MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES B.
Vol. XX. No.2, Article 1, 1951.

On the Error Involved in the Depth of Isostatic Compensation Derived from the Condition $\sum$ (Squared Residuals) = a Minimum.

By

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(Received July 1, 1951)

In the following table are shown various values of the depth of isostatic compensation obtained by different authors. In this table the gravity data used by HELMERT, BOWIE and the present author are based upon the same normal gravity derived from HELMERT's formula of 1901 in Potsdam system. It is worth noticing that the depth of compensation shown in the table varies markedly with the region where the depth has been found.

When considered from the view-points of geology and geophysics, the above mentioned dispersion of the depth of isostatic compensation with respect to region seems to be an important thing, because the dispersion of the depth may imply variation of the condition of the earth's crust ; this opinion is correct as will be proved later on. Nevertheless, it appears that none has ever considered this fact of dispersion of the depth. In order to clarify the reason why the depth of compensation does disperse with respect to region, we must pay attention to the method by which the depth has been determined. The case of HELMERT being put aside, HAYFORD, BOWIE and the present author have made use of the principle of least squares, and the depths obtained by them are the most probable ones, that is, those which make minimum the sum of squared residuals. Then, what is meant by the most probable depth ?

In the following it will be proved that the most probable depth of compensation is not such a one as we may assume to be really existing, but the one which involves an error as affected by two factors, one being topography and the other an unknown anomalous structure, that is, an invisible geological structure inside the earth's crust below the area under consideration. The author reported this conclusion in 1946 and its further developments in 1949 and 1950 at the meetings of several geophysical societies in Japan.[6] In 1950 C. TSUBOI[7] wrote a paper in which he showed by example that the criterion $\sum$(isostatic anomaly)^2 = a minimum is sometimes misleading in determining the thickness of the isostatic earth's crust, proposing a substitute criterion by which a more reasonable solution is obtained. It is to be noticed that his conclusion concerning the criterion coincides with mine.

Assuming that the true depth of compensation, say T_*, is really

Table 1

Author	year	Region	Data	Depth of compensation
HAYFORD[1]	1909	U.S.A.	Plumbline deflection	113.0 km
HAYFORD[2]	1910	〃	〃	122.2
HELMERT[3]	1910	Oceanic coasts	Gravity	120 (mean value)
BOWIE[4]	1917	U.S.A. as a whole	〃	57.1
		East part	〃	62
		West part	〃	48
		Mountainous region	〃	104
		Mountainous region above general level	〃	120
KUMAGAI, N.[5]	1940	North-East Honsyû and Nippon Trench, Japan	〃	170

existing but unknown, then the gravity g at a point of H meters in height above sea-level may be computed as follows :

$$g=\gamma_o'-0.0003086\,H+B+P+C_*+A \quad \text{gal} \qquad \text{............(1),}$$

where γ_o'= the theoretical gravity on sea-level corrected for geoid undulation, B= attraction of a spherical shell placed between the gravity station under consideration and sea-level, with density equal to the normal continental rock, P= topographical correction to gravity in order to replace the outer surface of the shell with actual topography, C_*=vertical attraction of the masses due to those defect and excess of density below the topography which are obtained under the assumption of complete compensation and assumed as extending down to the true depth of compensation after Pratt's hypothesis, and A = vertical attraction of the masses due to the excess and defect of density possessed by invisible rocks constituting the crust against the density for complete compensation with the true depth. From (1) we get,

$$g+0.0003086\,H-B-P-\gamma_o'=C_*+A \qquad \text{............(2)}$$

Present state of the world distribution of gravity stations is by no means satisfactory to enable one by means of Stokes' theorem to calculate the accurate values of geoid undulations. If we assume, however, that γ_o' may be approximately replaced with the normal gravity usually denoted by γ_o, then the left side of the formula (2) is a quantity of which the value can be found and let it be denoted by $\delta g_o''$. Then we have,

$$\delta g_o''=C_*+A \qquad \text{............(3)}$$

The above equation shows that the known quantity $\delta g_o''$ is equal to the sum of the two unknowns C_* and A. $\delta g_o''$ is not the same with BOUGUER's anomaly usually denoted by $\Delta g_0''$, the latter being based upon attraction of an infinitely extending and horizontal plate of thickness H and not upon that of a spherical shell of the same thickness, upon which $\delta g_0''$ is based. However, when H is not large, $\delta g_0''$ is nearly equal to $\Delta g_0''$.

Now for a given region with known topography, let us assume a certain value t for the depth of compensation, and then the vertical attraction of the isostatic excess and defect of density uniformly distributed down to this depth (Hayfordian distribution) may be expressed by a function $C(t)$. C_* is then the value of this function for $t = T_*$. The difference $\delta g_0'' - C(t)$ is the residual for depth t, which is denoted by r, and accordingly we get,

$$r = C(T_*) - C(t) + A \qquad \text{...........(4).}$$

In the above, $C(T_*)$ and A are independent of t. The condition $\sum r^2 =$ a minimum is $\partial \sum r^2 / \partial t = 0$, and, denoting the most probable depth by T, we obtain,

$$\sum [C(T_*) - C(T) + A] \partial C / \partial T = 0 \qquad \text{...........(5),}$$

where $\partial C / \partial T$ means the value of $\partial C / \partial t$ for $t = T$ and the summation is taken for all available gravity stations in the region under consideration.
Putting

$$T_* = T + \tau \qquad \text{...........(6),}$$

and expanding $C(T+\tau)$ in TAYLOR's series in the vicinity of T, in which higher terms will be omitted, the following result will be finally obtained :

$$\tau = - \frac{\sum A \dfrac{\partial C}{\partial T}}{\sum \left(\dfrac{\partial C}{\partial T} \right)^2} \qquad \text{...........(7)}$$

In deriving the most probable depth, BOWIE[8] uses a special procedure in which for each of the assumed depths he firstly finds residuals based upon HELMERT's formula of 1901 ; secondly he calculats the mean of the residuals for all the 219 stations m U.S.A. ; and thirdly he subtracts this mean value from each residual. According to BOWIE this subtraction means a correction to the equatorial gravity of HELMERT's formula of 1901. The most probable depth he has adopted is that which makes minimum the sum of squares of residuals thus treated. Thus he has found the most probable depths for the five different regions in U.S.A. (Table I). After computing τ for the most probable depth derived by Bowie in such a way, the present author has found that the formula to calculate τ is also a function of A and $\partial C / \partial T$, but somewhat different from the formula (7). Therefore, the data used by BOWIE are now under investigations by the present method by which author has obtained the most probable depth, 170 km, for North-East Honsyû and Nippon Trench (Table I).

$C(T)$ depends upon the topography of the area under consideration.

A is, as mentioned above, the effect of an invlsibie geological structure below this area. Therefore, τ is influenced by both the topography and the invisible geological structure. Hence come the followign conclusions :

(1) If it is assumed that the true value of the depth of compensation is uniform everywhere below the earth's surface, then it is possible that the most probable depth derived from the condition $\sum r^2=a$ minimum may take a value different from the true depth, and this value may vary with change of the region where the depth has been found, because an invisible geological structure or a topographical feature or both may change as the region does. And therefore the fact of dispersion of the most probable depth with respect to the region implies that either or both of an invisible geological structure and a topographical feature of the region for which the depth has been determined changes as the region does.

(2) When considered from the view-point just mentioned, it is interesting to note the dispersion of the depths obtained by BOWIE and KUMAGAI as shown in Table I.

(3) If it is assumed on the other hand that the true depth is not uniform and varies with region, then the most probable depth may not vary in parallel with the change of the true depth.

(4) When in a given region gravity stations are newly added, the most probable depth derived from the total data including the new observations may become different from that derived from the old data, because the addition of the new observations may change the values of the numerator and denominator on the right side of the formula (7). Moreover, when the region is widened by new observations, the most probable depth obtained for the widened region may take, for the similar reason, a different value as compared with that for the original region.

(5) The most probable depth T having been found, the values of $\partial C/\partial T$ at every station can be computed by interpolation from the results of isostatic calculations carried out for various depths. However, A being unknown for all stations, the value of τ and therefore the value of the true depth cannot be found.

(6) In the following two cases, τ takes a value equal to zero. One is the case in which the earth's crust were at a complete compensation, that is, A vanished at all stations, and the other the case in which the invisible geological structure and the topography were such that the summation $\sum A\ \partial C/\partial T$ takes a zero value. In the former case, each residual and therefore the minimum value of $\sum r^2$ is necessarily equal to zero. The latter case can be investigated as follows : Distribution of the anomaly A and that of the topographical height h may be expressed in the series of surface spherical harmonics, such as

$$A=\sum A_n \quad \text{and} \quad h=\sum H_m \qquad \text{...........(8)}$$

n and m being degrees of the harmonics. If C_m is the vertical attraction of the isostatic excess and defect of density due to a harmonic topography H_m, then we obtain without difficulty,

$$\frac{\partial C_m}{\partial t} = -\frac{4\pi k\sigma}{a} f\left(m, \frac{t}{a}\right) H_m \qquad \text{............(9)}$$

in which k, σ, a and f are respectively the gravitational constant, the density of the surface of continent, the earth's radius and a definite function. Consequently, if the summations on the right side of the formula (7) be substituted by integrations, the numerator becomes equal to

$$\iint A \frac{\partial C}{\partial T} d\lambda d\mu = -\frac{4\pi k\sigma}{a} \sum_{n,m} f\left(m, \frac{T}{a}\right) \iint A_n H_m d\lambda d\mu \qquad \text{............(10),}$$

where λ is logitude and μ cosine of polar distance. The domain in which the harmonics on the rignt sides of the formulas (8) are valid is, of course, the limitted area under consideration. Assuming, however, the harmonics to be extrapolated outside this domain, then the integrations on both sides of the formula (10) may be taken on the whole earth's surface, and therefore when n is not equal to m, that is, there is no-common degree between the two series, the integral on the right and accordingly that on the left side of the formula become zero. Consequently, if the condition $n \neq m$ were satisfied, the most probable depth would become equal to the true depth.

In order to see what order of magnitude τ may take, the author adopted an example in which were assumed a simple topography of a plateau having its width and density equal to $4T_*$ and 2.67 respectively and a distribution of the anomaly A falling between -25 and $+50$ milligals and vanishing at great distances, the problem being two-dimensional (Fig. 1). Then, by putting roughly $\partial C/\partial T \doteqdot \partial C/\partial T_*$, it was found that the ratio of τ/T_*

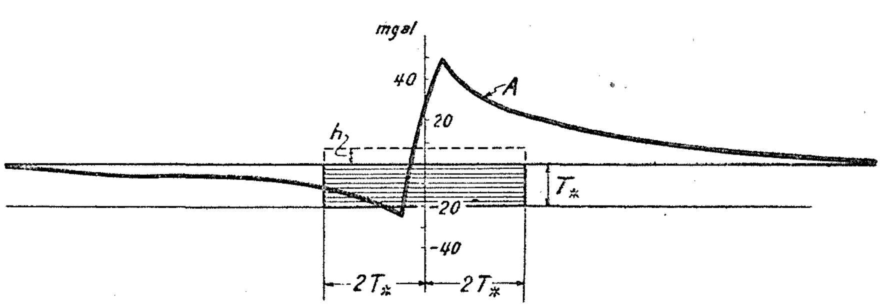

Fig. 1.

increased from 7.4 % to 74 %, as the height h of the plateau decreased from 2000 to 200 metres. Though this numerical result involves approximations, this example may show that the higher is the plateau, the nearer approaches the most probable depth towards the true one.

It is worth noticing that HELMERT's depth, 120 km, and HAYFORD's, 122.2 km, beautifully coincide with each other, despite the quite different materials used to derive these depths, and BOWIE's depth, 120 km, obtained in mountainous region above the general level practically coincides with the former two values ; and further the depth of 104 km obtained by BOWIE for mountainous region approximates these three depths. As to the enormously large depth, 170 km, found by KUMAGAI —— this value incited him to engage in the present research ——, it must be noticed that 1) the region where this depth has been obtained is of such a special topography including deep trench as none has ever approached, and 2) the residuals for this depth are very large, amounting at most to far more than 100 percent of the theoretical anomalies for complete compensation for this depth[9], suggesting large values of A.

Lastly, I will introduce another condition by which a depth of compensation may also be determined. This condition is $\sum r=0$. Let the depth derived from this condition be denoted by T' and put $T_{*}=T'+\tau'$, and then we get,

$$\tau' = -\frac{\sum A}{\sum \frac{\partial C}{\partial T}} \qquad \cdots\cdots\cdots(8)$$

This formula shows that τ' is also influenced by the invisible geological structure inside the earth's crust below and the topography of the region for which the depth of compensation is to be found.

The gravity anomaly the geologist desires to know is evidenly the anomaly A which is equal to $\delta g_0''-C_{*}$ from the formula (3). Therefore, in order to find this anomaly, we must know the value of the true depth T_{*}, bebause $C_{*}=C(T_{*})$. It is impossible, however, to execute this process, because for finding T_{*}, the values of A are required at all the stations, as stated in the above, and thus we confront a contradiction.

The reasoning stated avove can also be applied similarly to the case in which the thickness of the isostatic earth's crust is to be determined after Airy's hypothesis.

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