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Period-Two Trichotomies in Rational Equations

E. CAMOUZIS[1], G. LADAS[2,3], and E.P.QUINN[2]

[1]Department of Mathematics
The American College of Greece
6 Gravias Street Aghia Paraskevi,
15342 Athens, GREECE

[2]Department of Mathematics
University of Rhode Island
Kingston, Rhode Island, USA

ABSTRACT: We present some facts and pose several open problems and conjectures about period-two trichotomies in rational difference equations of the form

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \qquad n=0,1,\ldots$$

with nonnegative parameters and nonnegative initial conditions.

1. Introduction

We present some facts and pose several open problems and conjectures about period-two trichotomies in rational difference equations of the form

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots \tag{1.1}$$

with nonnegative parameters $\alpha,\beta,\gamma,\delta,A,B,C,D$, and nonnegative initial conditions x_{-2},x_{-1},x_0 such that the denominator is always positive. To avoid degenerate cases we will assume without further mention that

$$\alpha+\beta+\gamma+\delta,\ B+C+D\in(0,\infty).$$

By a period-two trichotomy result for Eq.(1.1), we mean a bifurcation result where at a certain value of a parameter all solutions converge to a period-two solution of the equation, and then for smaller values all solutions have limits, while for larger values there exist unbounded solutions.

[3]Corresponding author. email: gladas@math.uri.edu

For the Riccati equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n}, \quad n = 0, 1, \ldots$$

there are no periodic solutions unless

$$\beta = A = 0$$

in which case every solution is periodic with period 2.

The second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{1.2}$$

has been the subject of investigation in the Monograph [9]. For this equation, with $B + C > 0$, we have a period-two trichotomy result when

$$C = 0 \; and \; B > 0$$

and this result is summarized by the following theorem about the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n}, \quad n = 0, 1, \ldots \tag{1.3}$$

Theorem A (See [6],[7], and [9]). Assume that $B > 0$. Then the following period-two trichotomy result holds for Eq.(1.3):

(a) Every solution of Eq.(1.3) has a finite limit if and only if

$$\gamma < \beta + A.$$

(b) Every solution of Eq.(1.3) converges to a (not necessarily prime) period-two solution of Eq.(1.3) if and only if

$$\gamma = \beta + A.$$

(c) Eq.(1.3) has unbounded solutions if and only if

$$\gamma > \beta + A.$$

In addition to Theorem A the following two period-two trichotomy results have been established for the difference equations

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots \tag{1.4}$$

and

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n + Dx_{n-2}}, \quad n = 0, 1, \ldots \quad . \tag{1.5}$$

Theorem B (See [1],[3], and [8]). Assume that $\gamma + \delta + A > 0$. Then the following period-two trichotomy result holds for Eq.(1.4):

(a) Every solution of Eq.(1.4) has a finite limit if and only if

$$\gamma < \delta + A.$$

(b) Every solution of Eq.(1.4) converges to a (not necessarily prime) period-two solution of Eq.(1.4) if and only if

$$\gamma = \delta + A.$$

(c) Eq.(1.4) has unbounded solutions if and only if

$$\gamma > \delta + A.$$

Theorem C (See [4]). Assume that $\gamma + A + B > 0$. Then the following period-two trichotomy result holds for Eq(1.5):

(a) Every solution of Eq.(1.5) has a finite limit if and only if

$$\gamma < A.$$

(b) Every solution of Eq.(1.5) converges to a (not necessarily prime) period-two solution of Eq.(1.5) if and only if

$$\gamma = A.$$

(c) Eq.(1.5) has unbounded solutions if and only if

$$\gamma > A.$$

No other period-two trichotomy result is known at this time for Eq.(1) or any special cases of it.

Are there other period-two trichotomy results which are true for any special cases of Eq.(1)?

One can show that for Eq.(1) to have a period-two trichotomy it is neccessary that

$$C = 0 \quad \textit{and} \quad \gamma > 0$$

in which case the equation reduces to

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Dx_{n-2}}, \quad n = 0, 1, \dots . \tag{1.6}$$

Furthermore Eq.(1.6) has prime period-two solutions if and only if

$$\gamma = \beta + \delta + A.$$

Computer observations and analytic investigation suggest that if Eq.(1.6) has a period-two trichotomy it should be as follows with some of the parameters possibly equal to zero or restricted appropriately.

(a) Every solution of Eq.(1.6) converges to a finite limit if and only if

$$\gamma < \beta + \delta + A.$$

(b) Every solution of Eq.(1.6) converges to a (not necessarily prime) period-two solution if and only if

$$\gamma = \beta + \delta + A.$$

(c) Eq.(1.6) has unbounded solutions if and only if

$$\gamma > \beta + \delta + A.$$

Open Problem 1. Find special cases of Eq.(1.6) such that the above three statements (a), (b), and (c) are all true.

Conjecture 1. Show that the special case of Eq.(1.6) given by

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + x_{n-2}}{1 + x_n} \quad , n = 0, 1, \dots \tag{1.7}$$

with $\beta, \gamma \in (0, \infty)$ has a period-two trichotomy character. More precisely, show that the following three statements are true.

(a) Every solution of Eq.(1.7) has a finite limit if and only if

$$\gamma < \beta + 2 \quad .$$

(b) Every solution of Eq.(1.7) converges to a (not necessarily prime) period-two solution if and only if

$$\gamma = \beta + 2 \quad .$$

(c) Eq.(1.7) has unbounded solutions if and only if

$$\gamma > \beta + 2 \quad .$$

Conjecture 2. Show that a special case of Eq.(1.6) with a unique positive equilibrium $\overline{x}$ has a period-two trichotomy character if and only if the following two statements hold:

(i) $\overline{x}$ is locally asymptotically stable when

$$\gamma < \beta + \delta + A.$$

(ii) Every solution of Eq.(1.6) is bounded when

$$\gamma = \beta + \delta + A.$$

Conjecture 3. Show that the solutions of Eq.(1.6) have the following character:

(a) Eq.(1.6) has unbounded solutions when

$$\gamma > \beta + \delta + A.$$

(b) Every solution of Eq.(1.6) is bounded when

$$\gamma < \beta + \delta + A.$$

It is known that some special cases of Eq.(1.6) have unbounded solutions when

$$\gamma = \beta + \delta + A.$$

For example, one can show that the solutions of the equation

$$x_{n+1} = \frac{\alpha + x_{n-1} + x_{n-2}}{x_n}, \quad n = 0, 1, \ldots \tag{1.8}$$

with $\alpha \geq 0$ and with initial conditions x_{-2}, x_{-1}, x_0 such that

$$x_0 = x_{-2} \leq 1$$

are unbounded. See [2] and [5]. Indeed, it follows from Eq.(1.8) that

$$x_{n+2} - x_n = \frac{1}{x_{n+1}} \left(x_n - x_{n-2} \right), \quad n \geq 0.$$

Therefore, in this case,

$$x_{2n} = x_0, \quad n \geq 0$$

and so from Eq.(1.8) we see that

$$x_{2n+1} = \frac{\alpha + x_0}{x_0} + \frac{1}{x_0} x_{2n-1} \to \infty \quad as\ n \to \infty.$$

Open Problem 2. Obtain necessary and sufficient conditions on α, β, γ, δ, A, B, and D so that every solution of Eq.(1.6) is bounded when

$$\gamma = \beta + \delta + A.$$

2. Special Cases Remaining to be Investigated

On the basis of the above discussion one can see that there remain 28 special cases of Eq.(1.6) with positive parameters to be investigated for possible period-two convergence, for existence of unbounded solutions, and for conditions under which the equilibrium is globally asymptotically stable.

After a change of variables of the form

$$x_n = \lambda y_n$$

these twenty eight equations may be reparameterized to be as follows with all their parameters positive:

$$y_{n+1} = \frac{\alpha + y_n + \gamma y_{n-1} + \delta y_{n-2}}{y_n}, \qquad n = 0, 1, \ldots \qquad (2.1)$$

$$y_{n+1} = \frac{\alpha + \beta y_n + \gamma y_{n-1} + y_{n-2}}{y_{n-2}}, \qquad n = 0, 1, \ldots \qquad (2.2)$$

$$y_{n+1} = \frac{\alpha + y_n + \gamma y_{n-1} + \delta y_{n-2}}{A + y_n}, \qquad n = 0, 1, \ldots \qquad (2.3)$$

$$y_{n+1} = \frac{\alpha + \beta y_n + \gamma y_{n-1} + y_{n-2}}{A + y_{n-2}}, \qquad n = 0, 1, \ldots \qquad (2.4)$$

$$y_{n+1} = \frac{\alpha + \beta y_n + \gamma y_{n-1} + y_{n-2}}{B y_n + y_{n-2}}, \qquad n = 0, 1, \ldots \qquad (2.5)$$

$$y_{n+1} = \frac{\alpha + \beta y_n + \gamma y_{n-1} + y_{n-2}}{A + B y_n + y_{n-2}}, \qquad n = 0, 1, \ldots \qquad (2.6)$$

$$y_{n+1} = \frac{1 + \beta y_n + \gamma y_{n-1}}{y_{n-2}}, \qquad n = 0, 1, \ldots \qquad (2.7)$$

$$y_{n+1} = \frac{\alpha + \beta y_n + \gamma y_{n-1}}{1 + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.8}$$

$$y_{n+1} = \frac{\alpha + y_n + \gamma y_{n-1}}{y_n + D y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.9}$$

$$y_{n+1} = \frac{\alpha + y_n + \gamma y_{n-1}}{A + y_n + D y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.10}$$

$$y_{n+1} = \frac{1 + \gamma y_{n-1} + \delta y_{n-2}}{y_n}, \qquad n = 0, 1, \ldots \tag{2.11}$$

$$y_{n+1} = \frac{\alpha + \gamma y_{n-1} + \delta y_{n-2}}{1 + y_n}, \qquad n = 0, 1, \ldots \tag{2.12}$$

$$y_{n+1} = \frac{\alpha + \gamma y_{n-1} + y_{n-2}}{B y_n + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.13}$$

$$y_{n+1} = \frac{\alpha + \gamma y_{n-1} + y_{n-2}}{A + B y_n + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.14}$$

$$y_{n+1} = \frac{y_n + \gamma y_{n-1} + \delta y_{n-2}}{y_n}, \qquad n = 0, 1, \ldots \tag{2.15}$$

$$y_{n+1} = \frac{\beta y_n + \gamma y_{n-1} + y_{n-2}}{y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.16}$$

$$y_{n+1} = \frac{y_n + \gamma y_{n-1} + \delta y_{n-2}}{A + y_n}, \qquad n = 0, 1, \ldots \tag{2.17}$$

$$y_{n+1} = \frac{\beta y_n + \gamma y_{n-1} + y_{n-2}}{A + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.18}$$

$$y_{n+1} = \frac{\beta y_n + \gamma y_{n-1} + y_{n-2}}{B y_n + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.19}$$

$$y_{n+1} = \frac{\beta y_n + \gamma y_{n-1} + y_{n-2}}{A + By_n + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.20}$$

$$y_{n+1} = \frac{y_n + \gamma y_{n-1}}{y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.21}$$

$$y_{n+1} = \frac{\beta y_n + \gamma y_{n-1}}{1 + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.22}$$

$$y_{n+1} = \frac{y_n + \gamma y_{n-1}}{y_n + Dy_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.23}$$

$$y_{n+1} = \frac{y_n + \gamma y_{n-1}}{A + y_n + Dy_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.24}$$

$$y_{n+1} = \frac{\gamma y_{n-1} + y_{n-2}}{y_n}, \qquad n = 0, 1, \ldots \tag{2.25}$$

$$y_{n+1} = \frac{\gamma y_{n-1} + \delta y_{n-2}}{1 + y_n}, \qquad n = 0, 1, \ldots \tag{2.26}$$

$$y_{n+1} = \frac{\gamma y_{n-1} + y_{n-2}}{By_n + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.27}$$

$$y_{n+1} = \frac{\gamma y_{n-1} + y_{n-2}}{A + By_n + y_{n-2}}, \qquad n = 0, 1, \ldots \tag{2.28}$$

The following open problem actually contains 28 problems, each of which is quite a challenge.

Open Problem 3. Investigate each of the above twenty eight equations for boundedness, global stability, and period-two convergence.

3. Period-Two Convergence When $\gamma = \beta + \delta + A$

The characteristic polynomial $P(\lambda)$ of the linearized equation associated with Eq.(1.6) about its equilibrium evaluated at $\lambda = -1$ is

$$P(-1) = \frac{2(\gamma - A - \beta - \delta)}{A + \beta + \gamma + \delta + \sqrt{4\alpha(B+1) + (\beta + \gamma + \delta - A)^2}}. \qquad (3.1)$$

Therefore $\lambda = -1$ is a root of the characteristic equation if and only if $\gamma = \beta + \delta + A$. For the twenty eight third-order equations, Eqs.(2.1) through Eq.(2.28), it can be shown that when $\gamma = \beta + \delta + A$ the two remaining characteristic roots lie on the unit circle or inside the unit disk. Furthermore, for all except Eqs.(2.7) and (2.21) they lie inside the unit disk.

What is it that makes Eqs.(2.7) and (2.21) different from the rest of Eqs.(2.1)-(2.28)? No doubt this is due to the nature of the characteristic roots of their linearized equations.

For Eqs.(2.7) and (2.21) the condition

$$\gamma = \beta + \delta + A \qquad (3.2)$$

does not imply period-two convergence.

Also, for Eqs.(2.11) and (2.25) we do not have period-two convergence because as we saw, Eq.(1.8) has unbounded solutions.

For Eqs.(2.11) and (2.25) the characteristic roots of their corresponding linearized equations are all real numbers with the dominant characteristic root equal to -1, which is a second root of 1.

What caused the unboundedness of solutions of Eqs.(2.11) and (2.25)? Could it have been detected from the linearized equations about their positive equilibrium points?

A question of paramount importance for rational equations is to understand the extent to which the characteristic roots of the linearized equation about the equilibrium determine the periodic convergence of the equations, their boundedness character, or their global asymptotic stability.

Open Problem 4. Does the equation

$$x_{n+1} = \frac{x_{n-1} + x_{n-2}}{x_n + x_{n-2}} \quad , n = 0, 1, \ldots$$

have unbounded solutions?

Conjecture 4. Assume that (3.2) holds. Show that every **bounded** solution of Eq.(1.6) converges to a period-two solution when $\lambda = -1$ is the only characteristic root of the linearized equation on the unit circle.

Conjecture 5. Assume that the linearized equation about the equilibrium point has two characteristic roots which are complex conjugate and inside the unit disk. Show that every solution of Eq.(1.6) converges to a (not necessarily prime) period-two solution of Eq.(1.6) if and only if the third characteristic root is equal to -1, that is, if and only if (3.2) holds.

One can see that when $\gamma = \delta + 1$ the solutions of Eq.(2.1) satisfy the identity

$$y_{n+2} - y_n = \frac{1}{y_{n+1}} \left(y_{n+1} - y_{n-1} \right) + \frac{\delta}{y_{n+1}} \left(y_n - y_{n-2} \right), \quad n \geq 0.$$

From this if follows that the subsequences of even and odd terms of the solution of (2.1) are eventually monotonic and **bounded**. Therefore when $\gamma = \delta + 1$ every solution of Eq.(2.1) converges to a period-two solution.

What is it that makes Eqs.(2.1) and (1.8) different in their boundedness and period-two convergence character?

When (3.2) holds, -1 is a root of the characteristic equation of the linearized equation of Eq.(1.6) about the equilibrium of the equation. Table 1. below gives information about the nature of the other two roots of the linearized equation.

Equation	$\|\lambda_i\| < 1$	Real	Equation	$\|\lambda_i\| < 1$	Real
1	always	always	15	always	always
2	always	never	16	always	never
3	always	always	17	always	always
4	always	sometimes	18	always	sometimes
5	always	sometimes	19	always	sometimes
6	always	sometimes	20	always	sometimes
7	never	never	21	never	never
8	always	sometimes	22	always	sometimes
9	always	sometimes	23	always	sometimes
10	always	sometimes	24	always	sometimes
11	always	always	25	always	always
12	always	always	26	always	sometimes
13	always	sometimes	27	always	sometimes
14	always	sometimes	28	always	sometimes

Table 1.

4. Existence of Unbounded Solutions

It is clear from (3.1) that the characteristic polynomial $P(\lambda)$ of the linearized equation associated with Eq.(1.6) evaluated at $\lambda = -1$ is positive when $\gamma > \beta + \delta + A$, so in this case $P(\lambda)$ must have a real root less than -1 and therefore outside the unit disk.

We have conjectured that Eq.(1.6) has unbounded solutions when

$$\gamma > \beta + \delta + A.$$

The problem of the boundedness character of solutions of Eq.(1.1) is extremely difficult and only very few of the 225 possible special cases of Eq.(1.1) have been investigated so far. On the other hand of the 49 special cases of the second order rational equation (1.2) with positive parameters, the question of boundedness has been resolved for all but the following two equations:

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, \ldots \tag{4.1}$$

and

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{B x_n + x_{n-1}}, \quad n = 0, 1, \ldots \quad . \tag{4.2}$$

For these two solutions we offer the following conjectures.

Conjecture 6. Every positive solution of Eq.(4.1) is bounded.

Conjecture 7. Every positive solution of Eq.(4.2) is bounded.

When $B = C = 0$, Eq.(1.2) reduces to a linear equation and its boundedness is easy to describe. For the nonlinear case $B + C > 0$ we pose the following conjecture.

Conjecture 8. Every solution of Eq.(1.2) is bounded if and only if either $C > 0$ or

$$B > 0, \quad C = 0, \quad \textit{and} \quad \gamma \leq \beta + A.$$

Contrary to Eq.(1.2), when $C > 0$, it is not true that every solution of Eq.(1.1) is bounded. In fact we offer the following conjecture.

Conjecture 9. Every solution of each of the following equations

$$x_{n+1} = \frac{px_n + x_{n-2}}{x_{n-1}}, \quad n = 0, 1, \ldots$$

and

$$x_{n+1} = \frac{p + x_n}{x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots$$

is bounded if and only if $p \geq 1$.

Conjecture 10. Assume $\gamma > \delta + 1$. Show that for every unbounded solution of Eq.(2.1) the subsequence of its even terms and the subsequence of its odd terms converge, one of them to ∞ and the other to

$$\frac{\gamma}{\gamma - \delta}.$$

Conjecture 11. Assume $\gamma > \delta$. Show that for every unbounded solution of Eqs.(2.11) and (2.25) the subsequence of its even terms and the subsequence of its odd terms converge, one of them to ∞ and the other to zero. Note that in Eq.(2.25), $\delta = 1$.

Open Problem 5. Does the equation

$$x_{n+1} = \frac{\gamma x_{n-1} + x_{n-2}}{x_n + x_{n-2}} \quad , n = 0, 1, \ldots$$

with $\gamma > 1$ have a solution $\{x_n\}_{n=-2}^{\infty}$ such that

$$\lim_{n\to\infty} x_{2n} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{2n+1} = \infty?$$

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