Gap Number of Groups

Hiroshi Tanaka
Graduate School of Natural Science and Technology,
Okayama University

Finite gap number is introduced by J. C. Lennox and J. E. Roseblade in [LR]. We study groups of the small gap number.

1 ladder index

Let \( T \) be complete theory in an \( L, \phi(x, y) \) \( L \)-formula (\( x, y \) are free variables).

**Definition 1** An \( n \)-ladder for \( \phi \) is a sequence \((a_0, \ldots, a_{n-1}; b_0, \ldots, b_{n-1})\) of tuples in some model \( M \) of \( T \), such that

\[
\forall i, j < n, M \models \phi(a_i, b_j) \iff i \leq j.
\]

We say that \( \phi \) is stable formula if there exists \( n \) such that no \( n \)-ladder for \( \phi \) exists; otherwise it is unstable. The least such \( n \) is the ladder index of \( \phi \).

**Theorem 2** The theory \( T \) is unstable if and only if there exists an unstable formula in \( L \) for \( T \).

Henceforth we consider the ladder index for the commutativity formula "\( xy = yx \)". The ladder index of a group \( G \) for the commutativity formula is denoted by \( \ell(G) \).
2 gap number

Let $G$ be a group.

Definition 3 A group $G$ has a finite gap number if for any subgroups $H_0, H_1, \ldots, H_n, \ldots$ of $G$, among the sequence

$$C_G(H_0) \leq C_G(H_1) \leq \cdots \leq C_G(H_n) \leq \cdots$$

there exist at most $m$ many strict inclusions. The most such $m$ is the gap number of $G$, and denoted by $g(G)$.

Lemma 4 Let $g(G) = n$. Suppose that the sequence

$$C_G(H_0) > C_G(H_1) > \cdots > C_G(H_n)$$

gives gap number $n$. Then there exists $a_i (0 \leq i \leq n)$ in $G$ such that $C_G(H_i) = C_G(\{a_0, a_1, \ldots, a_i\})$ for each $i$. In particular we may do $a_0 = 1$. Henceforth we abbreviate as $C_G(\{a_0, a_1, \ldots, a_i\}) = (a_0, a_1, \ldots, a_i)$.

By Lemma 4, we can prove the following:

Theorem 5 [ITT] $\ell(G) = g(G) + 2$.

Lemma 6 Let $A, B \subset G$ with $A \subset B$. Then $(A) \supset (B)$.

Lemma 7 Let $A \subset G$. Then $((A)) = (A)$.

By the above lemma, the following holds:

Lemma 8 Let $g(G) = n$. Suppose $(a_0, \ldots, a_n; b_0, \ldots, b_n)$ is $(n+1)$-ladder. Then

$$((a_0)) = (b_n, \ldots, b_1, b_0);$$
$$((a_0, a_1)) = (b_n, \ldots, b_1);$$
$$\vdots$$
$$((a_0, \ldots, a_n)) = (b_n).$$

Lemma 9 Let $g(G) = n$. Suppose that the sequence

$$G > (a_1) > \cdots > (a_1, a_2, \ldots, a_n)$$

gives gap number $n$. Then $(a_1, a_2, \ldots, a_{n-1})$ is abelian.
3 Groups of gap number up to four

From now on we do not consider the ladder index but the gap number.

Theorem 10 [ITT] $g(G) = 0$ if and only if $G$ is abelian.

Theorem 11 [ITT] There exist no groups $G$ of $g(G) = 1$.

(proof) Let $g(G) \geq 1$. Then there exists $a \in G$ such that $G > (a)$. Since $(a) \neq G$, there exists $b \notin (a)$. Therefore, we have $G > (a) > (a, b)$. Thus $g(G) \geq 2$.

Theorem 12 [ITT] $g(G) = 2$ if and only if $G$ is not abelian, and for any $a, b \in G \setminus Z(G)$, if $(a) \neq (b)$ then $(a, b) = Z(G)$.

Example 13 $g(S_3) = g(D_n) = 2$ ($D_n$ is a dihedral group).

Example 14 $g(SL(2, F)) = 2$ ($F$ is a field).

Theorem 15 [ITT] There exist no groups $G$ of $g(G) = 3$.

(proof) Let $g(G) \geq 3$. Then there exist $a_1, a_2 \in G$ such that $G > (a_1) > (a_1, a_2) > Z(G)$.

Case 1: $a_1a_2 = a_2a_1$.

Since $(a_1) \neq (a_2)$, we may assume $(a_1) \setminus (a_2) \neq \emptyset$. Let $b \in (a_1) \setminus (a_2)$. As $a_1 \notin G$, there exists a $c \in G \setminus (a_1)$. Therefore, we have $G > (a_1) > (a_1, a_2) > (a_1, a_2, b) > (a_1, a_2, b, c)$.

Thus $g(G) \geq 4$.

Case 2: $a_1a_2 \neq a_2a_1$.

There exists a $d \in (a_1, a_2) \setminus Z(G)$. Since $d \notin Z(G)$, we can find $e \notin G \setminus (d)$.

Then we have $G > (d) > (d, a_1) > (d, a_1, a_2) > (d, a_1, a_2, e)$.

Thus $g(G) \geq 4$.

Example 16 $g(S_4) = g(S_5) = 4$. 
4 Groups of gap number five

In this section, we investigate whether a group $G$ of gap number 5 exists or not.

Let $g(G) = 5$ and let $(1, a_1, \ldots, a_5; b_0, \ldots, b_4, 1)$ be 6-ladder.

**Case 1:** $a_1a_2 = a_2a_1$, $a_1a_3 = a_3a_1$ and $a_2a_3 = a_3a_2$.

Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_2) > (a_1, a_2, a_3, b_2, b_1) > Z(G).$$

Thus, $g(G) \geq 6$.

**Case 2:** $a_1a_2 = a_2a_1$, $a_1a_3 = a_3a_1$, $a_2a_3 = a_3a_2$ and $a_1a_4 \neq a_4a_1$.

Then we have

$$G > (b_4) > (b_4, a_1) > (b_4, a_1, a_2) > (b_4, a_1, a_2, a_3) > (b_4, a_1, a_2, a_3, a_4) > Z(G).$$

Thus, $g(G) \geq 6$.

**Case 3:** $a_1a_2 = a_2a_1$, $a_1a_3 = a_3a_1$, $a_2a_3 = a_3a_2$ and $a_1a_4 = a_4a_1$.

Then we have

$$G > (a_1) > (a_1, b_3) > (a_1, b_3, a_2) > (a_1, b_3, a_2, a_3) > (a_1, b_3, a_2, a_3, a_4) > Z(G).$$

Thus, $g(G) \geq 6$.

**Case 4:** $a_1a_2 = a_2a_1$, $a_1a_3 \neq a_3a_1$ and $a_2a_3 = a_3a_2$.

Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover $a_2a_1 = a_1a_2$, $a_2a_3 = a_3a_2$ and $a_1a_3 \neq a_3a_1$. By case 2, 3, $g(G) \geq 6$.

**Case 5:** $a_1a_2 = a_2a_1$ and $a_1a_3 \neq a_3a_1$.

Then we have

$$G > (b_4) > (b_4, b_3) > (b_4, b_3, a_1) > (b_4, b_3, a_1, b_1) > Z(G).$$

Moreover $b_4a_1 = a_1b_4$ and $b_3a_1 = a_1b_3$. By case 1, 4, $g(G) \geq 6$.

Therefore, by case 1 through 5, we hold $a_1a_2 \neq a_2a_1$. 

Case 6: $a_1a_2 \neq a_2a_1$ and $a_1a_3 = a_3a_1$.

Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_2) > (a_1, a_3, a_2, a_4) > Z(G).$$

Moreover $a_1a_3 = a_3a_1$. Thus, $g(G) \geq 6$.

Case 7: $a_1a_2 \neq a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 = a_3a_2$.

Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover $a_2a_3 = a_3a_2$. By case 6, $g(G) \geq 6$.

Therefore, by case 1 through 7, we hold $a_1a_2 \neq a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 \neq a_3a_2$.

Case 8: all of $a_1, a_2, a_3, a_4$ are noncommutative except $a_1a_4 = a_4a_1, a_2a_4 = a_4a_2$.

Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_4) > (a_1, a_2, a_4, a_3) > Z(G).$$

Moreover $a_1a_4 = a_4a_1$. By case 6, $g(G) \geq 6$.

Case 9: all of $a_1, a_2, a_3, a_4$ are noncommutative except $a_1a_4 = a_4a_1, a_3a_4 = a_4a_3$.

Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_4) > (a_1, a_3, a_4, a_2) > Z(G).$$

Moreover $a_1a_4 = a_4a_1$. By case 6, $g(G) \geq 6$.

Case 10: all of $a_1, a_2, a_3, a_4$ are noncommutative except $a_2a_4 = a_4a_2, a_3a_4 = a_4a_3$.

Then we have

$$G > (a_2) > (a_2, a_3) > (a_2, a_3, a_4) > (a_2, a_3, a_4, a_1) > Z(G).$$

Moreover $a_2a_4 = a_4a_2$. By case 6, $g(G) \geq 6$. 
In the cases of remaining we understand the following:

**Lemma 17** Let $G > (a_1) > (a_1, a_2) > \cdots > (a_1, a_2, a_3, a_4, a_5) = Z(G)$. Then we can do as follows: all of $a_1, a_2, a_3, a_4$ are noncommutative except $a_1 a_4 = a_4 a_1, a_2 a_4 = a_4 a_2, a_3 a_4 = a_4 a_3, a_1 a_5 = a_5 a_1, a_2 a_5 = a_5 a_2, a_3 a_5 = a_5 a_3$.

(proof) We have

$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_4) > (a_1, a_2, a_3, b_4, b_5)$. Moreover all of $a_1, a_2, a_3$ are noncommutative, and all of $b_4, b_3, b_2$ are noncommutative, as desired.

**Question 18** Does there exist a group $G$ of $g(G) = 5$?

**References**


