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On Generic Predicates and Automorphisms

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Abstract

We prove that the class of the generic automorphisms of unstable structures constructed from stable structures by adding a generic predicate is not elementary. We also give Some discussion on generic automorphisms of a generic automorphism.

Introduction

Given a complete, model complete theory $T$ in a language $\mathcal{L}$, we consider the theory $T_{\sigma} = T \cup \{ \sigma \text{ is an } \mathcal{L}\text{-automorphism} \}$ in the language $\mathcal{L} \cup \{ \sigma \}$. For $M$ a model of $T$, and $\sigma \in \text{Aut}_{\mathcal{L}}(M)$ we call $\sigma$ a generic automorphism of $M$ if $(M, \sigma)$ is an existentially closed model of $T_{\sigma}$.

It is known that the class of generic automorphism of $T$ is not elementary if $T$ is unstable with the PAPA [4], has the strict order property [5], or $T$ does not eliminate the $\exists^{\infty}$ [4]. We conjecture that this class is not elementary if $T$ is unstable. But we don’t even know how to handle the general simple unstable case. We will consider simple unstable theories constructed from stable theories by adding a generic predicate or a generic automorphism. We try to show that the class of the generic automorphisms of the models of a theory constructed this way is not elementary. We have succeeded to show it in case of generic predicates but not in case of generic automorphisms. Nevertheless, we will give some arguments concerning two commuting automorphisms.

1 Preliminaries

In this paper, we work in a big model of some theory. $a, b,$ etc. denote tuples of elements of the universe, $A, B,$ etc. denote a small subset of the universe, and $x, y,$ etc. denote tuples of variables. If $a$ is a tuple and $A$ is a set, $a \in A$ means that each entry of $a$ belongs to $A$. We don’t usually distinguish by notation between a tuple $a$ and the set of the entries of $a$. 
Suppose \( \mathcal{L} \) is a language. \( \text{acl}_\mathcal{L}(A) \) denote the set of the elements satisfying some algebraic formula in \( \mathcal{L} \) with parameters in \( A \). We write \( \text{acl}(A) \) for \( \text{acl}_\mathcal{L}(A) \) if it is clear from the context. \( \text{dcl}_\mathcal{L}(A) \) denote the set of the elements satisfying some algebraic formula in \( \mathcal{L} \) with parameters in \( A \) with only one solution.

If \( \mathcal{L} \) is a language and \( \sigma, \tau, P \) are new non-logical symbols, \( \mathcal{L}_P = \mathcal{L} \cup \{P\} \), \( \mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\} \), and \( \mathcal{L}_{\sigma, \tau} = \mathcal{L} \cup \{\sigma, \tau\} \).

We list some known facts needed later.

**Definition 1.1** Let \( T \) be a theory in a language \( \mathcal{L} \). We say that \( T \) has the PAPA (la propriété d’amalgamation pour les automorphismes) if \( M_0, M_1, M_2 \models T \), \( \sigma_1 \in \text{Aut}_\mathcal{L}(M_1) \), \( \sigma_2 \in \text{Aut}_\mathcal{L}(M_2) \), and \( \sigma_1|_{M_0} = \sigma_2|_{M_0} \) then there are \( M_3 \models T \), \( \sigma_3 \in \text{Aut}_\mathcal{L}(M_3) \), and \( h : M_2 \to M_3 \) such that \( h|M_0 \) is the identity on \( M_0 \), \( \sigma_3|M_1 = \sigma_1 \) and \( \sigma_3 h(M_2) = h \sigma_2 h^{-1} \).

**Fact 1.2** ([4]) Let \( T \) be a complete theory in a language \( \mathcal{L} \). If \( T \) is model complete, unstable and has the PAPA, then \( T_{\sigma} \) has no model companion in \( \mathcal{L}_{\sigma} \).

**Fact 1.3** (Chatzidakis, Pillay [2]) Let \( T \) be a complete theory in a language \( \mathcal{L} \) and suppose \( T \) is model complete. Then the model companion \( T^*_{\mathcal{L}} \) of \( T \) in the language \( \mathcal{L}_{P} \) exists if and only if \( T \) eliminates the quantifier \( \exists^\infty \). If \( T^*_{\mathcal{L}} \) exists then \((M, P) \models T^*_{\mathcal{L}} \) if and only if (i) \( M \models T \) and (ii) for every \( \mathcal{L} \)-formula \( \varphi(x, z) \) where \( x \) is a \( \ell \)-tuple of variables, for every subset \( I \) of \( \{1, \ldots, n\} \), for any tuple \( b \in M \), if there is \( a = (a_1, \ldots, a_n) \in M \) such that \( a \cap \text{acl}_\mathcal{L}(b) = \emptyset \) and \( a_i \neq a_j \) for \( i \neq j \), then there is \( a' = (a'_1, \ldots, a'_n) \in M \) such that \( \varphi(a', b), P(a'_i) \) for \( i \in I \), and \( \neg P(a'_i) \) for \( i \notin I \).

## 2 Theories with a Predicate and an Automorphism

The following lemma is a well-known fact.

**Lemma 2.1** Let \( T \) be a complete theory. Let \( a \) be a tuple and \( A \) a set such that \( a \cap \text{acl}(A) = \emptyset \) then for any \( B \supset A \) there is a tuple \( a' \models \text{tp}(a/A) \) such that \( a' \cap \text{acl}(B) = \emptyset \).

**Proof.** We prove this by induction on the length of a tuple \( a \). We can assume that \( A = \text{acl}(A) \). If \( a \) is a single element, the conclusion follows by compactness.

Let \( a = (a_1, a_2) \) where \( a_1 \) is a single element and \( a_2 \) a tuple. Suppose \( \varphi(x, y) \in \text{tp}(a_1, b/A) \) where \( x \) is a single variable, and \( b_1, \ldots, b_n \) are elements in \( \text{acl}(B) \setminus \text{acl}(A) \). We show that there are \( a'_1, a'_2 \) such that \( \varphi(a'_1, a'_2) \) and \( (a'_1, a'_2) \cap \{b_1, \ldots, b_n\} = \emptyset \). Then the conclusion follows by compactness.

We can choose \( a'_2 \models \text{tp}(a_2/A) \) such that \( a'_2 \cap \{b_1, \ldots, b_n\} = \emptyset \) by induction hypothesis. If there is \( a'_1 \notin \{b_1, \ldots, b_n\} \) such that \( \varphi(a'_1, a'_2) \), we are done.

By way of contradiction, suppose for any \( c \) and \( d \), \( \varphi(c, d) \) and \( d \cap \{b_1, \ldots, b_n\} = \emptyset \) implies \( c \in \{b_1, \ldots, b_n\} \). Consider a formula \( \psi(x) \) over \( A \) expressing that there are
pairwise disjoint tuples $d_1, \ldots, d_{n+1}$ such that $\varphi(x, d_j)$ for $j = 1, \ldots, n+1$. We show that $\psi(x)$ is algebraic. Let $b$ satisfy $\psi(x)$. Then there are pairwise disjoint tuples $d_1, \ldots, d_{n+1}$ such that $\varphi(b, d_j)$ for $j = 1, \ldots, n+1$. Since they are disjoint, some $d_j$ is disjoint from $\{b_1, \ldots, b_n\}$. Therefore, $b \in \{b_1, \ldots, b_n\}$ by the hypothesis.

Hence, $b_i$ satisfying $\psi(x)$ belongs to $\text{acl}(A) = A$, and for $b_i$ satisfying $\neg\psi(x)$, the number of pairwise disjoint solutions of $\varphi(b_i, y)$ is bounded by $n$.

By an iterated use of induction hypothesis, there are tuples $d_1, \ldots, d_{n^2+1}$ such that $d_j \models \text{tp}(a_2/A)$ and $d_j \cap \text{acl}(Ab_1, \ldots, b_n d_0 \ldots d_{j-1}) = \emptyset$ for $j = 1, \ldots, n^2 + 1$. In particular, the $d_j$'s are disjoint each other. For each $b_i$ satisfying $\neg\psi(x)$, at most $n$ tuples among the $d_j$'s satisfy $\varphi(b_i, y)$. Therefore, for some $d_j$, $\neg\varphi(b_i, d_j)$ holds for any $b_i$ satisfying $\neg\psi(x)$. Let $d = d_j$. Since $d$ and $a_2$ are conjugate over $A$, there is an element $c$ such that $(c, d)$ and $(a_1, a_2)$ are conjugate over $A$. Therefore, $\varphi(c, d)$ and $c \not\in A$. Hence, $c \neq b_i$ for any $b_i$. This is a contradiction. 

\begin{theorem}
Let $T$ be a complete theory in a language $\mathcal{L}$. Suppose $T$ is model complete and the model companion $T^*_p$ of $T$ in the language $L_P$ exists. Then any model of $T_{P, \sigma} = T \cup \{\sigma \text{ is an } L_P\text{-automorphism}\}$ embeds in a model of $(T^*_p)_\sigma = T^*_p \cup \{\sigma \text{ is an } L_P\text{-automorphism}\}$. In particular, they have the same class of the existentially closed models. Therefore, $T_{P, \sigma}$ has a model companion if and only if $(T^*_p)_\sigma$ has one, and they are the same if they exist.

\end{theorem}

\begin{proof}
We work in a big model $\mathcal{M}$ ($\mathcal{L}$-structure) of $T$.

\begin{claim}
Suppose $(\mathcal{M}, \sigma_{\mathcal{M}})$ is a model of $T_{\sigma}$ and $a, b$ are tuples in $\mathcal{M}$ such that $a \cap \text{acl}_{\mathcal{L}}(b) = \emptyset$. Then there is a sequence $\langle a_i : 0 \leq i < \omega \rangle$ such that $\sigma(\text{tp}_{\mathcal{L}}(\langle a_i : 0 \leq i < \omega \rangle/M)) = \text{tp}_{\mathcal{L}}(\langle a_i : 1 \leq i < \omega \rangle/M)$, $a_i \cap \text{acl}_{\mathcal{L}}(Ma_0 \ldots a_{i-1}) = \emptyset$ for each $i$, and $a_0 \models \text{tp}(a/b)$.

We construct such a sequence by induction. By Lemma 2.1, there is $a_0 \models \text{tp}_{\mathcal{L}}(a/b)$ such that $a_0 \cap M = \emptyset$. Again by Lemma 2.1, there is $a_1 \models \sigma_\mathcal{M}(\text{tp}_{\mathcal{L}}(a_0/M))$ such that $a_1 \cap Ma_0 = \emptyset$.

Suppose we have constructed a sequence $\langle a_i : 0 \leq i < n \rangle$ such that

$$
\sigma_\mathcal{M}(\text{tp}_{\mathcal{L}}(a_0, \ldots, a_{n-2}/M)) = \text{tp}_{\mathcal{L}}(a_1, \ldots, a_{n-1}/M) \text{ and } a_i \cap \text{acl}_{\mathcal{L}}(Ma_0 \ldots a_{i-1}) = \emptyset
$$

for $i < n$. Let $\sigma' \in \text{Aut}_{\mathcal{L}}(\mathcal{M})$ be an extension of $\sigma_\mathcal{M}$ such that $\sigma'(a_0, \ldots, a_{n-2}) = (a_1, \ldots, a_{n-1})$. By Lemma 2.1, we can choose $a_n \models \sigma'(\text{tp}_{\mathcal{L}}(a_{n-1}/Ma_0 \ldots a_{n-2}))$ such that $a_n \cap \text{acl}(Ma_0 \ldots a_{n-1}) = \emptyset$. Therefore, there is an $\mathcal{L}$-automorphism $\sigma_n$ of $\mathcal{M}$ such that $\sigma_n$ extends $\sigma'$ and $\sigma_n(a_{n-1}) = a_n$. We have Claim 2.2.1.

\begin{claim}
Suppose $(\mathcal{M}, P^\mathcal{M}, \sigma_{\mathcal{M}})$ is a model of $T_{P, \sigma}$, $a, b$ are tuples from $\mathcal{M}$ such that $a \cap \text{acl}_{\mathcal{L}}(b) = \emptyset$, $a = (a_1, \ldots, a_l)$, $1 \leq i < j \leq l$ implies $a_i \neq a_j$, and $I \subseteq \{1, \ldots, l\}$. Then there is an extension $(N, P^N, \sigma_N) \models T_{P, \sigma}$ of $(\mathcal{M}, P^\mathcal{M}, \sigma_{\mathcal{M}})$ satisfying that there is $a^i = (a^i_1, \ldots, a^i_l) \in N \setminus M$ realizing $\text{tp}_{\mathcal{L}}(a/b)$ such that $P(a^i_l)$ for $i \in I$ and $\neg P(a^i_l)$ for $i \notin I$.

\end{claim}

\end{proof}
Choose a sequence \( \langle a_i : 0 \leq i < \omega \rangle \) as in Claim 2.2.1. Then there is an extension \((N, \sigma_N) \models T_\sigma \) of \((M, \sigma_M)\) such that \(N\) contains the \(a_i\)'s for \(0 \leq i\). Let \(a_k = \sigma_N^k(a_0)\) for each integer \(k < 0\). Then \(a_k = \sigma_N^k(a_0)\) for any \(k \in \mathbb{Z}\). Since \(a_0 \cap a_i = \emptyset\) for \(i > 0\), we have \(a_i \cap a_j = \emptyset\) for any \(i, j \in \mathbb{Z}\) such that \(i < j\). Now let \(a_0 = (a'_1, \ldots, a'_\ell)\). Let \(P^N = P^M \cup \{\sigma_N^k(a'_i) : k \in \mathbb{Z}, i \in I\}\). Then \(\sigma_N\) is an \(L_P\)-automorphism. We have Claim 2.2.2.

Now, suppose \((M, P^M, \sigma_M)\) is a model of \(T_{P,\sigma}\). With Claim 2.2.2, a standard argument shows that there is an extension \((N, P^N, \sigma_N) \models T_{P,\sigma} \) of \((M, P^M, \sigma_M)\) such that \((N, P^N) \models T_\sigma\) using Fact 1.3.

**Theorem 2.3** Let \(T\) be a complete theory in a language \(\mathcal{L}\). Suppose \(T\) is model complete, stable, and the model companion \(T^*_P\) of \(T\) in the language \(L_P\) exists. Then \(T^*_P\) has the PAPA.

**Proof.** Let \((M_0, P_0, \sigma_0), (M_1, P_1, \sigma_1), (M_2, P_2, \sigma_2)\) be models of \(T_{P,\sigma}\) and suppose that \((M_1, P_1, \sigma_1)\) and \((M_2, P_2, \sigma_2)\) are extensions of \((M_0, P_0, \sigma_0)\). We can assume that \(M_1\) and \(M_2\) are independent over \(M_0\) in a big model of \(T\). Since \(T\) is stable, \(\sigma_1 \cup \sigma_2\) is an \(\mathcal{L}\)-elementary map on \(M_1 \cup M_2\) and there is \((M_3 \models T\) and \(\sigma_3 \in \text{Aut}_\mathcal{L}(M_3)\) such that \((M_3, \sigma_3)\) is an extension of both \((M_1, P_1, \sigma_1)\) and \((M_2, P_2, \sigma_2)\). Let \(P_3 = P_1 \cup P_2\). Then \((M_3, P_3, \sigma_3) \models T_{P,\sigma}\). By Theorem 2.2, it embeds in a model of \((T^*_P)_{\sigma}\). \(\square\)

**Theorem 2.4** Let \(T\) be a complete theory in a language \(\mathcal{L}\). Suppose \(T\) is model complete, stable, and the model companion \(T^*_P\) of \(T\) in the language \(L_P\) exists. If \(T^*_P\) is unstable then \((T^*_P)_{\sigma}\) and \(T_{P,\sigma}\) has no model companion.

**Proof.** By Fact 1.2 and Theorem 2.3. \(\square\)

In Theorem 2.3, it is sufficient to assume that \(T\) has the PAPA and any \((M, \sigma) \models T_\sigma\) is a strong amalgamation base for \(T_\sigma\). In general, a subset \(A\) of a model of a theory \(U\) is a strong amalgamation base for \(U\) if \(A \subseteq M_1, M_2\) are models of \(U\) then there is a \(M_3\) of \(U\) and an embedding \(h : M_2 \rightarrow M_3\) such that \(M_1 \subseteq M_3, h\) fixes \(A\) pointwise, and \(M_1 \cap h(M_2) = A\). Also, we can conclude that \(T^*_P\) has the PAPA and \((M, P, \sigma) \models T_{P,\sigma}\) is a strong amalgamation base for \(T_{P,\sigma}\). Therefore, we can repeatedly use Theorem 2.3 to show that a theory with several generic predicates (the model companion of a theory with several new predicates) has the PAPA.

### 3 Two Commuting Automorphisms

Let \(T\) be a complete theory in a language \(\mathcal{L}\) and \(\sigma, \tau\) new unary function symbols. Let \(\mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\}\) and \(\mathcal{L}_{\sigma,\tau} = \mathcal{L} \cup \{\sigma, \tau\}\). Suppose the model companion \(T^*_\sigma\) of \(T \cup \{\{\sigma\text{ is an }\mathcal{L}\text{-automorphism}\}\}\) exists. If \(T\) is stable and admits quantifier elimination, Chatzidakis and Pillay showed that \(T^*_\sigma\) is simple if it exists. They gave a mild assumption under which \(T^*_\sigma\) will be unstable. We tried to show that there is no
model companion for \((T^*_\sigma)\cup \{ \tau \text{ is an } \mathcal{L}_\sigma\text{-automorphism}\}\). Note that \(\tau\) is an \(\mathcal{L}_\sigma\)-automorphism if and only if \(\tau\) and \(\sigma\) are two commuting \(\mathcal{L}\)-automorphisms. Although we have not succeed to show it, we present some argument towards the proof. Main obstacle is that it is not clear if we can expand two commuting automorphisms to commuting automorphisms over some algebraic extensions.

First of all, we give a proof for the fact that there is no model companion for the theory of fields with two commuting automorphisms based on [1]. Note that the theory of fields is essentially the universal part of the theory of algebraically closed fields, which is stable.

**Lemma 3.1** Let \(T\) be the theory of fields with two commuting automorphisms. If \((F, \sigma, \tau)\) is an existentially closed model of \(T\) then for any integer \(n \geq 2\) there is \(c\) in \(F\) such that \(\sigma(c) = \tau(c), c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^{n-1}(c) = 0\), and \(c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^{k-1}(c) \neq 0\) for any \(k < n\).

**Proof.** Let \(t_0, t_1, \ldots, t_{n-2}\) be transcendental and algebraically independent over \(F\). Let \(t_{n-1} = -(t_0 + t_1 + \cdots + t_{n-2})\) Then \(t_1, \ldots, t_{n-2}, t_{n-1}\) are also transcendental and algebraically independent over \(F\). Hence we can expand \(\sigma\) and \(\tau\) so that \(\sigma(t_i) = \tau(t_i) = t_{i+1}\) for \(i = 0, 1, \ldots, n - 2\). Then we have \(\sigma(t_0) = \tau(t_0)\) and \(t_0 + \sigma(t_0) + \cdots + \sigma^{n-1}(t_0) = 0\). \(\sigma\) and \(\tau\) commute on \(F(t_0, t_1, \ldots, t_{n-2})\). Since \((E, \sigma, \tau)\) is an existentially closed model of \(T\), we can pull down \(t_0\) in \(F\) to find \(c\) satisfying the conclusion of the lemma.

**Theorem 3.2 (Hrushovski)** There is no model companion of the theory of fields with two commuting automorphisms.

**Proof.** Let \(\zeta\) be a primitive third root of unity and suppose that \(\zeta\) does not belong to the prime field (characteristic \(2 \pmod{3}\), or 0). Let \(K_0\) be an algebraic closure of the prime field and \(\sigma_0\) be an automorphism of \(K_0\) such that \(\sigma_0(\zeta) = \zeta^2\).

Now, suppose that \(T^*\) is a model companion of the theory of fields with two commuting automorphisms. Extend \((K_0, \sigma_0, \sigma_0)\) to \((K, \sigma, \tau) \models T^*\). We can assume that \((K, \sigma, \tau)\) is \(\aleph_1\)-saturated.

**Claim 3.2.1** In \((K, \sigma, \tau)\),

\[
\sigma(z) = \tau(z), \quad z + \sigma(z) + \sigma^2(z) + \cdots + \sigma^k(z) \neq 0 \quad \text{for } k < \omega
\]

\[
\vdash \exists x \exists y [\sigma(x) = \tau(x) = x + z \land y^3 = x \land \tau(y) = \zeta \sigma(y)]
\]

Let \(c \in K\) be such that \(\sigma(c) = \tau(c), c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^k(c) \neq 0\) for \(k < \omega\). Note that such \(c\) exists by Lemma 3.1.

Let \(E\) be a countable subfield of \(K\) such that \(c \in E\) and \((E, \sigma|_E, \tau|_E)\) is an elementary substructure of \((K, \sigma, \tau)\). Let \(a\) be a transcendental element over \(E\). Then we can expand \(\sigma|_E\) and \(\tau|_E\) to automorphisms \(\sigma'\) and \(\tau'\) respectively on \(E(a)\) so that \(\sigma'(a) = \tau'(a) = a + c\). Then \(\sigma^n(a) = \tau^m(a) = a + c + \sigma'(c) + \cdots + \sigma^{k-1}(c)\) and
\(\sigma^i(a) \neq \sigma^j(a)\) if \(i \neq j\). Since \(a\) has no third root in \(E(a)\) and \(\zeta \in E\), \(X^3 - \sigma^i(a)\) is irreducible over \(E(a)\). For each \(i\), let \(b_i\) be a third root of \(\sigma^i(a)\). Then we can expand \(\sigma'\) and \(\tau'\) so that

\[
\sigma(b_i) = b_{i+1}, \\
\tau(b_i) = \zeta b_{i+1} \quad (i \text{ is even}), \\
\tau(b_i) = \zeta^2 b_{i+1} \quad (i \text{ is odd}).
\]

Let \(E'\) be a field obtained by adjoining all \(b_i\) for \(i \in \mathbb{Z}\) to \(E(a)\). If \(i\) is even then \(\sigma \tau(b_i) = \sigma(\zeta b_{i+1}) = \zeta^2 b_{i+2}\), \(\tau \sigma(b_i) = \tau(b_{i+1}) = \zeta b_{i+2}\). If \(i\) is odd then \(\sigma \tau(b_i) = \sigma(\zeta^2 b_{i+1}) = \zeta b_{i+2}\), \(\tau \sigma(b_i) = \tau(b_{i+1}) = \zeta^2 b_{i+2}\). Therefore, we have \(\sigma \tau = \tau \sigma\) on \(E'\). Hence, the RHS of the claim holds in \(E'\). Since \((E, \sigma, \tau)\) is existentially closed, the RHS of the claim holds in \(E\). We have the claim.

By compactness, there is \(n_0\) such that

\[
\sigma(z) = r(z), \quad z + \sigma(z) + \sigma^2(z) + \cdots + \sigma^k(z) \neq 0 \text{ for } k < n_0 \\
\Rightarrow \exists x \exists y [\sigma(x) = r(x) = x + z \land y^3 = x \land r(y) = \zeta \sigma(y)]
\]

in \((K, \sigma, \tau)\). By Lemma 3.1, we can choose \(c\) such that \(\sigma(c) = \tau(c), c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^k(c) \neq 0\) for \(k < n_0\) but \(c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^{n-1}(c) = 0\) for some odd number \(n\). Let \(a, b\) be such that \(\sigma(a) = \tau(a) = a + c, b^3 = a\), and \(\tau(b) = \zeta \sigma(b)\). Then \(\sigma^n(a) = \tau^n(a) = a\). Since \(\sigma^n(b)\) is a third root of \(a\), we can write \(\sigma^n(b) = \zeta^i b\) for some \(i\).

Calculate \(\sigma^n \tau(b)\) in two ways:

\[
\begin{align*}
\sigma^n \tau(b) &= \sigma^n(\zeta \sigma(b)) \\
&= \sigma^n(\zeta) \sigma^{n+1}(b) \\
&= \sigma^n(\zeta) \sigma \sigma^n(b) \\
&= \sigma^n(\zeta) \sigma(\zeta b) \\
&= \sigma^n(\zeta) \sigma(\zeta^i b). \\
\sigma^n \tau(b) &= \tau \sigma^n(b) \\
&= \tau(\zeta^i b) \\
&= \sigma(\zeta^i) \zeta \sigma(b).
\end{align*}
\]

Therefore, \(\sigma^n(\zeta) = \zeta\) and thus \(n\) must be even. This is a contradiction. \(\square\)

Since the fields are essentially the substructures of algebraically closed fields and the theory of algebraically closed fields is stable, we can conjecture that if \(T\) is stable (with some additional assumption) then there is no model companion for \(T_{\sigma} \cup \{\sigma\text{ and }\tau\text{ are commuting automorphisms}\}\).

Suppose \(T\) is stable, admits quantifier elimination, and there is a model \(M\) of \(T\) and tuples \(a, b\) in a big model of \(T\) such that \(a \perp_M b\) and \(\text{acl}(M, a, b) \neq \text{dcl}(\text{acl}(M, a) \cup \text{acl}(M, b))\). Chatzidakis and Pillay [2] showed that the model companion of \(T_{\sigma}\) is
unstable in this case. With the same assumption, we will try to show that there is no model companion for $T_v$ \cup \{\sigma$ and $\tau$ are commuting automorphisms\}.

For the sake of simplicity, we assume that $T$ is countable. We can assume that $M$ is countable. Let $\epsilon \in acl(M, a, b) \setminus acl(acl(M, a) \cup acl(M, b))$. Let $\varphi(x, a, b)$ be a formula isolating $tp(c/Mab)$. Let $\overline{e}$ be an enumeration of all realizations of $\varphi(x, a, b)$.

Let $\{b_i : i \in Z\}$ be a Morley sequence for $tp(b/acl(aM))$ and let $\overline{e}_i$ be an enumeration of all realizations of $\varphi(x, a, b_i)$ for each $i$ in $Z$. Then $\{b_i \overline{e}_i : i \in Z\}$ is independent over $acl(aM)$. For each $i$ in $Z$, let $\sigma_i$ be an automorphism such that it is identity on $acl(aM) b_i, \sigma_i(\overline{e}_i) = e_i$ if $i \geq 0$ and $\sigma_i(\overline{e}_i) \neq e_i$ if $i < 0$. Since $tp(b_i \overline{e}_i/acl(aM))$ is stationary by the elimination of imaginaries, there is an automorphism $\sigma$ such that $\sigma$ is an extension of all $\sigma_i$ for $i$ in $Z$. Therefore, we have a countable extension $N \supset Ma \cup \{b_i, e_i : i \in Z\}$ and an $L$-automorphism of $N$ such that $\sigma$ fixes $Ma \cup \{b_i : i \in Z\}$ pointwise and $\sigma(e_i) = e_i$ (as tuples) if and only if $i \geq 0$.

Let $\tau$ be an $L$-automorphism such that $\tau$ fixes $M$ pointwise and $\tau(b_i) = b_{i+1}$ for $i \in Z$. Let $N_0 = N$, and for $i > 0$, let $N_i$ be a model of $T$ such that $N_i$ is independent from $M \cup \bigcup_{j<i} N_j$ over $acl(M \cup \{b_i : i \in Z\})$ and realizes $\sigma tp(N_{i-1}/M \cup \bigcup_{j<i-1} N_j)$. $\tau$ can be extended to an $L$-automorphism such that $\tau(N_i) = N_{i+1}$. Let $N_i = \tau^i(N)$ for $i < 0$. Then $\{N_i : i \in Z\}$ is an independent set over $acl(M \cup \{b_i : i \in Z\})$. Extend $\sigma$ to every $N_i$ for $i \in Z$ through $\tau$. Then $\sigma$ is an elementary map on $\bigcup_{i \in Z} N_i$ and $\sigma$ and $\tau$ commute on $\bigcup_{i \in Z} N_i$. Let $K = acl(\bigcup_{i \in Z} N_i)$. Then $K \models T_v$, and $\sigma$ and $\tau$ can be extended to $L$-automorphisms of $K$ that they are commuting.

Note that $(K, \sigma, \tau)$ has the order property. Let $a_i = \tau^i(a)$ for $i \in Z$. Consider a formula $r(x, y, x', y')$ expressing that $\tau$ pointwise fixes every realization of $\varphi(x, x', y)$. Then $r(a_i, b_i, a_j, b_j)$ if and only if $i \leq j$. Note that $r(a_i, b_i, a_j, b_j)$ and $\neg r(a_j, b_j, a_i, b_i)$ if and only if $i < j$.

Now, assume that there is a model companion $T^*$ of

$$T_v \cup \{\sigma$ and $\tau$ are commuting automorphisms\}.

By extending, we can assume that $(K, \sigma, \tau)$ is a model of $T^*$. Also, we can assume that $(K, \sigma, \tau)$ is $N_i$-saturated.

Let $R(x, y, x', y') \equiv (r(x, y, x', y') \land \neg r(x', y', x, y))$.

We want to show the following claim in $(K, \sigma, \tau)$:

$$\{R(a_i, b_i, u, v) : i < \omega\} \vdash \exists x, y [R(a_0, b_0, x, y) \land R(x, y, u, v) \land \tau(x, y) = (x, y)]$$

If we have this claim, then we get a contradiction by compactness and the fact that $\tau$ is an $L$-automorphism.

Let $(x_0, y_0)$ realize a non-forking extension of $tp_L(a_0, b_0/M)$ to $K$. Since $tp_L(a_0, b_0/M)$ is stationary, $tp(x_0, b_0/K)$ is fixed by $\tau$. If we can extend $\sigma$ and $\tau$ to some extension of $K$ so that they are commuting, $R(a_i, b_i, x_0, y_0)$ for $i < \omega$ and $R(x_0, y_0, u, v)$, we are done since $(K, \sigma, \tau)$ is existentially closed. But, it seems very difficult to do this in an abstract situation like this.
References

[1] Z. Chatzidakis, “Udi’s example that the theory of fields with two commuting automorphisms does not have a model companion”, (unpublished).


