before stating our results, we should recall several Besov type function spaces used in this paper; see [25].

Definition 1. *Let $n \geq 1$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. An inhomogeneous Besov space is defined by*

$$B^s_{p,q}(\mathbb{R}^n) \equiv \{ f \in \mathcal{S}'; ||f; B^s_{p,q}|| < \infty \},$$

$$||f; B^s_{p,q}|| \equiv \begin{cases} ||\psi * f; L^p|| + \left[\sum_{j=1}^\infty 2^{jsq} ||\phi_j * f; L^p||^q \right]^{1/q} & if \quad q < \infty, \\ ||\psi * f; L^p|| + \sup_{j \geq 1} 2^{js} ||\phi_j * f; L^p|| & if \quad q = \infty. \end{cases}$$

Here, (ψ, ϕ_j) is the Littlewood–Paley dyadic decomposition of unity, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all tempered distributions. Throughout this paper we suppress $n \geq 1$ and $\mathbb{R}^n$. Following J. Johnsen [14], we call s the differentiability–exponent, p the integral–exponent and q the sum–exponent. We next define its homogeneous version.

Definition 2. *Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. A homogeneous Besov space is defined by*

$$\dot{B}^s_{p,q} \equiv \{f \in \mathcal{Z}'; \|f; \dot{B}^s_{p,q}\| < \infty\},$$
$$\|f; \dot{B}^s_{p,q}\| \equiv \begin{cases} \left[\sum_{j=-\infty}^{\infty} 2^{jsq}\|\phi_j * f; L^p\|^q\right]^{1/q} & if \quad q < \infty, \\ \sup_{-\infty \leq j \leq \infty} 2^{js}\|\phi_j * f; L^p\| & if \quad q = \infty, \end{cases}$$

where $\mathcal{Z}'$ is the topological dual space of

$$\mathcal{Z} \equiv \{f \in \mathcal{S};\ D^\alpha \hat{f}(0) = 0,\ \forall \alpha \in \mathbb{N}_0^n\}.$$

Here, $\hat{f}$ is denoted by the Fourier transform, and we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers. It is well known that the homogeneous Besov space can be regarded as subspace of $\mathcal{S}'$ if either $s < n/p$ or $s = n/p$ and $q = 1$; see [6] or [20]. We hereafter only treat these spaces with exponents satisfying this condition.

We also define several associated spaces. We set that $e^{t\Delta} = G_t*$ denotes the solution–operator of the heat equation; G_t is Gauss kernel denoted by $G_t(x) = (4\pi t)^{-n/2}\exp(-\frac{|x|^2}{4t})$. One extends $e^{t\Delta}$ from $\mathcal{S}$ to $\mathcal{S}'$ in usual way. Unfortunately, $e^{t\Delta}$ is not a continuous (C_0)–semigroup in Besov spaces if integral–exponent or sum–exponent is infinity. We note that $e^{t\Delta}f \to f$ in $B^s_{p,q}$ need not hold for general element of $B^s_{p,q}$. Thus, in order to construct the solution which is continuous up to initial time, we have to set the small space.

Definition 3 (small Besov spaces). *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. A small inhomogeneous Besov space is the subspace defined by*

$$b^s_{p,q} \equiv \{f \in B^s_{p,q};\ e^{t\Delta}f \to f \ in\ B^s_{p,q}\ as\ t \downarrow 0\}.$$

Assume in addition that (in order to operate $e^{t\Delta}$) these exponents satisfy the condition of either $s < n/p$ or $s = n/p$ and $q = 1$. A small homogeneous Besov space is defined by

$$\dot{b}^s_{p,q} \equiv \{f \in \dot{B}^s_{p,q};\ e^{t\Delta}f \to f \ in\ \dot{B}^s_{p,q}\ as\ t \downarrow 0\}.$$

It is easy to see that the small Besov space is a closed subspace of Besov space, so it is Banach space. Let $\overset{\circ}{B}{}^s_{p,q}$ be the closure of $\mathcal{S}$ with respect to the norm of $B^s_{p,q}$ (see e.g. [25]). By definition our spaces satisfy

$$\overset{\circ}{B}{}^s_{p,q} \subset b^s_{p,q} \subset B^s_{p,q}.$$

Of course, these three spaces agree each other if p and q are finite. But otherwise these spaces are different from each other, for example, if $s \le 0$, $p = \infty$ and $q < \infty$, then

$$\overset{\circ}{B}{}^s_{\infty,q} \subsetneqq b^s_{\infty,q} = B^s_{\infty,q}.$$

Indeed, non-zero constant function belongs to $b^s_{\infty,q}$, however, it does not belong to $\overset{\circ}{B}{}^s_{\infty,q}$. It is also easy to see that $b^s_{p,q} \ne B^s_{p,q}$ if and only if $q = \infty$. Moreover, one can prove that small Besov space is equivalent to the space of closure of $B^{s+1}_{p,q}$ with respect to the norm of $B^s_{p,q}$, i.e. $b^s_{p,q} = \overline{B^{s+1}_{p,q}}^{\|\cdot;B^s_{p,q}\|}$. The space $\overline{B^{s+1}_{p,q}}^{\|\cdot;B^s_{p,q}\|}$ is called *little Besov space.* In [2] H. Amann characterizes the little Besov spaces, see also [23, Appendix]. However, in the homogeneous version $\dot{b}^s_{p,q}$ is new space.

(Main Result). Our goal is to prove the existence and uniqueness of locally–in–time smooth solution to (NS) when the initial velocity u_0 belongs to $b^s_{p,q}$ or $\dot{b}^s_{p,q}$ with $s \le 0$. We are now in position to state our main results.

Theorem 1. *Assume that $n \ge 2$, $n < p \le \infty$, $1 \le q \le \infty$ and $0 \le \varepsilon < 1 - n/p$, and assume that the initial data $u_0 \in b^{-\varepsilon}_{p,q}(\mathbb{R}^n)$ satisfying* $\operatorname{div} u_0 = 0$. *Then there exists a positive constant T_0 and a unique u satisfying*

$$t^{\gamma/2} u \in C([0,T_0]; b^{\gamma-\varepsilon}_{p,q}(\mathbb{R}^n)) \quad \text{for all} \quad 0 \le \gamma \le 1,$$
$$t^{\delta/2} u \in C([0,T_0]; L^p(\mathbb{R}^n)) \quad \text{for all} \quad \varepsilon < \delta < 1,$$

such that $(u(t), \nabla p(t))$ is a unique classical solution to (NS), provided that

$$\nabla p(t) = \sum_{i,j=1}^{n} \nabla R_i R_j u^i(t) u^j(t),$$

where $R_i = \partial_i(-\Delta)^{-1/2}$ is the Riesz transform.

Remark 1. (i) In our result $q = \infty$ is included, the space $b^{-\varepsilon}_{p,\infty}$ includes L^p spaces for $p < \infty$ and BUC for $p = \infty$ for any $\varepsilon \ge 0$. Here, BUC represents the space of all bounded and uniformly continuous functions.

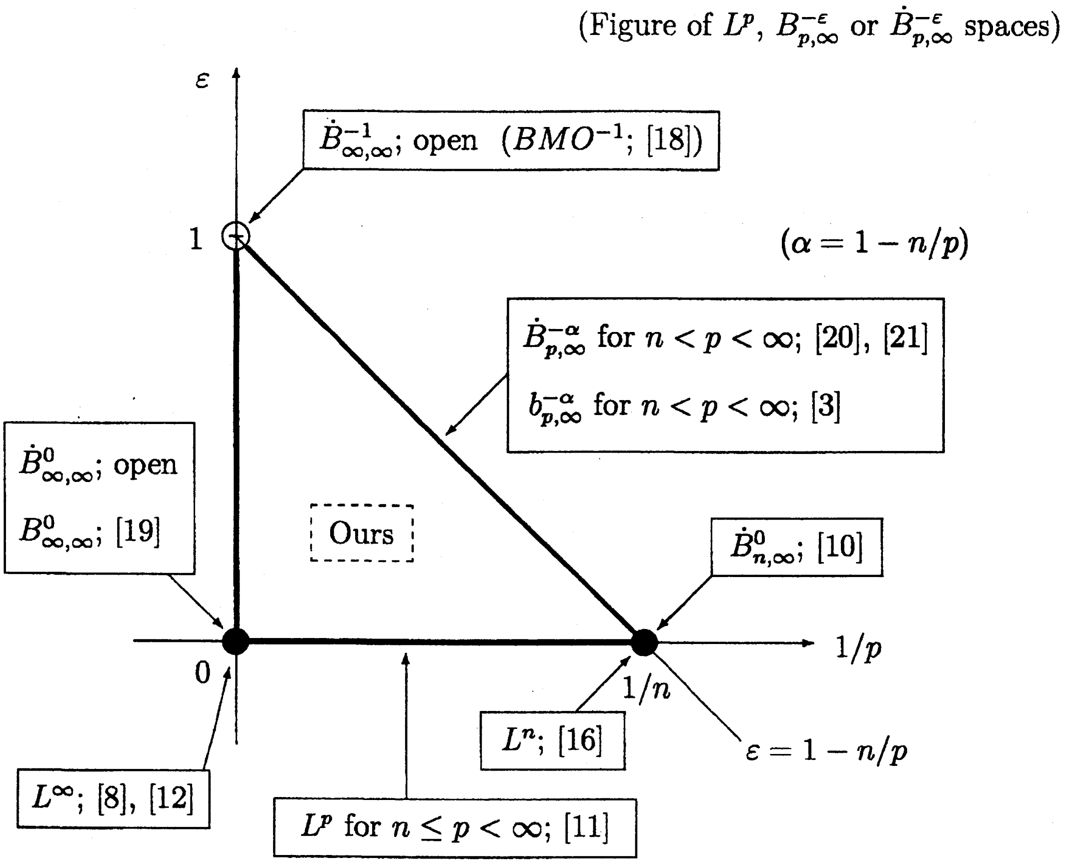

Figure 1:

(ii) Similarly, one can also construct the locally–in–time solution in $\dot{b}^{-\varepsilon}_{p,q}$ with assumption of $n < p \le \infty$, $1 \le q \le \infty$ and $0 < \varepsilon < 1 - n/p$. Of course, we get the properties of the solution by replacing function spaces by their homogeneous version.

(iii) In [3, Theorem 6.1] H. Amann shows the local solvability of Navier–Stokes equations in $b^{-1+n/p}_{p,\infty}$ for $n < p < \infty$. So our results on this paper for $n < p < \infty$ is given by interpolation theory easily. In the case of $p = \infty$ Theorem is new.

2 Known Results.

We mention several known results on the solvability for the Navier–Stokes equations in L^p. Previous work by T. Kato [16] in 1984, in whole spaces he showed the local existence with initial data in $L^n(\mathbb{R}^n)$, and Y. Giga [11] also obtained the local existence with initial data in $L^p(\mathbb{R}^n)$ for $n \le p < \infty$; see Figure 1. The local existence for L^∞ initial data (or BUC initial data) is also constructed by M. Cannone [8] and Giga–Inui–Matsui [12] in general dimension. Our results include of theirs, in the sense that the space of initial data contains theirs.

There have already been several results on solvability in Besov spaces. In 1994 Kozono–Yamazaki [20] obtained the solution in $\dot{B}^{-\alpha}_{p,\infty}$ for $n < p < \infty$ with $\alpha = 1 - n/p$. The spaces $\dot{B}^{-\alpha}_{p,\infty}$ are important since these spaces are scaling invariant. Cannone–Planchon [10] showed that in $\dot{B}^{0}_{3,\infty}$, and they also obtained that in same spaces as Kozono–Yamazaki's results. By the way, in the inhomogeneous case H. Amann [3] showed that in $b^{-\alpha}_{p,\infty}$. Although Kobayashi–Muramatu [17] also obtained that in $\mathring{B}^{-1/2}_{\infty,\infty}$, there seem to be no results when the space of initial data does not decay at space infinity. Our results is the first results handling nondecaying Besov space as the space of initial data.

Recent work by Koch–Tataru [18] introduce the new space of $BMO^{-1}(\mathbb{R}^n)$ which is the space of all first derivatives of BMO function, and related localized space BMO_T^{-1}. They show the existence of time–local solution of (NS) in this space, and they also construct the time–global solution with small data. We note that BMO^{-1} is very closed to $\dot{B}^{-1}_{\infty,\infty}$, and $\dot{B}^{-1}_{\infty,\infty}$ is important for us to investigate the self–similar solution, see [8]. The present work is inspired by their work.

The author guesses that those researchers who obtained the local existence of the solution with initial data in $\dot{B}^{-\alpha}_{p,\infty}$ wanted to get the solution in $\dot{B}^{-1}_{\infty,\infty}$. Then they studied that along this line, but they could not achieve it. While we intended to achieve it along the axis $\dot{B}^{-\varepsilon}_{\infty,\infty}$ tending $\varepsilon \to 1$ since we have already obtained L^∞ solution, however, we could not. The solvability in $\dot{B}^{-1}_{\infty,\infty}$ is still open. The author was informed of a recent work of Kozono–Ogawa–Taniuchi [19] closely related to ours. They also proved the existence of a unique solution to (NS) with initial data in $B^{0}_{\infty,\infty}$, but which space is contained by ours. However, the solvability in $\dot{B}^{0}_{\infty,\infty}$ is also still open.

3 Estimate for products.

We consider the integral equation:

$$\text{(INT)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s)ds,$$

where $u \otimes u$ is a tensor whose ij–component is $u^i u^j$; $\mathbf{P}$ denotes by $n \times n$ matrix operator, its ij–component is $\delta_{ij} + R_i R_j$, where δ_{ij} is Kronecker's delta. We call the solution of (INT) *mild solution.* Once we get the mild solution, it is easy to see that the mild solution satisfies (NS) in classical sense.

A crucial step in getting the mild solution is to estimate for bilinear terms, that is, we have to estimate the Besov norm of the integrant of (INT). Herenow, we shall establish a Hölder type inequality to state it in the next proposition.

Proposition 1. *Let* $\alpha > 0$, $1 \le p, q \le \infty$, *and let* $1 \le r, s \le \infty$ *satisfying* $1/p = 1/r + 1/s$. *Let* $\sigma > 0$, $\theta \ge 0$. *Then there exists a positive constant* $C = C(\alpha, p, q, r, s, \sigma, \theta)$ *such that*

$$\begin{aligned}||fg; B^{\alpha}_{p,q}|| \le C\Big[&(N^2+1)\{||f; B^{\theta+\alpha}_{r,q}||\ ||g; B^{-\theta}_{s,q}|| + ||f; B^{-\theta}_{s,q}||\ ||g; B^{\theta+\alpha}_{r,q}||\}\\ &+ 2^{-N\delta}(N+1)\{||f; B^{\sigma+\alpha+\delta}_{r,q}||\ ||g; B^{-\sigma}_{s,q}|| + ||f; B^{-\sigma}_{s,q}||\ ||g; B^{\sigma+\alpha+\delta}_{r,q}||\}\Big]\end{aligned}$$

for all $N \in \mathbb{N}_0$, $0 < \delta \le \alpha$, f *and* g *belong to intersection of all inhomogeneous Besov spaces in right–hand–side, respectively.*

Remark 2. (i) In the last term of above inequality the sum of differentiability–exponents do not coincide with those in other terms. It is too strong in appearance, but it is compensated by coefficients $2^{-N\delta}$ of the inequality. We shift differential to dyadic decomposition, then this term appear.
(ii) One can prove similar inequality in the homogeneous Besov spaces. Let exponents be the same as in Proposition 1. Then

$$\begin{aligned}||fg; \dot{B}^{\alpha}_{p,q}|| \le C\Big[&(N^2+1)\{||f; \dot{B}^{\theta+\alpha}_{r,q}||\ ||g; \dot{B}^{-\theta}_{s,q}|| + ||f; \dot{B}^{-\theta}_{s,q}||\ ||g; \dot{B}^{\theta+\alpha}_{r,q}||\}\\ &+ 2^{-N\delta}(N+1)\{||f; \dot{B}^{\sigma+\alpha+\delta}_{r,q}||\ ||g; \dot{B}^{-\sigma}_{s,q}|| + ||f; \dot{B}^{-\sigma}_{s,q}||\ ||g; \dot{B}^{\sigma+\alpha+\delta}_{r,q}||\}\\ &+ 2^{-N\delta}(N+1)\{||f; \dot{B}^{\sigma+\alpha-\delta}_{r,q}||\ ||g; \dot{B}^{-\sigma}_{s,q}|| + ||f; \dot{B}^{-\sigma}_{s,q}||\ ||g; \dot{B}^{\sigma+\alpha-\delta}_{r,q}||\}\Big].\end{aligned}$$

(iii) Hölder type estimates, for example

$$||fg; B^{\alpha}_{p,q}|| \le C\{||f; B^{\beta}_{p_1,q_1}||\ ||g; B^{\gamma}_{p_2,q_2}|| + ||f; B^{\gamma}_{p_2,q_2}||\ ||g; B^{\beta}_{p_1,q_1}||\}$$

have been proved by [22, §4.4.3 Theorem 1, §4.5.2 Corollary, and so on] with several restriction of exponents. However, we want to use such estimate for $p = p_1 = p_2 = \infty$ and $\alpha > 0$ which is unfortunately excluded. So we prepare the present version of the Hölder type inequality.

For the proof of Proposition 1 we prepare two lemmas. Next is paraproduct lemma which is similar as Bony's paraproduct lemma [5]. We shall use the convention that $f_k = \phi_k * f$, $g_l = \phi_l * g$, $f_\sharp = \psi * f$ and $g_\sharp = \psi * g$ as well as $a \vee b = \max(a, b)$.

Lemma 1 (paraproduct lemma). *Let $j \in \mathbb{N}$. Let $f, g, fg \in \mathcal{S}'$. Then*

$$\psi * \{(f_\natural + \sum_{k=1}^{\infty} f_k) \cdot (g_\natural + \sum_{l=1}^{\infty} g_l)\}$$

$$= \psi * \{ \sum_{\{k,l \geq 1;\ |k-l| \leq 2\}} f_k g_l \} + \psi * \{ \sum_{k=1}^{2} f_k g_\natural \} + \psi * \{ \sum_{l=1}^{2} f_\natural g_l \} + \psi * \{ f_\natural g_\natural \},$$

and then

$$\phi_j * \{(f_\natural + \sum_{k=1}^{\infty} f_k) \cdot (g_\natural + \sum_{l=1}^{\infty} g_l)\}$$

$$= \phi_j * \{ \sum_{(k,l) \in S_j} f_k g_l \} + \phi_j * \{ \sum_{k=1 \vee (j-2)}^{j+2} f_k g_\natural \} + \phi_j * \{ \sum_{l=1 \vee (j-2)}^{j+2} f_\natural g_l \}$$
$$+ (\delta_{j1} + \delta_{j2}) \phi_j * \{ f_\natural g_\natural \},$$

where $S_j = S_j^1 + S_j^2 + S_j^3$;

$$S_j^1 = \{(k,l) \in \mathbb{N}^2;\ k, l \geq j,\ |k-l| \leq 2\},$$
$$S_j^2 = \{(k,l) \in \mathbb{N}^2;\ k \leq j,\ |l-j| \leq 2\},$$
$$S_j^3 = \{(k,l) \in \mathbb{N}^2;\ l \leq j,\ |k-j| \leq 2\}.$$

Proof. We shall verify whether $\phi_j * (f_k g_l) \equiv 0$ for given j, k and l. We consider its Fourier transforms and obtain

$$\mathcal{F}[\phi_j * \{(\phi_k * f) \cdot (\phi_l * g)\}] = \hat{\phi}_j \cdot \{(\hat{\phi}_k \hat{f}) * (\hat{\phi}_l \hat{g})\}.$$

Then it is enough to estimate the support of $\hat{\phi}_j \cdot (\hat{\phi}_k * \hat{\phi}_l)$. We have

$$\Phi_{jkl} = \left(\hat{\phi}_j \cdot (\hat{\phi}_k * \hat{\phi}_l)\right)(\xi) = \hat{\phi}_j(\xi) \int_{\mathbb{R}^n} \hat{\phi}_k(\xi - \eta) \hat{\phi}_l(\eta) d\eta,$$

and observe that Φ_{jkl} equals zero if (j, k, l) satisfies the following conditions:

$$\text{either} \quad 2^{l+1} + 2^{j+1} \leq 2^{k-1}, \tag{3.1}$$
$$\text{or} \quad 2^{j+1} + 2^{k+1} \leq 2^{l-1}, \tag{3.2}$$
$$\text{or} \quad 2^{k+1} + 2^{l+1} \leq 2^{j-1}. \tag{3.3}$$

The proof is now complete. □

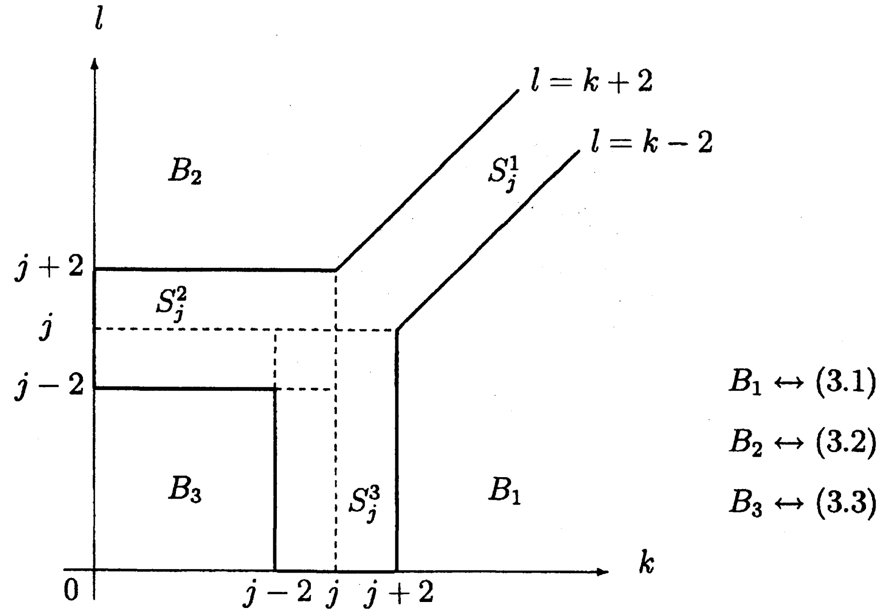

Figure 2:

Similar paraproduct lemma is found in [Bon]. He calculates the support of $(\phi_k * f) \cdot (\phi_l * g)$ to show that Φ_{jkl} equals zero for the indices in B_1 and B_2; see Figure 2. We also calculate $\phi_j * \{(\phi_k * f) \cdot (\phi_l * g)\}$ and show $\Phi_{jkl} = 0$ in B_3. This procedure is not included in [5], so our lemma is different from his results. In order to state the next lemma it is necessary to study the part corresponding B_3.

Its homogeneous version are essentially known by those who study nonlinear wave equations in several papers, e.g. [22]. The authors of these papers calculate $\Phi_{jkl} = 0$ in some indices, after using Bony's paraproduct lemma. However, they do not write $\Phi_{jkl} = 0$ in B_3 explicitly. We fix j and prove that $\Phi_{jkl} = 0$ for arbitrary k and l. Thus we are able to describe the situation clearly in Figure 2.

The next lemma yields Proposition 1. This is one of the most general form of Hölder type inequality in inhomogeneous Besov spaces.

Lemma 2 (Hölder inequality). *Let $1 \le p, q \le \infty$ and $\alpha > 0$. Let $i = 1, 2, \ldots, 12$; $1 \le r_i, s_i \le \infty$ satisfying $1/p = 1/r_i + 1/s_i$, and let $\rho_i \in \mathbb{R}$, $\theta_i \ge 0$ and $\sigma_i > 0$. Then there exists a positive constant $C = C(\alpha, p, q, r_i, s_i, \rho_i, \theta_i, \sigma_i)$ such that*

$$||fg; B^{\alpha}_{p,q}|| \le C(N^2+1)\Pi_1(f,g) + C(N+1)2^{-N\delta}\Pi_2(f,g),$$

where

$$\begin{aligned}
\Pi_1(f,g) = {} & ||f; B^{\rho_1+\alpha}_{r_1,\infty}||\ ||g; B^{-\rho_1}_{s_1,\infty}|| + ||f; B^{-\theta_2}_{r_2,\infty}||\ ||g; B^{\theta_2+\alpha}_{s_2,\infty}|| \\
& + ||f; B^{\theta_3+\alpha}_{r_3,\infty}||\ ||g; B^{-\theta_3}_{s_3,\infty}||, \\
\Pi_2(f,g) = {} & ||f; B^{\rho_4+\alpha+\delta}_{r_4,\infty}||\ ||g; B^{-\rho_4}_{s_4,\infty}|| + ||f; B^{-\sigma_5}_{r_5,\infty}||\ ||g; B^{\sigma_5+\alpha+\delta}_{s_5,\infty}|| \\
& + ||f; B^{\sigma_6+\alpha+\delta}_{r_6,\infty}||\ ||g; B^{-\sigma_6}_{s_6,\infty}|| + ||f; B^{\rho_7+\sigma_7}_{r_7,\infty}||\ ||g; B^{-\rho_7}_{s_7,\infty}|| \\
& + ||f; B^{\sigma_8+\alpha}_{r_8,\infty}||\ ||g; B^{\mu_8}_{s_8,\infty}|| + ||f; B^{\mu_9}_{r_9,\infty}||\ ||g; B^{\sigma_9+\alpha}_{s_9,\infty}|| \\
& + ||f; B^{\sigma_{10}}_{r_{10},\infty}||\ ||g; B^{\mu_{10}}_{s_{10},\infty}|| + ||f; B^{\mu_{11}}_{r_{11},\infty}||\ ||g; B^{\sigma_{11}+\alpha}_{s_{11},\infty}|| \\
& + ||f; B^{\mu_{12}}_{r_{12},\infty}||\ ||g; B^{\bar{\mu}_{12}+\alpha}_{s_{12},\infty}||
\end{aligned}$$

for all $N \in \mathbb{N}_0$, $0 < \delta \leq \alpha$, $\mu_i, \tilde{\mu}_i \in \mathbb{R}$, f and g belong to intersection of all Besov spaces in right–hand–side, respectively.

Proof. We may assume that q is finite without loss of generality, since we give the proof for the case $q = \infty$ is obtained by a standard modification of that for finite q.

By the definition we have

$$\begin{aligned}
||fg; B^{\alpha}_{p,q}|| = {} & \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} ||\phi_j * (fg); L^p||^q\Big]^{1/q} + ||\psi * (fg); L^p|| \\
= {} & \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} ||\phi_j * \{(\sum_{k=1}^{\infty} f_k + f_\natural) \cdot (\sum_{l=1}^{\infty} g_l + g_\natural)\}; L^p||^q\Big]^{1/q} \\
& + ||\psi * \{(\sum_{k=1}^{\infty} f_k + f_\natural) \cdot (\sum_{l=1}^{\infty} g_l + g_\natural)\}; L^p||.
\end{aligned}$$

Applying Lemma 3, we observe that

$$\begin{aligned}
\leq & \Big[\sum_{j=1}^{\infty} 2^{j\alpha q} ||\phi_j * \{ \sum_{(k,l)\in S_j} f_k g_l + \sum_{k=1\vee(j-2)}^{j+2} f_k g_\natural + \sum_{l=1\vee(j-2)}^{j+2} f_\natural g_l \\
& + (\delta_{j1} + \delta_{j2}) f_\natural g_\natural\}; L^p||^q\Big]^{1/q} \\
& + ||\psi * \{ \sum_{\{k,l\in\mathbb{N};|k-l|\leq 2\}} f_k g_l + \sum_{k=1}^{2} f_k g_\natural + \sum_{l=1}^{2} f_\natural g_l + f_\natural g_\natural\}; L^p||.
\end{aligned}$$

We set $||\phi_j; L^1|| = C_0$ (independent of j) and $||\psi; L^1|| = C_1$. By using l^q–Minkowski and

L^p–Young inequalities, we get

$$
\begin{aligned}
&\|fg; B^{\alpha}_{p,q}\| \\
&\le C_0\Big\{\Big[\sum_{j=1}^{\infty} 2^{j\alpha q}\|\sum_{(k,l)\in S_j} f_k g_l; L^p\|^q\Big]^{1/q} + \Big[\sum_{j=1}^{\infty} 2^{j\alpha q}\|\sum_{k=1\vee(j-2)}^{j+2} f_k g_\sharp; L^p\|^q\Big]^{1/q} \\
&\quad + \Big[\sum_{j=1}^{\infty} 2^{j\alpha q}\|\sum_{l=1\vee(j-2)}^{j+2} f_k g_\sharp; L^p\|^q\Big]^{1/q} + \Big[\sum_{j=1}^{2} 2^{j\alpha q}\|f_\sharp g_\sharp; L^p\|^q\Big]^{1/q}\Big\} \\
&\quad + C_1\Big\{\|\sum_{\{k,l\in\mathbb{N};|k-l|\le 2\}} f_k g_l; L^p\| + \|\sum_{k=1}^{2} f_k g_\sharp; L^p\| + \|\sum_{l=1}^{2} f_\sharp g_l; L^p\| + \|f_\sharp g_\sharp; L^p\|\Big\} \\
&\equiv C_0(\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4) + C_1(\mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4).
\end{aligned}
$$

We shall estimate each term.

We present estimates for I_1 and I_2 only, since other terms can be estimated in a similar (and easier) way. First we estimate I_1. We divide S_j into three sets, we have $\mathrm{I}_1 \le \sum_{m=1}^{3} \mathrm{J}_m$ with $\mathrm{J}_m = \Big[\sum_{j=1}^{\infty} 2^{j\alpha q}\|\sum_{(k,l)\in S_j^m} f_k g_l; L^p\|^q\Big]^{1/q}$.

We start to estimate J_1 by recalling definition of S_j^1:

$$
\mathrm{J}_1 \le [\sum_{j=1}^{\infty} 2^{j\alpha q}\{\sum_{k\ge j}\sum_{l=1\vee(k-2)}^{k+2} \|f_k g_l; L^p\|\}^q]^{1/q}.
$$

We divide the sum into three parts with respect to indices j and k of middle–middle, middle–high, high–high frequency. For all positive integer N

$$
\begin{aligned}
\mathrm{J}_1 \le& [\sum_{1\le j\le N} 2^{j\alpha q}\{\sum_{j\le k\le N}\sum_{l=1\vee(k-2)}^{k+2} \|f_k g_l; L^p\|\}^q]^{1/q} \\
&+ [\sum_{1\le j\le N} 2^{j\alpha q}\{\sum_{k\ge N}\sum_{l=1\vee(k-2)}^{k+2} \|f_k g_l; L^p\|\}^q]^{1/q} \\
&+ [\sum_{j\ge N+1} 2^{j\alpha q}\{\sum_{k\ge j}\sum_{l=1\vee(k-2)}^{k+2} \|f_k g_l; L^p\|\}^q]^{1/q}, \\
\equiv& \mathrm{J}_{MM} + \mathrm{J}_{MH} + \mathrm{J}_{HH}.
\end{aligned}
$$

[J_{MM} estimate]. We use exponents $1 \le r, s \le \infty$, $1/p = 1/r + 1/s$ and $\rho \in \mathbb{R}$ to get

$$
\mathrm{J}_{MM} \le \Big[\sum_{1\le j\le N} 2^{j\alpha q}\{\sum_{j\le k\le N} 2^{-k\rho}2^{k\rho}\|f_k; L^r\| \sum_{l=1\vee(k-2)}^{k+2} \|g_l; L^s\|\}^q\Big]^{1/q}.
$$

Since $j \leq k$ and $k-2 \leq l \leq k+2$, we obtain that $2^{j\alpha} \leq 2^{k\alpha}$ and $2^{-k\rho} \leq 2^{2|\rho|} \cdot 2^{-l\rho}$. We also observe that $2^{k(\alpha+\rho)}||f_k; L^r|| \leq \sup_k 2^{k(\alpha+\rho)}||f_k; L^r|| = ||f; B^{\rho+\alpha}_{r,\infty}||$ and similarly $2^{-l\rho}||g_l; L^s|| \leq ||g; B^{-\rho}_{s,\infty}||$. Combining these estimates yields

$$\begin{aligned}\mathrm{J}_{MM} \leq & C||f; B^{\rho+\alpha}_{r,\infty}||\ ||g; B^{-\rho}_{s,\infty}||\Big[\sum_{1\leq j\leq N} 1\Big]^{1/q} \cdot \{\sum_{1\leq k\leq N} 1\}\\ \leq & C(N^2+1)||f; B^{\rho+\alpha}_{r,\infty}||\ ||g; B^{-\rho}_{s,\infty}||.\end{aligned}$$

[J_{MH} estimate] Let r, s and ρ be as the same exponents as in J_{MM} estimate, and let $\delta > 0$. We obtain

$$\begin{aligned}\mathrm{J}_{MH} \leq & \Big[\sum_{1\leq j\leq N} 2^{j\alpha q}\{\sum_{k\geq N} 2^{-k(\alpha+\delta+\rho)}2^{k(\alpha+\delta+\rho)}||f_k; L^r|| \sum_{l=1\vee(k-2)}^{k+2} ||g_l; L^s||\}^q\Big]^{1/q}\\ \leq & C\Big[\sum_{1\leq j\leq N} 1\Big]\{\sum_{k\geq N} 2^{-k\delta}||f; B^{\rho+\alpha+\delta}_{r,\infty}|| \sum_{l=1\vee(k-2)}^{k+2} ||g; B^{-\rho}_{s,\infty}||\}\\ \leq & C2^{-N\delta}(N+1)||f; B^{\rho+\alpha+\delta}_{r,\infty}||\ ||g; B^{-\rho}_{s,\infty}||.\end{aligned}$$

[J_{HH} estimate] Let r, s, ρ, δ be as the same exponents as in J_{MH} estimate. We obtain

$$\begin{aligned}\mathrm{J}_{HH} \leq & \Big[\sum_{j\geq N} 2^{-j\delta q/2}2^{j(\alpha+\delta/2)q}\{\sum_{k\geq j} ||f_k; L^r|| \sum_{l=1\vee(k-2)}^{k+2} ||g_l; L^s||\}^q\Big]^{1/q}\\ \leq & C\Big[\sum_{j\geq N} 2^{-j\delta q/2}\{\sum_{k\geq j} 2^{-k\delta/2}||f; B^{\rho+\alpha+\delta}_{r,\infty}|| \sum_{l=1\vee(k-2)}^{k+2} ||g; B^{-\rho}_{s,\infty}||\}\Big]^{1/q}\\ \leq & C||f; B^{\rho+\alpha+\delta}_{r,\infty}||\ ||g; B^{-\rho}_{s,\infty}||\Big[\sum_{j\geq N} 2^{-j\delta q/2}\Big]^{1/q}\{\sum_{k\geq N} 2^{-j\delta/2}\}\\ \leq & C2^{-N\delta}||f; B^{\rho+\alpha+\delta}_{r,\infty}||\ ||g; B^{-\rho}_{s,\infty}||.\end{aligned}$$

The estimates for J_2 and J_3 are basically the same as that for J_1, so we do not present the details.

We next estimate I_2. Let $1 \leq r, s \leq \infty$; $1/p = 1/r + 1/s$, $\sigma > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. We observe that

$$\mathrm{I}_2 \leq \Big[\sum_{j\geq 1} 2^{-j\sigma q}\{\sum_{k=1\vee(j-2)}^{j+2} 2^{k(\sigma+\alpha)}||f_k; L^r||\ ||g_{\natural}; L^s||\}^q\Big]^{1/q}.$$

Note that $||g_{\natural}; L^s|| \leq ||g; B^{\mu}_{s,\infty}||$ for all $\mu \in \mathbb{R}$ to get

$$\leq C||f; B^{\sigma+\alpha}_{r,\infty}||\ ||g; B^{\mu}_{s,\infty}||.$$

Similarly one can estimate all of other terms. The proof is now complete. □

We note that if q is infinite, above estimates holds with (N^2+1) and $(N+1)$ replaced by 1. We mention the proof of Remark 2–(ii). We can also obtain its homogeneous version by dividing the sum into six parts with respect to frequencies of j and k, these are low–frequencies, middle–frequencies and high–frequencies. This proof parallels that of Lemma 2.

4 Sketch of proof of Theorem 1.

In this section we describe the sketch of the proof of Theorem 1. The local existence of the solutions for this type is often proved by the method called *iteration*, saying, successive approximation. The method is standard when we construct an L^p solution, see [16] and [11], and also by using this method L^∞ solutions are constructed by [12]. Since we handle the small Besov space, we can prove the continuity of approximate sequence in time, in particular continuity up to initial time with values in small Besov space.

Let $n \geq 2$, $0 < \varepsilon < 1/2$ and $1 \leq q < \infty$ since other cases can be proved by a similar argument. Assume that an initial velocity u_0 belongs to $b^{-\varepsilon}_{\infty,q}$. We define the successive approximation by setting $\{u_j(t)\}_{j\geq 1}$ inductively as $u_1(t) \equiv e^{t\Delta}u_0$ and

$$u_{j+1}(t) \equiv e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u_j \otimes u_j)(s)ds.$$

We shall conclude that the approximation $\{u_j(t)\}_{j\geq 1}$ have a unique limit function by a priori estimate. It is easy to see that $u(t)$ satisfies (INT) for $t \in [0, T_0]$. The uniqueness is obtained by Gronwall's inequality (see [13]) easily.

On this paper we only make sure that u_j belongs to $B^{-\varepsilon}_{\infty,q}$ since it is key estimate in this proof. We show the following lemma:

Lemma 3. *There exists a positive constant T such that*

$$t^{\gamma/2}u_j(t) \in B^{\gamma-\varepsilon}_{\infty,q} \quad \text{with} \quad \sup_{0\leq t\leq T} \|u_j(t); B^{-\varepsilon}_{\infty,q}\| \leq 2K_0$$
$$\text{and} \quad \sup_{0\leq t\leq T} t^{\gamma/2}\|u_j(t); B^{\gamma-\varepsilon}_{\infty,q}\| \leq 2CK_0$$

for all $t \in [0,T]$, $j \geq 1$ and $\gamma \in (0,1]$. Here C is a constant independent of j, u_0 and T.

Proof. Let $0 < t \leq T \leq 1$, $\gamma \in [0,1]$ and we put $K^\gamma_j = K^\gamma_j(T)$ defined by

$$K^\gamma_j \equiv \sup_{0\leq t\leq T} t^{\gamma/2}\|u_j(t); B^{\gamma-\varepsilon}_{\infty,q}\|.$$

We start to estimate the linear terms. By Young's inequality we have

$$\|e^{t\Delta}u_0; B^{\gamma-\varepsilon}_{\infty,q}\| = \|\psi * (G_t * u_0); L^\infty\| + \Big[\sum_{j=1}^{\infty} 2^{j(\gamma-\varepsilon)q}\|\phi_j * G_t * u_0; L^\infty\|^q\Big]^{1/q}$$

$$\leq \|G_t; L^1\|\ \|\psi * u_0; L^\infty\| + C\Big[\sum_{j=1}^{\infty} \|(-\Delta)^{\gamma/2}G_t; L^1\|^q 2^{-j\varepsilon q}\|\phi_j * u_0; L^\infty\|^q\Big]^{1/q}.$$

By $L^p - L^q$ estimate (see e.g. [12]) we have

$$\leq \|\psi * u_0; L^\infty\| + Ct^{-\gamma/2}\Big[\sum_{j=1}^{\infty} 2^{-j\varepsilon q}\|\phi_j * u_0; L^\infty\|^q\Big]^{1/q}$$

$$\leq Ct^{-\gamma/2}K_0,$$

since $t \leq 1$. In particular, we note that if $\gamma = 0$ then we can choose these constants $C = 1$. We thus obtain

$$K_1^0 \leq K_0 \quad \text{and} \quad K_1^\gamma \leq CK_0$$

for all $\gamma \in (0, 1]$.

The next is to estimate the bilinear terms. To begin with, we prepare as follows; there exists a positive constant C such that

$$\|\nabla \cdot f; B^s_{p,q}\| \leq C\|f; B^{s+1}_{p,q}\|,$$

for all $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $f \in B^{s+1}_{p,q}$. Thus, for all $\gamma \in [0, 1]$ and $0 < s < t < T$ we have

$$\|\nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u_j \otimes u_j)(s); B^{\gamma-\varepsilon}_{\infty,q}\| \leq C\|e^{(t-s)\Delta}\mathbf{P}(u_j \otimes u_j)(s); B^{1+\gamma-\varepsilon}_{\infty,q}\|$$

$$\leq C|||(I-\Delta)^{(\gamma+\varepsilon)/2}\mathbf{P}e^{(t-s)\Delta}|||\ \|(u_j \otimes u_j)(s); B^{1-2\varepsilon}_{\infty,q}\|.$$

Here, $|||\cdot|||$ stands for an operator norm from L^∞ to L^∞. Using Proposition, we get

$$\leq C(t-s)^{-(\gamma+\varepsilon)/2}\|(u_j \otimes u_j)(s); B^{1-2\varepsilon}_{\infty,q}\|$$

$$\leq C(t-s)^{-(\gamma+\varepsilon)/2}\Big[(N^2+1)\|u_j(s); B^{1-\varepsilon}_{\infty,q}\|\ \|u_j(s); B^{-\varepsilon}_{\infty,q}\|$$

$$+ (N+1)2^{-N\varepsilon/2}\|u_j(s); B^{1-\varepsilon}_{\infty,q}\|\ \|u_j(s); B^{-\varepsilon/2}_{\infty,q}\|\Big].$$

Here we may choose arbitrary number $N \sim \varepsilon^{-1}\log(\|u_j(s); B^{1-\varepsilon}_{\infty,q}\| + 1)$, whose setting is similar to [7] and [13], thus we obtain

$$\leq C(t-s)^{-(\gamma+\varepsilon)/2}\Big[\Big(\big\{\log(\|u_j(s); B^{1-\varepsilon}_{\infty,q}\| + 1)\big\}^2 + 1\Big)\|u_j(s); B^{1-\varepsilon}_{\infty,q}\|\ \|u_j(s); B^{-\varepsilon}_{\infty,q}\|$$

$$+ \big\{\log(\|u_j(s); B^{1-\varepsilon}_{\infty,q}\| + 1) + 1\big\}\|u_j; B^{-\varepsilon/2}_{\infty,q}\|\Big]$$

$$\leq C\tilde{K}_j(t-s)^{-(\gamma+\varepsilon)/2}s^{-1/2}(\log s^{-1})^2,$$

where $\tilde{K}_j = K_j^0 K_j^1 \{ \log(K_j^1+1)+1\}^2 + K_j^{\varepsilon/2}\log(K_j^1+1)$. The last inequality is yielded by the definition of K_j^γ and the assumption of $T<1$.

Therefore we obtain

$$\begin{aligned} K_{j+1}^\gamma \leq & CK_0 + C\tilde{K}_j \sup_{0\leq t\leq T} t^{\gamma/2}\int_0^t (t-s)^{-(\gamma+\varepsilon)/2}(\log s^{-1})^2 s^{-1/2} ds \\ \leq & CK_0 + C\tilde{K}_j(\log T^{-1})^2 T^{1/2-\varepsilon/2}. \end{aligned}$$

Since $\varepsilon<1$, we now take T enough small, so we obtain Lemma 3. □

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