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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1390: 18-22</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25827">http://hdl.handle.net/2433/25827</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On CM-triviality

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November 18, 2003

Abstract
Hrushovski’s generic construction yields CM-trivial structures with weak elimination of imaginaries. Here, we would like to give questions on CM-triviality.

1 CM-triviality of generic structures.

In this section, we show weak elimination of imaginaries and CM-triviality of well-known stable generic structures.

Definition 1 We say that $T$ is CM-trivial if

$$A_1 \downarrow_{A_2} B \Rightarrow A_1 \downarrow_{\text{acl}^{eq}(A_1 A_2) \cap B} A_2$$

for $A_1, A_2, B \subset \mathcal{M}$ algebraically closed sets.

Fact 2 Let $T$ be any theory of well-known stable generic structure.

Then we have;

1. $(\mathcal{M}: \text{big model of } T)$ For any $A, B \subset \mathcal{M}$ algebraically closed sets,

$$A \downarrow_{A \cap B} B \Leftrightarrow AB = A \otimes_{A \cap B} B \leq \mathcal{M},$$

2. any type over algebraically closed sets in real sort is stationary.
Explanations

1. The language is $L = \{R_i(X_1 \ldots X_{n_i}) : i < \omega\}$, where $R_i(X_1 \ldots X_{n_i})$ is an $n_i$-ary predicate.

2. We assume that any predicate is closed under permutations and $R_i(a_1 \ldots a_{n_i}) \Rightarrow a_i \neq a_j (i \neq j \leq n_i)$.

3. We defined predimension on finite $L$-structures.
   \[ \delta(A) = |A| - \sum_{i<\omega} \alpha_i \cdot |R_i^A| \]
   , where $A$ is a finite $L$-structure, $R_i^A$ is the set of tuples of $A$ satisfying $R_i$ (up to permutations) and $\alpha_0 > \alpha_1 > \ldots > \alpha_i (i < \omega)$ are fixed positive real numbers .

4. For finite $L$-structures $A, B$, we say $A$ is closed in $B$ (write $A \leq B$) if
   \[ \delta(XA) - \delta(A) \geq 0 \]
   for any $X \subseteq B$. “$A$ is closed in $B$” means that there are only suitably many (depending on $\alpha$) sequences intersecting $A$ and $B \setminus A$, and satisfying some predicates.

   For possibly infinite $L$-structures $A, N$, we say that $A$ is closed in $N$ (write $A \leq N$) if
   \[ A_0 \leq A_0X \]
   for any $A_0 \subset_\omega A, X \subset_\omega N \setminus A$.

   There exists the smallest closed subset $\text{cl}_N(A)$ of $N$ containing $A$.

   In any well-known stable generic structure, $\text{cl}_N(A) \subseteq \text{acl}_N(A)$, in particular $\text{acl}_N(A) \leq N$.

5. For $L$-structure $A, B, C$ with $A \cap B \subseteq C$, we say that $A$ and $B$ are freely joined over $C$ if there are no $i < \omega$ and $\bar{d} \in ABC$ such that $R_i(\bar{d})$, $\bar{d} \cap (A \setminus C) \neq \emptyset$ and $\bar{d} \cap (B \setminus C) \neq \emptyset$, and we write $ABC = A \otimes_C B$.

From now on, let $T$ be as in Fact 2.
Proposition 3 $T$ has weak elimination of imaginaries.

**proof** First we show the following claim.

Claim Let $A, B, B_1, B_2$ be algebraically closed. Suppose that $B_i \subseteq B$ and $A \perp_{B_i} B$ for $i = 1, 2$. Then $A \perp_{B_1 \cap B_2} B$.

The proof of this claim: Put $A_i = \text{acl}(AB_i)$. Then $A_i B = A_i \otimes_{B_i} B \leq \mathcal{M}$ by Fact 2, for $i = 1, 2$. Intersecting the two sets yields $(A_1 \cap A_2) \otimes_{B_1 \cap B_2} B \leq \mathcal{M}$. Note that $A_1 \cap A_2$ and $B_1 \cap B_2$ are algebraically closed.

So by Fact 2, $A_1 \cap A_2 \perp_{B_1 \cap B_2} B$; since $A \subseteq A_1 \cap A_2$ we get $A \perp_{B_1 \cap B_2} B$, as desired.

Now we show the weak elimination of imaginaries. Let $E(x, y)$ be a definable equivalence relation over $\emptyset$, and consider $e = a_E$, where $a_E$ is the $E$-class of $a$.

Take $\bar{b}_1, \bar{b}_2$ such that $\bar{a}, \bar{b}_1, \bar{b}_2$ are independent over $e$, and $\bar{a}_E = (\bar{b}_1)_E = (\bar{b}_2)_E$.

As $e \in \text{acl}^{eq}(\bar{b}_i)$ we have $\bar{a} \downarrow_{\bar{b}_i} \bar{b}_1 \bar{b}_2$, for $i = 1, 2$.

Put $B = \text{acl}(b_1) \cap \text{acl}(b_2)$, where the algebraic closure is taken in the real sort.

Then $\bar{a} \downarrow_B \bar{b}_1 \bar{b}_2$ by claim. As $\text{tp}(\bar{a}/B)$ is stationary and $e \in \text{dcl}^{eq}(\bar{a}) \cap \text{dcl}^{eq}(\bar{b}_1 \bar{b}_2)$, we get $e \in \text{dcl}^{eq}(B)^*$. On the other hand, as $\bar{b}_1 \downarrow e \bar{b}_1$, we have $B \subseteq \text{acl}(e)$.

By compactness we can find a finite tuple $\bar{b} \in B$ with $e \in \text{dcl}^{eq}(\bar{b})$; clearly $\bar{b} \in \text{acl}^{eq}(e)$, as desired.

Proposition 4 $T$ is CM-trivial.

**proof** We use the following fundamental property.

1. If $ABC = A \otimes_C B, A \cap B \subseteq C \subset C$, $A \setminus C = A \setminus C', B \setminus C = B \setminus C'$, then $ABC' = A \otimes_C B$.

2. If $ABC = A \otimes_C B, B' \subset B, B' \subset B C \leq BC$, then $AB'C = A \otimes_C B' \leq ABC = A \otimes_C B$.

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*If $\text{tp}(a/A)$ is stationary and $e \in \text{dcl}^{eq}(a)$, then $\text{tp}(e/A)$ is also stationary. Suppose $e \equiv_A e', e \downarrow_A B$ and $e' \downarrow_A B$. We need to show $e \equiv_B e'$. By $e \in \text{dcl}^{eq}(a)$ and compactness, there exists a definable function $f$ such that $f(a) = e$. We may assume $a \downarrow_{Ae} B$, so we have $a \downarrow_A B$. Take $a'$ with $ea \equiv_A e'a'$. Again we may assume $a' \downarrow_{Ae'} B$, so $a' \downarrow_A B$ follows. As $a \equiv_A a'$, $a \equiv_B a'$ follows. On the other hand $e = f(a), e' = f(a')$, we see $e \equiv_B e'$.

If $\text{tp}(a/A)$ is stationary, $a \downarrow_A B$ and $a \in \text{dcl}^{eq}(B)$, then $a \in \text{dcl}^{eq}(A)$: Note that $a \in \text{acl}^{eq}(A)$. So, if $a' \equiv_A a$, then $a' \downarrow_A B$. By stationarity, we see $a \equiv_B a'$, so $a = a'$ follows.
By weak elimination of imaginaries, we may work in $\mathcal{M}$ not in $\mathcal{M}^{eq}$ to show the CM-triviality. Put $D = acl(A_1A_2), \tilde{A}_i = acl(A_iE)$. We need to show $A_1 \downarrow_E A_2 \Rightarrow A_1 \downarrow_{E \cap D} A_2$. By Fact 2 we see $\tilde{A}_1 \tilde{A}_2 = \tilde{A}_1 \otimes_E \tilde{A}_2 \leq \mathcal{M}$. So, by 1 $D = (\tilde{A}_1 \cap D) \otimes_{D \cap E} (\tilde{A}_2 \cap D) \leq \mathcal{M}$ (†). Put $A'_i = acl(A_i(D \cap E)) \subset \tilde{A}_i$. As $A_1 \downarrow_E A_2$, we see $A'_1 \cap A'_2 = D \cap E$. By $D \cap E \leq A'_i \leq \tilde{A}_i \cap D$, (†) and 2,

$$A'_1 A'_2 = A'_1 \otimes_{D \cap E} A'_2 \leq (\tilde{A}_1 \cap D) \otimes_{D \cap E} (\tilde{A}_2 \cap D)D \leq \mathcal{M}.$$ 

By Fact 2 again, we see $A_1 \downarrow_{E \cap D} A_2$.

## 2 Questions

We say that a theory (or structure) is strictly CM-trivial if it is CM-trivial but not one-based.

**Question 1** Every strictly CM-trivial stable structure we know has weak elimination of imaginaries. On the other hand, Evans had CM-trivial SU-rank 1 structure without weak elimination of imaginaries in [E].

Does strictly CM-trivial strongly minimal set has weak elimination of imaginaries?

(It is well-known that if $D$ is a strongly minimal sets with infinite acl($\emptyset$), then $D$ has weak elimination of imaginaries.)

**Question 2** Is there strictly CM-trivial stable structure except stable generic structures?

**Question 3** One-basedness coinsides with local modularity among strongly minimal sets. (This is not true among SU-rank 1 sets. See [V].)

Is there combinatorial geometric notion equivalent to CM-triviality among strongly minimal sets?

The following is a famous question.

**Question 4** Evans showed supersimple $\aleph_0$-categorical CM-trivial theory must have finite SU-rank in [EW].

Is there supersimple $\aleph_0$-categorical theory with infinite SU-rank? Is there $\aleph_0$-categorical simple non-CM-trivial theory?
References.


