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FATOU AND LITTLEWOOD THEOREMS FOR POISSON INTEGRALS WITH RESPECT TO NON-INTEGRABLE KERNELS (Harmonic Analysis and Nonlinear Partial Differential Equations)

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FATOU AND LITTLEWOOD THEOREMS FOR POISSON INTEGRALS WITH RESPECT TO NON-INTEGRABLE KERNELS

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1. Fatou Theorem and Littlewood Theorem

In 1906 Fatou [5] proved the following:

**Theorem** (Fatou Theorem). Let $f$ be a bounded analytic function on the unit disk $U = \{|z| < 1\}$ in $\mathbb{C}$. Then $f$ has non-tangential limit at a.e. $e^{i\theta} \in \partial U$.

![Figure 1. Fatou Theorem.](image1)

![Figure 2. Littlewood Theorem.](image2)

In 1927 Littlewood [9, 10] proved the sharpness of non-tangential approach.

**Theorem** (Littlewood Theorem). Let $\gamma \subset U$ be a tangential curve at 1 and let $\gamma_\theta$ be the rotation. Then there exists a bounded analytic function $f$ on $U$ such that the limit of $f$ along $\gamma_\theta$ does not exists for a.e. $e^{i\theta} \in \partial U$.

There are many generalizations of Fatou theorem as follows:

- Hardy space $H^p$
- Harmonic functions
- Local Fatou theorem
• Square root of the Poisson kernel. Sjögren (1983) [18, 19, 20]
• Harmonic functions on trees
• Symmetric spaces

On the other hand, there are rather few works for Littlewood theorem:
• Zygmund (1949) [21]. (Blaschke product/Real Analysis)
• Lohwater-Piranian (1957) [11]. (Blaschke product. Everywhere divergence)
• Hakim-Sibony (1983) [6]. (Invariant harmonic functions)
• Aikawa (1990) [1, 2]. (Everywhere divergence)
• Salvatori-Vignati (1997) [17]. (Homogeneous tree).
• Di Biase (1998) [4]. (General tree)
• Hirata (2003) [7]. (Invariant harmonic functions in the unit ball of \( \mathbb{C}^n \))

In this note, we would like to observe that Fatou Theorem and Littlewood Theorem should go hand in hand.

2. FATOU AND LITTLEWOOD THEOREMS FOR HARMONIC FUNCTIONS ON \( \mathbb{R}^{n+1}_+ \)

Let \( \Psi(x) = (1 + |x|^2)^{-(n+1)/2} \) for \( x \in \mathbb{R}^n \) and put \( \Psi_t(x) = \frac{1}{t^n} \Psi(\frac{x}{t}) \) for \( t > 0 \).

Then \( \Psi_t * f(x) \) and \( \Psi_t * 1 = c_n \) and

\[
\frac{\Psi_t * f(x)}{\Psi_t * 1} = \frac{1}{c_n} \int_{\mathbb{R}^n} \frac{t f(y) dy}{(|x-y|^2 + t^2)^{(n+1)/2}}
\]

is the Poisson integral \( Pf(x,t) \) for the half space \( \mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0 \} \). By \( A \) we denote a positive constant whose value may change from occurrence to the next. If two positive functions \( f \) and \( g \) satisfy \( f \leq A g \) for some \( A \geq 1 \), then we write \( f \preceq g \). If \( f \preceq g \) and \( g \preceq f \), then we write \( f \sim g \). Let \( h(t) \) be a positive function for \( t > 0 \). Define the approach region

\[
\mathcal{A}_h(\xi) = \{(x,t) : |x-\xi| < h(t) \} \quad \text{for} \quad \xi \in \mathbb{R}^n.
\]

If \( h(t) \sim t \), then \( \mathcal{A}_h(\xi) \) gives a nontangential approach to \( \xi \). We say that a function \( u \) in \( \mathbb{R}^{n+1}_+ \) has a nontangential limit at \( \xi \) if the limit of \( u \) along \( \mathcal{A}_h(\xi) \) exists for every nontangential approach \( \mathcal{A}_h(\xi) \).

**Theorem A** (Fatou Theorem). Let \( 1 \leq p \leq \infty \). If \( f \in L^p(\mathbb{R}^n) \), then \( Pf(x,t) \) has nontangential limit \( f(\xi) \) at a.e. \( \xi \in \mathbb{R}^n \).
Theorem B (Littlewood Theorem). If \( \limsup_{t \to 0} h(t)/t = \infty \), then there exists \( f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) such that
\[
\lim_{t \to 0} (x, t) \notin d_h(\xi) \lim_{t \to 0} Pf(x, t)
\]
fails to exist at every \( \xi \in \mathbb{R}^n \).

If \( \gamma \) is a tangential curve in \( \mathbb{R}_{+}^{n+1} \) ending at \( \partial \mathbb{R}_{+}^{n+1} \), then there exists \( f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) such that
\[
\lim_{t \to 0} (x, t) \notin \gamma + \xi \lim_{t \to 0} Pf(x, t)
\]
fails to exist at every \( \xi \in \mathbb{R}^n \).

The above theorems suggest that the higher integrability of the boundary function \( f \) does not improve the admissible tangency.

3. Non-integrable Kernel

Sjörgen [18, 19, 20] gave extensions of the Fatou theorem for fractional Poisson integrals. Let
\[
P(z, \xi) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - \xi|^2}
\]
be the Poisson kernel for the unit disk \( U \). Then the classical Poisson integral
\[
Pf(z) = \int_{\partial U} P(z, e^{i\theta}) f(e^{i\theta}) d\theta
\]
is, of course, harmonic, i.e., \( \Delta Pf = 0 \).

Consider the fractional integral, or the \( \lambda \)-Poisson integral
\[
u = P_\lambda f(z) = \int_{\partial U} P(z, e^{i\theta})^{\lambda+1/2} f(e^{i\theta}) d\theta.
\]
Then, with the invariant or hyperbolic Laplacian
\[ \Delta = \frac{1}{4}(1 - |z|^2)^2 \Delta, \]
u enjoys \[ \Delta u = (\lambda^2 - \frac{1}{4})u. \] Sjögren studied the boundary behavior of the normalization
\[ P_\lambda f(z) = \frac{P_\lambda f(z)}{P_\lambda 1(z)}. \]
If \( \lambda > 0 \), then the Fatou theorem holds for \( P_\lambda f \) almost verbatim.

**Theorem C.** If \( f \in L^1(\partial U) \), then \( P_\lambda f(z) \) has nontangential limit \( f(e^{i\theta}) \) at a.e. \( e^{i\theta} \in \partial U \).

If \( \lambda = 0 \), then suddenly tangential limits appear (Sjögren [18, 19, 20] and Rönning [14, 15, 16]).

**Theorem D.** Suppose \( f \in L^p(\partial U) \) with \( 1 \leq p \leq \infty \). Then \( P_0 f(z) \) has limit \( f(e^{i\theta}) \) along \( \mathcal{A}_h(e^{i\theta}) \) at a.e. \( e^{i\theta} \in \partial U \), where
\[
h(t) \leq \begin{cases} 
  t(\log 1/t)^p & \text{if } 1 \leq p < \infty, \\
  t^{1-\epsilon} & \text{for all } \epsilon > 0 \text{ if } p = \infty.
\end{cases}
\]

*How should we understand the tangential nature?* It seems that the tangential nature is caused by the non-integrability of the kernel.

\[
P(z, \zeta)^{1/2} = \sqrt{\frac{1}{2\pi} \frac{1 - |z|^2}{|z - \zeta|^2}} \sim \frac{1}{|z - \zeta|}.
\]

Let us observe this phenomenon with the half space version due to Brundin [3] and Mizuta-Shimomura [12]. Define \( (P_0 f)(x, t) \) by
\[
\int_{\mathbb{R}^n} \left[ \frac{t}{c_n(|x - y|^2 + t^2)^{(n+1)/2}} \right]^{n/(n+1)} f(y) dy.
\]
Then \( (P_0 1)(x, t) \equiv \infty \) (non-integrable). Fix a bounded open set \( \Omega \subset \mathbb{R}^n \) and regard \( (P_0 \chi_\Omega)(x, t) \) as a substitute of \( (P_0 1)(x, t) \). Let us study the normalization \( (P_0 f)(x, t)/(P_0 \chi_\Omega)(x, t) \).

**Theorem E.** Let \( 1 \leq p \leq \infty \). Suppose, for small \( t > 0 \),
\[
(3.1) \quad h(t) \leq t(\log 1/t)^{p/n} \quad \text{if } 1 \leq p < \infty,
\]
\[
(3.2) \quad h(t) \leq t^{1-\epsilon} \text{ for all } \epsilon > 0 \text{ if } p = \infty.
\]
If $f \in L^p(\mathbb{R}^n)$, then
\[
\lim_{t \to 0} \frac{(P_0 f)(x, t)}{(P_0 \chi_\Omega)(x, t)} = f(\xi) \quad \text{for a.e. } \xi \in \Omega.
\]

Observe that
- For the critical power $n/(n + 1)$, certain tangential limits exist.
- Possible tangency depends on the Lebesgue exponent $p$ for which $f \in L^p(\mathbb{R}^n)$.

The tangential nature in Theorem E is caused by the non-integrability of the kernel. Let $\Phi(x) = \Psi(x)^{n/(n+1)} = (1 + |x|^2)^{-n/2}$. Then
\[
\frac{(P_0 f)(x, t)}{(P_0 \chi_\Omega)(x, t)} = \frac{\Phi_t * f(x)}{\Phi_t * \chi_\Omega(x)}.
\]

Observe that $\Phi \notin L^1(\mathbb{R}^n)$; $\Phi \in L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$; and $\Phi_t * \chi_\Omega(x) \sim \log 1/t$ as $t \to 0$ for $x \in \Omega$. This is a sharp contrast between $\Psi$ and $\Phi$.

From now on we need not the explicit form $(1 + |x|^2)^{-n/2}$. Instead we suppose
- $\Phi(x) > 0$ is a doubling function of $|x|$.
- $\Phi \notin L^1(\mathbb{R}^n)$, $\Phi \in L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$.

Let
\[
\varphi(r) = \int_{|x|<r} \Phi(x) dx.
\]
Then $\varphi(r) \uparrow \infty$ is doubling. Assume
\[
(3.3) \quad \lim_{r \to \infty} \frac{\varphi(2r)}{\varphi(r)} = 1.
\]

This condition looks technical; but it turns out to be crucial as observed in Proposition 1 below. Fix a bounded open set $\Omega \subset \mathbb{R}^n$. Study the boundary behavior of the normalization
\[
(\mathcal{P}_0 f)(x, t) = \frac{\Phi_t * f(x)}{\Phi_t * \chi_\Omega(x)}.
\]

**Proposition 1.** Condition (3.3) holds if and only if
\[
\lim_{t \to 0} (\mathcal{P}_0 f)(x, t) = f(x) \quad \text{for } x \in \Omega
\]

for all $f \in C_0(\mathbb{R}^n)$.

With (3.3) we obtain the following Fatou theorem for $(\mathcal{P}_0 f)(x, t)$. 
Theorem 1. Let $1 \leq p \leq \infty$. Suppose, for small $t > 0$,

(3.4) \[ h(t) \lesssim t \varphi(1/t)^{p/n} \quad \text{if } 1 \leq p < \infty, \]
(3.5) \[ \lim_{t \to 0} \frac{\varphi(h(t)/t)}{\varphi(1/t)} = 0 \quad \text{if } p = \infty. \]

If $f \in L^p(\mathbb{R}^n)$, then

\[ \lim_{(x,t)\to d} (\mathcal{P}_0 f)(x, t) = f(\xi) \quad \text{for a.e. } \xi \in \Omega. \]

Remark 1. Theorem 1 extends Theorem E.

- (3.4) $\Rightarrow$ (3.5).
- If $\Phi(x) = (1 + |x|^2)^{-n/2}$, then
  (i) $\varphi(r) \sim \log r$ for large $r > 0$;
  (ii) (3.1) $\iff$ (3.4), (3.2) $\iff$ (3.5).

What is a Littlewood type theorem? The cases $1 \leq p < \infty$ and $p = \infty$ are different.

Theorem 2. Let $1 \leq p < \infty$. If (3.4) does not hold, i.e.,

(3.6) \[ \lim_{t \to 0} \sup_{(x,t)\in\Omega(\xi)} \frac{h(t)}{t \varphi(1/t)^{p/n}} = \infty. \]

then there exists $f \in L^p(\Omega)$ such that for all $\xi \in \Omega$,

\[ -\infty = \liminf_{t \to 0} (\mathcal{P}_0 f)(x, t) < \limsup_{t \to 0} (\mathcal{P}_0 f)(x, t) = \infty. \]

Theorem 3. If (3.5) does not hold, i.e.,

(3.7) \[ \lim_{t \to 0} \sup_{(x,t)\in\Omega(\xi)} \frac{\varphi(h(t)/t)}{\varphi(1/t)} > 0. \]

then there exists $f \in L^\infty(\Omega)$ such that for all $\xi \in \Omega$,

\[ \liminf_{t \to 0} (\mathcal{P}_0 f)(x, t) < \limsup_{t \to 0} (\mathcal{P}_0 f)(x, t). \]

Let us close this section with the proof of Proposition 1. Let $B(x, r)$ be the open ball with center at $x$, radius $r$ and $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$. By $\text{diam} \Omega$ we denote the diameter of $\Omega$. 
Proof of Proposition 1. For simplicity we assume that \( \Omega \) is a bounded Lipschitz domain. For all \( x \in \Omega \), there exists a cone \( \Gamma(x) \subset \Omega \) with vertex at \( x \) and fixed aperture \( \alpha \) and radius \( r_0 \). Change of variable gives

\[
A \varphi \left( \frac{r_0}{t} \right) \leq \Phi_t * \chi_\Omega(x) \leq \varphi \left( \frac{\text{diam } \Omega}{t} \right),
\]

where \( A > 0 \) depends only on the aperture \( \alpha \). Since \( \varphi \) is doubling, it follows that

\[
\Phi_t * \chi_\Omega(x) \sim \varphi \left( \frac{1}{t} \right) \quad \text{for } x \in \Omega.
\]

Let \( x \in \Omega \) and let \( 0 < \varepsilon < \delta_\Omega(x) \). Then (3.8) and the doubling of \( \varphi \) gives

\[
\frac{\varphi(\delta_\Omega(x)/t) - \varphi(\varepsilon/t)}{\varphi(\varepsilon/t)} \leq (\mathcal{P}_0 \chi_\Omega \mathbb{B}(x,t))(x, t) \leq \frac{\varphi(\text{diam } \Omega/t) - \varphi(\varepsilon/t)}{\varphi(\varepsilon/t)}.
\]

Hence \( \lim_{t \to 0} (\mathcal{P}_0 \chi_{\Omega \setminus B(x,t)}) (x, t) = 0 \) if and only if (3.3) holds. Proposition 1 follows from this. \( \square \)

4. Ingredients of Proof of Theorem 1

We state some estimates needed for the proof of Theorem 1. The complete proof will be given elsewhere. First we estimate the influence of the local part of \( f \). If \( p = \infty \), this is stated as follows.

Lemma 1. Suppose \( h \) satisfies (3.5). Then

\[
\lim_{(x,t) \to (\xi,0)} (\mathcal{P}_0 \chi_{B(x,4h(t))})(x, t) = 0 \quad \text{for } \xi \in \Omega.
\]

If \( 1 \leq p < \infty \), then the Lebesgue point argument gives an estimate at almost every boundary point.

Lemma 2. Let \( 1 \leq p < \infty \) and \( f \in L^p(\mathbb{R}^n) \). Suppose \( h \) satisfies (3.4). Then for a.e. \( \xi \in \Omega,

\[
\lim_{t \to 0} (\mathcal{P}_0 [\chi_{B(x,4h(t))}f])(x, t) = 0.
\]

On the other hand the influence of the global part is controlled by maximal functions. Define the truncated maximal function by

\[
M_t f(x) = \sup_{r > t} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy
\]
with \( t \geq 0 \). \( M f(x) = M_0 f(x) \) is the classical Hardy-Littlewood maximal function. Define another maximal function \( \mathcal{M}_h f(\xi) \) by

\[
\sup_{(x,t) \in \mathcal{A}_h(\xi)} \left| \frac{1}{\Phi_t * \chi_\Omega(x)} \int_{|x-y| \geq 4h(t)} \Phi_t(x-y) f(y) dy \right|
\]

associated with the approach region \( \mathcal{A}_h(\xi) \).

**Lemma 3.** There is \( A \) such that

\( \mathcal{M}_h f(\xi) \leq A M f(\xi) \) for \( \xi \in \Omega \)

for arbitrary \( h(t) > 0 \).

**Lemma 4.** Let \( f \in L^p(\Omega) \) with \( 1 \leq p < \infty \). Then

\[
\lim_{t \to 0} \| (\mathcal{P}_0 f)(\cdot, t) - f \|_p = 0.
\]

As a result, for a.e. \( x \in \Omega \), some subsequence \( \{(\mathcal{P}_0 f)(x, t_j)\}_j \) converges to \( f(x) \).

5. **Outline of Proof of Theorem 2**

Let us prove Theorem 2 with the aid of the following two lemmas, whose proof will be given elsewhere.

**Lemma 5** (Lower Estimate). We find \( 0 < \exists A_0 < 1 \) such that

\[
(\mathcal{P}_0 \chi_{B(x, r)})(x, t) \geq A_0 \frac{\varphi(r/t)}{\varphi(1/t)}
\]

for \( x \in \Omega, t > 0, r > 0 \) small.

**Lemma 6** (Upper Estimate). If \( f \in L^1(\Omega) \), then

\[
| (\mathcal{P}_0 f)(x, t) | \leq M_f(x) \text{ for } x \in \Omega.
\]

**Proof of Theorem 2.** By (3.6) we find \( t_j \downarrow 0 \) such that

\[
\frac{t_j \varphi(1/t_j)^{p/n}}{h(t_j)} \to 0.
\]

Let \( \{x_j^\prime\} \) be lattice points \((h(t_j)/\sqrt{n})\mathbb{Z}^n\). Observe \( x_j^\prime \) are vertices of cubes of side length \( h(t_j)/\sqrt{n} \). Hence we have \( x_j^\prime \in B(\xi, h(t_j)) \).

If \( \xi \in \Omega \), then

\[
(x_j^\prime, t_j) \in \mathcal{A}_h(\xi) \text{ with } x_j^\prime \in \Omega,
\]

provided \( j \) is sufficiently large.
Put vertical line segments connecting $(x_j^v, 0)$ and $(x_j^v, t_j)$. We obtain a bed of thorns. We observe that $\mathcal{A}_h(\xi)$ cannot touch $\Omega$ without being pierced by some thorn. Now we construct $f_j$ such that $(\mathcal{P}_0 f_j)(x, t)$ is large on each "thorn". Put

$$f_j = \varphi\left(\frac{1}{t_j}\right)\chi_{D_j} \quad \text{with } D_j = \bigcup_{v} B(x_j^v, t_j) \cap \Omega.$$ 

Extract subsequence, find $c_j \uparrow \infty$ and let

$$f = \sum_{j=1}^{\infty} (-1)^j c_j f_j \in L^p(\mathbb{R}^n).$$

If $j$ is even and $j \to \infty$, then

$$(\mathcal{P}_0 f)(x_j^v, t_j) \to \infty;$$

if $j$ is odd and $j \to \infty$, then

$$(\mathcal{P}_0 f)(x_j^v, t_j) \to -\infty.$$ 

Since $\mathcal{A}_h(\xi)$ cannot touch $\Omega$ without being pierced by some thorn, we obtain

$$-\infty = \liminf_{t \to 0} (\mathcal{P}_0 f)(x, t) < \limsup_{t \to 0} (\mathcal{P}_0 f)(x, t) = \infty.$$ 

\hfill $\square$

6. Oscillating limits along curves

If $p = \infty$, then a result stronger than Theorem 3 can be obtained. Let $\gamma$ be a curve in $\mathbb{R}^{n+1}_+$ ending at the boundary. Let $\gamma[a]$ be the connected component of $\gamma \cap \{(x, t) : 0 \leq t \leq a\}$ containing the end point of $\gamma$. 
Theorem 4. Assume $\varphi(2r)/\varphi(r)$ is nonincreasing of $r$. Suppose $\gamma$ is more tangential than (3.5), i.e.,

\begin{equation}
\lim_{t \to 0} \sup_{r} \frac{\varphi(\mathrm{diam}(\gamma(t))/t)}{\varphi(1/t)} > 0.
\end{equation}

Then there exists $f \in L^\infty(\Omega)$ such that for every $\xi \in \Omega$,

\[
\lim_{t \to 0} \inf_{(x,t) \in \gamma + \xi} (\operatorname{REJECT}_0 f)(x,t) < \lim_{t \to 0} \sup_{(x,t) \in \gamma + \xi} (\operatorname{REJECT}_0 f)(x,t).
\]

The proof of this theorem will be given elsewhere.

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