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On minimal fields

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Abstract

An infinite structure $M$ is minimal if every definable subset (using parameters in $M$) is finite or cofinite. An algebraically closed field is minimal, since it allows quantifier elimination. Podewski [Po] conjectured that minimal fields are algebraically closed. Wagner [Wa] has shown that a minimal field of non-zero characteristics is algebraically closed. We discuss minimal fields in all characteristics.

1 Minimal fields with infinite bases

It is well-known that a minimal structure $M$ with an infinite basis is strongly minimal. For we can easily show that $M$ is saturated, and it follows that $M$ is strongly minimal. Since strongly minimal fields are algebraically closed, we get that a minimal field with an infinite basis is algebraically closed. Nevertheless we give a direct proof that a minimal field with an infinite basis is algebraically closed in order to use the arguments of that proof in the next section.

The following facts are well-known. See Hodges [Ho], and Wagner [Wa].

Fact 1 Let $M$ be a minimal structure. Then

1. $(M, acl_M)$ is a pregeometry.

2. Every subset of $M$ has a dimension over any subset.

3. For every independent (over $\emptyset$) subsets $A,B$ of $M$ with $|A|=|B|$, any bijective map of $A$ to $B$ can be extended to an automorphism of $M$.

4. If $X$ is an algebraically closed (in the sense of model theory) infinite subset of $M$, then $X \prec K$.

Every infinite field with quantifier elimination is algebraically closed. Direct proofs were given by several authors. We prove the following lemma in the same manner as its proof given by Wheeler [Wh].
Lemma 2 A minimal field $K$ with an infinite basis is algebraically closed.

Proof. Consider the definable set $A = \{a \in K : K \models \exists y (a = y^n)\}$ for $n \in \mathbb{N}^*$. Then $A$ must be infinite, hence cofinite. Suppose $K \setminus A \neq \emptyset$, and let $b \in K \setminus A$. Then $\{bc^n : c \in K^*\} \subseteq K \setminus A$, a contradiction. Therefore $A = K$. It follows that $K$ is perfect and $\text{acl}_K(\emptyset)$ is infinite, hence $\text{acl}_K(\emptyset) \prec K$.

Claim A Let $n \in \mathbb{N}^*$ and $a_1, \ldots, a_n$ be independent elements of $K$ (over $\emptyset$). Then the polynomial $X^n + a_1X^{n-1} + \cdots + a_n$ has a root in $K$.

For suppose that $b_1, \ldots, b_n$ be independent elements of $K$, and let $p(X)$ be the polynomial $\prod_{1 \leq i \leq n}(X - b_i)$. Then $p(X) = X^n + s_1X^{n-1} + \cdots + s_n$, where $s_i$ is the $i$-th elementary symmetric polynomial. $s_1, \ldots, s_n$ are independent, since $\text{acl}_K(\{b_1, \ldots, b_n\}) = \text{acl}_K(\{s_1, \ldots, s_n\})$. In minimal structures there is an automorphism which takes $s_i$ to $a_i$, hence the polynomial $X^n + a_1X^{n-1} + \cdots + a_n$ has a root in $K$.

Now suppose that $K$ is not algebraically closed. Then $\text{acl}_K(\emptyset)$ is not algebraically closed, since $\text{acl}_K(\emptyset) \prec K$. Let $K_0 = \text{acl}_K(\emptyset)$. Then there is $\alpha \in \overline{K}_0 \setminus K_0$, where $\overline{K}_0$ is the algebraic closure of $K_0$. $\alpha$ is separable over $K_0$, since $K_0$ is perfect. Let $\alpha$ has degree $n$ over $K_0$ ($n > 1$) and $\alpha = \alpha_0, \ldots, \alpha_{n-1}$ be the distinct conjugates of $\alpha$. Choose independent elements $t_0, \ldots, t_{n-1}$ of $K$ (over $\emptyset$), and form the polynomial $F(X) = \prod_{i \leq n}(X - \sum_{j \leq n} t_j \alpha_j^i)$. We can write this polynomial as $F(X) = X^n + g_1X^{n-1} + \cdots + g_n$ where each $g_i$ is in $K_0[\bar{\alpha}] \subset K$.

Claim B The $g_i$ are independent (over $\emptyset$).

Let the roots of $F(X)$ be $r_0, \ldots, r_{n-1}$, that is, $r_j = t_0 + t_1\alpha_j + \cdots + t_{n-1}\alpha_j^{n-1}$. Then

$$
\begin{pmatrix}
1 & \alpha_0 & \cdots & \alpha_0^{n-1} \\
1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \alpha_{n-1} & \cdots & \alpha_{n-1}^{n-1}
\end{pmatrix}
\begin{pmatrix}
t_0 \\
t_1 \\
\vdots \\
t_{n-1}
\end{pmatrix}
= 
\begin{pmatrix}
r_0 \\
r_1 \\
\vdots \\
r_{n-1}
\end{pmatrix}
$$

Since the matrix $M$ on the left is invertible, we get the $t_j$ as linear combinations of the roots $r_j$, with coefficients in $P(\bar{\alpha})$ where $P$ is the prime field of $K$. Then the $t_j$ are algebraic over $K_0(\bar{\alpha})$ in the sense of field theory, since each $r_j$ is algebraic over $P(\bar{\alpha})$ and each $\alpha_i$ is algebraic over $K_0$ in the sense of field theory. It follows that $\text{acl}_K(\bar{\alpha}) = \text{acl}_K(\{\bar{\alpha}\})$, and we conclude that the $g_i$ are independent (over $\emptyset$).

By Claim A, $F(X) = 0$ has a root in $K$. Therefore some $r_j = t_0 + t_1\alpha_j + \cdots + t_{n-1}\alpha_j^{n-1}$ is in $K$, hence $\alpha_j$ has degree at most $n - 1$ over $K$. 
On the other hand, $\alpha$ must have degree $n$ over $K$. For $K$ is separable over $K_0$ since $K_0$ is perfect, and $K_0$ is algebraically closed in $K$ in the sense of field theory. Then $K$ is a regular extension of $K_0$, hence $\alpha$ has degree $n$ over $K$. Thus we get a contradiction.

Uncountable minimal fields have infinite bases over $\emptyset$, hence we get:

**Corollary 3** An uncountable minimal field is algebraically closed.

**Remark.** Note that the proof of lemma 2 also gives a direct proof that strongly minimal fields are algebraically closed fields.

## 2 Minimal elementary extensions

In this section, assuming some common property of minimal fields we show that any minimal field $K$ has a proper minimal elementary extension. Then, under this assumption, we get a minimal field $L \succ K$ which has a dimension $> n$ (over $\emptyset$) for any $n \in \mathbb{N}$.

We assume the following property.

(†) For any minimal field, there is no $L(K)$-formula $\psi(x, y)$ such that $\exists x^n \psi(x, K)$ and $\exists^{\geq n} x - \psi(x, K)$ are cofinite for all $n \in \mathbb{N}$.

Note that algebraically closed fields have the property (†) since they admit quantifier elimination.

**Lemma 4** Assuming (†), any minimal field $K$ has a proper minimal elementary extension.

**Proof.** Let $T = \text{Th}(K)_{a \in K}$, $c$ a new constant symbol other than $L(K)$, and $T_0 = T \cup \{ x \neq a \}_{a \in K}$. Then $T_0$ is a consistent $L(K \cup \{ c \})$-theory. Let $L \models T_0$, and $L_0 = \text{acl}_L(K \cup \{ c \})$. We show that $L_0$ is a desired structure.

First we show that $L_0 \prec L$, which implies that $K \prec L_0$. Consider a sentence $\exists x \varphi(x, \bar{b})$ with $\bar{b} \in L_0$ where $\varphi(x, \bar{y})$ is an $L$-formula, and suppose that $L \models \exists x \varphi(x, \bar{b})$. We show that there exists $a \in L_0$ such that $L \models \varphi(a, \bar{b})$. Since each $b_i$ is algebraic over $K \cup \{ c \}$, there is an algebraic $L(K \cup \{ c \})$-formula $\psi(\bar{x})$ such that $L \models \psi(\bar{b}) \land \exists^{=n_0} \overline{\varphi}(\bar{x})$ for some $n_0 \in \mathbb{N}$. We choose $\psi$ to make $n_0$ as small as possible. Clearly $L \models \exists x \varphi(x, \bar{b}) \iff \exists x \exists \overline{\varphi}(\varphi(x, \bar{y}) \land \psi(\bar{y}))$, and hence $L \models \exists x \exists \overline{\varphi}(\varphi(x, \bar{y}) \land \psi(\bar{y}))$ since $\exists \overline{\varphi}(\varphi(x, \bar{y}) \land \psi(\bar{y}))$ is an $L(K \cup \{ c \})$-formula, we write it as $\varphi_0(x, c)$ where $\varphi_0(x, y)$ is an $L(K)$-formula.
We show that there exists \( a \in L_0 \) such that \( L \models \varphi_0(a, c) \). If \( \varphi_0(x, c) \) does not involve \( c \), then we are done since \( K \prec L \). Suppose that \( \varphi_0(x, c) \) actually involves \( c \).

If \( \varphi_0(x, c) \) is algebraic in \( L \), then we are done. Suppose not. Then \( L \models \exists \geq n x \varphi_0(x, c) \) for all \( n \). It follows that \( \exists \geq n x \varphi_0(x, K) \) is cofinite for all \( n \), since \( c \) does not satisfy algebraic formulas of \( K \). By (†), \( \exists \geq n x \varphi_0(x, K) \) is finite for some \( n \), which implies that \( L \models \exists^{< n} x \varphi_0(x, c) \) for some \( n \). Therefore \( -\varphi_0(L, c) \) is finite. It follows that there exists \( a \in L_0 \) such that \( L \models \varphi_0(a, c) \).

Now take \( \bar{d} \in L_0 \) such that \( L \models \varphi(a, \bar{d}) \land \psi(\bar{d}) \). Since \( a \in L_0 \), there is an algebraic \( L(K \cup \{c\}) \)-formula \( \eta(x) \) such that \( L \models \eta(a) \land \exists^{= n_1} x \eta(x) \) for some \( n_1 \in \mathbb{N} \). Again we choose \( \eta \) to make \( n_1 \) as small as possible. Consider \( L(K \cup \{c\}) \)-formula \( \theta(\bar{y}) = \exists x (\varphi(x, \bar{y}) \land \psi(\bar{y}) \land \eta(x)) \). Then \( L \models \theta(\bar{d}) \), which implies that \( L \models \theta(\bar{b}) \). It follows that there exists \( a' \in L_0 \) such that \( L \models \varphi(a', \bar{b}) \).

The above argument also shows that \( L_0 \) is minimal. This completes the proof.

\[ \square \]

**Theorem 5** Assuming (†), any minimal field \( K \) is algebraically closed.

Proof. Suppose that \( K \) is not algebraically closed. Again there is \( \alpha \in \overline{K_0} \setminus K_0 \), where \( \overline{K_0} \) is the algebraic closure of \( K_0 = \mathrm{acl}_K(\emptyset) \). Let \( \alpha \) has degree \( n \) over \( K_0 \) (\( n > 1 \)). By lemma 4, there is a minimal elementary extension \( L \) of \( K \) which has dimension \( > n \) (over \( \emptyset \)). Noting that \( \mathrm{acl}_K(\emptyset) = \mathrm{acl}_L(\emptyset) \), we get a desired contradiction as before.

\[ \square \]

**References**


