On weak dividing in n-simple theories

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(1390 巻 2004 年 65-68)

Weak dividing was originally defined by Shelah in [1]. After a long time, Dolich characterized that notion in simple context in [2]. Then Kim and Shi continued the investigation, in particular they proved that a theory $T$ is stable if and only if weak dividing is symmetric in [3]. Recently, the class of simple theory was split into $\omega + 1$ subclasses by Kolesnikov in [4]. He used the notion of n-simplicity for $n \leq \omega$. I studied his paper and had some consideration about the relation with weak dividing.

At first, we recall some definitions in [4].

For $n \geq 2$, let the symbol $\text{Ind}(x; y_0, \ldots, y_{n-1})$ denote the type expressing that $y_0, \ldots, y_{n-1}$ are indiscernible over $x$.

**Definition 1** Fix $1 \leq n \leq k < \omega$. Take a formula $\varphi(x, y_0, \ldots, y_{n-1})$ and a partial type $p(x)$. Define $D_n[p, \varphi, k] \geq \alpha$ by induction on $\alpha$.

1. $D_n[p, \varphi, k] \geq 0$ if $p$ is consistent.
2. (2) for $\alpha$ limit, $D_n[p, \varphi, k] \geq \alpha$ if $D_n[p, \varphi, k] \geq \beta$ for all $\beta < \alpha$;
3. (3) $D_n[p, \varphi, k] \geq \alpha + 1$ if for every finite $r \subseteq p(x)$ there is a sequence $\{a_i|i < \omega\}$ such that for all $\overline{i} \in [\omega]^n$:
   
   $$D_n[r \cup \{\varphi(x, \overline{a}_{\overline{i}})\}] \cup \text{Ind}(x; \overline{a}_{\overline{i}}), \varphi, k] \geq \alpha$$

and the set $\{\varphi(x, \overline{a}_{\overline{i}})|\overline{i} \in [\omega]^n\}$ is $[k]^n$-contradictory.

The expressions $D_n[p, \varphi, k] = \alpha, D_n[p, \varphi, k] = -1$, and $D_n[p, \varphi, k] = \infty$ are defined as usual.

**Definition 2** Let $\alpha \leq \omega$. We say that a complete thoery $T$ is $\alpha - simple$ if for all $n < \alpha$, for all $\varphi(x, y_0, \ldots, y_n)$ and $k > n + 1$ the rank $D_{n+1}[x = x, \varphi, k]$ is bounded.(i.e. is less than $\infty$.)

**Definition 3** (1) A formula $\varphi(x, y_0, \ldots, y_{n-1})$, a set of sequences $\{J_\eta|\eta \in ([\omega]^n)^{<\omega}\}$, and $k < \omega$ witness the $n - tree$ property if for every $\eta \in ([\omega]^n)^{<\omega}$, the type $\{\varphi(x, \overline{a}_{\eta[l]}^\eta)|l < \omega\}$ is realized by $\overline{b}_\eta$ such that sequences $\overline{a}_{\eta[l]}^\eta$ are
indiscernible over $b_{n}$ for each $l < \omega$ and for every $\eta \in ([\omega]^{n})^{<\omega}$ the set
\(\{\varphi(x, \overline{a}_{\eta}^{i})\bar{b} \in [\omega]^{n}\}\) is $[k]^{n}$-contradictory where $\overline{a}_{\eta[i]} := \{a_{\eta[i]}^{0}, \ldots, a_{\eta[i]}^{n-1}\}$ for $\eta[i] = i_{0} \cdots i_{n-1}$.

(2) A theory $T$ has the $n$-tree property if there exist a formula, a set of parameters, and a number $k$ witnessing the $n$-tree property.

The next proposition is proved by the definitions.

**Proposition 4** ([4]) A theory $T$ is $\alpha$-simple if and only if it does not have an $(n + 1)$-tree property for any $n < \alpha$.

Kolesnikov defined some notion of dividing for $n$-simple context.

**Definition 5** For $n < \omega$, we say that a formula $\varphi(x, a_{0}, \ldots, a_{n-1})$ $n$-divides over $A$ if there is an indiscernible sequence $\{a_{i}|i < \omega\}$ over $A$ and $b \models \varphi(x, a_{0}, \ldots, a_{n-1})$ such that $\{a_{0}, \ldots, a_{n-1}\}$ are indiscernible over $b$ and the set $\{\varphi(x, \overline{a}_{\eta})\bar{b} \in [\omega]^{n}\}$ is $[k]^{n}$-contradictory for some $k$.

**Remark 6** It is clear that for $n = 1$ the definition is the same as that of dividing.

We recall the definition of weak dividing to make sure.

**Definition 7** We say that $p(x) = \text{tp}(a/B)$ weakly divides over $A (\subseteq B)$ if there is a formula $\psi(x_{1}, \ldots, x_{n})$ over $A$ such that $[p]^{\psi} := p(x_{1}) \cup \ldots \cup p(x_{n}) \cup \{\psi(x_{1}, \ldots, x_{n})\}$ is inconsistent while $[q]^{\psi}$ is consistent where $q(x) = \text{tp}(a/A)$.

The next facts are easily checked.

**Fact 8** Let $A \subset B$ and $\varphi(x_{0}, \ldots, x_{n-1}, b)$ be a formula over $B$. Suppose that there is an indiscernible sequence $\{a_{i}|i < \omega\}$ over $A$ satisfying:
\(\models \varphi(a_{0}, \ldots, a_{n-1}, b)\) and $\{a_{i}|i < n\}$ are indiscernible over $b$. If the type $\{\varphi(x_{0}, \ldots, x_{n-1}, b)\} \cup \text{Ind}(B; \{x_{i}|i < \omega\})$ is inconsistent, then there is a formula $\psi(a_{0}, \ldots, a_{n-1}, z)$ such that $\psi(a_{0}, \ldots, a_{n-1}, z)$ $n$-divides over $A$.

**Fact 9** Let $A \subset B$ and $p(x) = \text{tp}(a/B)$. Suppose that there is a formula $\varphi(x_{0}, \ldots, x_{n-1})$ over $A$ and an infinite indiscernible sequence $\{a_{i}|i < \omega\}$ over $A$ with $\text{tp}(a_{0}/A) = p[A]$ such that
\(\models \varphi(a_{0}, \ldots, a_{n-1})\) and
the type $"\{\varphi(x_{0}, \ldots, x_{n-1})\} \cup \text{Ind}(A; \{x_{i}|i < \omega\}) \cup \bigcup_{i<\omega} p(x_{i})"$ is inconsistent.
Then $p$ weakly divides over $A$.
Moreover if $T$ is simple, then $p$ divides over $A$. 
The case is problematic when realizations of the formula cannot not be extended to an infinite indiscernible sequence over the original parameters. I tried to use the facts above for the argument of weak dividing in n-simple theories, but I have no result to show here.

We can define an analogy of weak dividing for n-dividing.

**Definition 10** Let $A \subseteq B$. And let $p(x_0, \ldots, x_{n-1})$ be a complete type over $B$ such that $p(x_0, \ldots, x_{n-1}) \vdash \text{Ind}(A; x_0, \ldots, x_{n-1})$. We say that $p(x_0, \ldots, x_{n-1})$ is \textit{weakly }$n$\textit{-divides over }$A$\textit{"} if there are $k < \omega$ and a formula $\psi(x_0, \ldots, x_{k-1})$ over $A$ such that $\{\psi(x_0, \ldots, x_{k-1})\} \cup \bigcup_{i \in [k]^n} p^i(x)$ is inconsistent where $p^i(x) = p(x_{i_0}, \ldots, x_{i_n-1})$ for $i_0 < i_1 < \cdots < i_{n-1} < k$ and $k > n$.

**Remark 11** When $n = 1$, "weak 1-dividing" is the same as "weak dividing".

**Notation**

From now we denote $[p]^\psi$ for the type $\{\psi(x_0, \ldots, x_{k-1})\} \cup \bigcup_{i \in [k]^n} p^i(x)$.

**Fact 12** Let $A \subseteq B \subseteq C$.

1. If $\text{tp}(a/C)$ does not weakly $n$-divide over $B$, then $\text{tp}(a/B)$ does not weakly $n$-divide over $A$.

2. If $\text{tp}(a/C)$ does not weakly $n$-divide over $B$ and $\text{tp}(a/B)$ does not weakly $n$-divide over $A$, then $\text{tp}(a/C)$ does not weakly $n$-divide over $A$.

**Fact 13** Weak $n$-dividing has the local character.

**Fact 14** If $\text{tp}(b/Aa_0 \ldots a_{n-1})$ $n$-divides over $A$, then $\text{tp}(a_0 \ldots a_{n-1}/Ab)$ weakly $n$-divides over $A$.

**Remark 15** Naturally, we define weak $n$-dividing for complete types as follows:

a complete type $p$ weakly $n$-divides over $A$ if it implies a formula which weakly $n$-divides over $A$.

**Lemma 16** Let $A \subseteq B$. And let $p(x_0, \ldots, x_{n-1})$ be a complete type over $B$ such that $p(x_0, \ldots, x_{n-1}) \vdash \text{Ind}(A; x_0, \ldots, x_{n-1})$.

Then the following are equivalent:

1. $p$ does not weakly $n$-divide over $A$.

2. For any set $C := \{a_i | i \in I\}$ satisfying that for any $n$-sequence $a_{i_0}, \ldots, a_{i_{n-1}}$ in $C$ with $i_0 < i_1 < \cdots < i_{n-1}$, $p[A(a_{i_0}, \ldots, a_{i_{n-1}})]$, there is $B'$ such that $\text{tp}(B/A) = \text{tp}(B'/A)$ and for any $a_{i_0}, \ldots, a_{i_{n-1}}$ in $C$ with $i_0 < i_1 < \cdots < i_{n-1}$, $\text{tp}(B'/a_{i_0} \ldots a_{i_{n-1}}A) = \text{tp}(B/a_0 \ldots a_{n-1}A)$.
The further characterization needs to investigate the relation between $n$–simple theories and $n$–dividing more.

References