On CM-triviality

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Abstract
Hrushovski's generic construction yields CM-trivial structures with weak elimination of imaginaries. Here, we would like to give questions on CM-triviality.

1 CM-triviality of generic structures.

In this section, we show weak elimination of imaginaries and CM-triviality of well-known stable generic structures.

Definition 1 We say that $T$ is CM-trivial if

$$A_1 \downarrow_B A_2 \Rightarrow A_1 \downarrow_{acl^e(A_1 A_2) \cap B} A_2$$

for $A_1, A_2, B \subset M^{eq}$ algebraically closed sets.

Fact 2 Let $T$ be any theory of well-known stable generic structure. Then we have;

1. (M: big model of $T$) For any $A, B \subset M$ algebraically closed sets,

$$A \downarrow_{A \cap B} B \Leftrightarrow AB = A \otimes_{A \cap B} B \leq M,$$

2. any type over algebraically closed sets in real sort is stationary.
Explanations

1. The language is $L = \{R_i(X_1 \ldots X_{n_i}) : i < \omega\}$, where $R_i(X_1 \ldots X_{n_i})$ is an $n_i$-ary predicate.

2. We assume that any predicate is closed under permutations and $R_i(a_1 \ldots a_{n_i}) \Rightarrow a_i \neq a_j (i \neq j \leq n_i)$.

3. We defined predimension on finite $L$-structures.

$$\delta(A) = |A| - \sum_{i<\omega} \alpha_i \cdot |R_i^A|$$

4. For finite $L$-structures $A, B$, we say $A$ is closed in $B$ (write $A \leq B$) if

$$\delta(XA) - \delta(A) \geq 0$$

for any $X \subseteq B$. "$A$ is closed in $B$" means that there are only suitably many (depending on $\alpha$) sequences intersecting $A$ and $B \setminus A$, and satisfying some predicates.

For possibly infinite $L$-structures $A, N$, we say that $A$ is closed in $N$ (write $A \leq N$) if

$$A_0 \leq A_0 X$$

for any $A_0 \subset_{\omega} A, X \subset_{\omega} N \setminus A$.

There exists the smallest closed subset $\text{cl}_N(A)$ of $N$ containing $A$.

In any well-known stable generic structure, $\text{cl}_N(A) \subseteq \text{acl}_N(A)$, in particular $\text{acl}_N(A) \leq N$.

5. For $L$-structure $A, B, C$ with $A \cap B \subseteq C$, we say that $A$ and $B$ are freely joined over $C$ if there are no $i < \omega$ and $\bar{d} \in ABC$ such that $R_i(\bar{d})$, $\bar{d} \cap (A \setminus C) \neq \emptyset$ and $\bar{d} \cap (B \setminus C) \neq \emptyset$, and we write $ABC = A \otimes_C B$.

From now on, let $T$ be as in Fact 2.
Proposition 3 \( T \) has weak elimination of imaginaries.

proof First we show the following claim.
Claim Let \( A, B, B_1, B_2 \) be algebraically closed. Suppose that \( B_i \subseteq B \) and \( A \downarrow B_i \) for \( i = 1, 2 \). Then \( A \downarrow B_1 \cap B_2 \).

The proof of this claim: Put \( A_i = \text{acl}(AB_i) \). Then \( A_i B = A_i \otimes_{B_i} B \leq \mathcal{M} \) by Fact 2, for \( i = 1, 2 \). Intersecting the two sets yields \( (A_1 \cap A_2) \otimes_{B_1 \cap B_2} B \leq \mathcal{M} \). Note that \( A_1 \cap A_2 \) and \( B_1 \cap B_2 \) are algebraically closed. So by Fact 2, \( A_1 \cap A_2 \downarrow B_1 \cap B_2 \); since \( A \subseteq A_1 \cap A_2 \) we get \( A \downarrow B_1 \cap B_2 \), as desired.

Now we show the weak elimination of imaginaries. Let \( E(x, y) \) be a definable equivalence relation over \( \emptyset \), and consider \( e = \overline{a}_E \), where \( \overline{a}_E \) is the \( E \)-class of \( a \).

Take \( \overline{b}_1, \overline{b}_2 \) such that \( \overline{a}, \overline{b}_1, \overline{b}_2 \) are independent over \( e \), and \( \overline{a}_E = (\overline{b}_1)_E = (\overline{b}_2)_E \). As \( e \in \text{acl}^{eq}(\overline{b}_i) \) we have \( \overline{a} \downarrow_{\overline{b}_i} \overline{b}_1 \overline{b}_2 \), for \( i = 1, 2 \).

Put \( B = \text{acl}(b_1) \cap \text{acl}(b_2) \), where the algebraic closure is taken in the real sort. Then \( \overline{a} \downarrow_{E} \overline{b}_1 \overline{b}_2 \) by claim. As \( \text{tp}(\overline{a}/B) \) is stationary and \( e \in \text{dcl}^{eq}(a) \cap \text{dcl}^{eq}(\overline{b}_1 \overline{b}_2) \), we get \( e \in \text{dcl}^{eq}(B)^* \). On the other hand, as \( \overline{b}_1 \downarrow_e \overline{b}_2 \), we have \( B \subseteq \text{acl}(e) \).

By compactness we can find a finite tuple \( \overline{b} \in B \) with \( e \in \text{dcl}^{eq}(\overline{b}) \); clearly \( \overline{b} \in \text{acl}^{eq}(e) \), as desired.

Proposition 4 \( T \) is CM-trivial.

proof We use the following fundamental property.

1. If \( ABC = A \otimes_C B, A \cap B \subseteq C' \subset C, A \setminus C = A \setminus C', B \setminus C = B \setminus C' \), then \( ABC' = A \otimes_{C'} B \).

2. If \( ABC = A \otimes_C B, B' \subset B, B'C \leq BC, \) then \( AB'C = A \otimes_C B' \leq ABC = A \otimes_C B \).

\*If \( \text{tp}(a/A) \) is stationary and \( e \in \text{dcl}^{eq}(a) \), then \( \text{tp}(e/A) \) is also stationary: Suppose \( e \equiv_A e', e \downarrow_A B \) and \( e' \downarrow_A B \). We need to show \( e \equiv_B e' \). By \( e \in \text{dcl}^{eq}(a) \) and compactness, there exists a definable function \( f \) such that \( f(a) = e \). We may assume \( a \downarrow_{Ae} B \), so we have \( a \downarrow_A B \). Take \( a' \) with \( ea \equiv_A e'a' \). Again we may assume \( a' \downarrow_{Ae'} B \), so \( a' \downarrow_A B \) follows. As \( a \equiv_A a', a \equiv_B a' \) follows. On the other hand \( e = f(a), e' = f(a') \), we see \( e \equiv_B e' \).

If \( \text{tp}(a/A) \) is stationary, \( a \downarrow_A B \) and \( a \in \text{dcl}^{eq}(B) \), then \( a \in \text{dcl}^{eq}(A) \): Note that \( a \in \text{acl}^{eq}(A) \). So, if \( a' \equiv_A a, \) then \( a' \downarrow_A B \). By stationarity, we see \( a \equiv_B a' \), so \( a = a' \) follows.
By weak elimination of imaginaries, we may work in $\mathcal{M}$ not in $\mathcal{M}^eq$ to show the CM-triviality. Put $D = acl(A_1A_2), \tilde{A}_i = acl(A_iE)$. We need to show $A_1 \downarrow E A_2 \Rightarrow A_1 \downarrow_{E \cap D} A_2$. By Fact 2 we see $\tilde{A}_1\tilde{A}_2 = \tilde{A}_1 \otimes_E \tilde{A}_2 \leq \mathcal{M}$. So, by 1

$$D = (\tilde{A}_1 \cap D) \otimes_{D \cap E} (\tilde{A}_2 \cap D) \leq \mathcal{M} \quad (\dagger).$$

Put $A'_i = acl(A_i(D \cap E)) \subset \tilde{A}_i$. As $A_1 \downarrow_E A_2$, we see $A'_1 \cap A'_2 = D \cap E$. By $D \cap E \leq A'_i \leq \tilde{A}_i \cap D,(\dagger)$ and 2,

$$A'_1A'_2 = A'_1 \otimes_{D \cap E} A'_2 \leq (\tilde{A}_1 \cap D) \otimes_{D \cap E} (\tilde{A}_2 \cap D) D \leq \mathcal{M}.$$

By Fact 2 again, we see $A_1 \downarrow_{E \cap D} A_2$.

2 Questions

We say that a theory (or structure) is strictly CM-trivial if it is CM-trivial but not one-based.

Question 1 Every strictly CM-trivial stable structure we know has weak elimination of imaginaries. On the other hand, Evans had CM-trivial SU-rank 1 structure without weak elimination of imaginaries in [E].

Does strictly CM-trivial strongly minimal set has weak elimination of imaginaries?

(It is well-known that if $D$ is a strongly minimal sets with infinite $acl(\emptyset)$, then $D$ has weak elimination of imaginaries.)

Question 2 Is there strictly CM-trivial stable structure except stable generic structures?

Question 3 One-basedness coincides with local modularity among strongly minimal sets. (This is not true among SU-rank 1 sets. See [V].)

Is there combinatorial geometric notion equivalent to CM-triviality among strongly minimal sets?

The following is a famous question.

Question 4 Evans showed supersimple $\aleph_0$-categorical CM-trivial theory must have finite SU-rank in [EW].

Is there supersimple $\aleph_0$-categorical theory with infinite SU-rank? Is there $\aleph_0$-categorical simple non-CM-trivial theory?
References.