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A Dynamic Analysis of an Economy with a Zero Interest Rate Bound*

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Abstract

For more than a decade the Japanese economy has been in the serious slump with very low interest rates and low price inflation or even deflation. The traditional IS-LM models analyze this kind of situation as a special case of macroeconomic phenomena known as a "liquidity trap" assuming constant price levels. This paper tries to explain the dynamics of inflation and interest rates incorporating the Phillips curve into an IS-LM model. If the economy is around the steady state with not-low nominal interest rates, it may converge to its steady state or exhibit the cyclical behavior of inflation and interest rates. In contrast, if its nominal interest rates are close to zero, the economy's behavior becomes very unstable. The economy may fall into a so-called deflationary spiral. The effects of fiscal and monetary policies are also examined in the economy with deflation and low interest rates.

1. Introduction

Most recent macroeconomic models assume economic agents' intertemporal maximizing behavior: utility-maximization by households, and profit- or value-maximization by producers. Building models based upon rigorous and explicit microfoundations has of course various merits. For example, one can examine the economic effects of tax policies more precisely because a change in the tax system affects the behavior of each agent at the micro level. As Akerlof (2002) points out, however, those models cannot explain such important macroeconomic phenomena as (1) the existence of involuntary unemployment, (2) the impact of monetary policy on output and employment, (3) the failure of deflation to accelerate when

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unemployment is high.

As is well-known, the Japanese economy has been in the serious slump with the above phenomena for more than a decade. According to Akerlof's insight, one may not employ the macroeconomic model based upon maximizing behavior to explain the recent Japanese economy. Instead of intertemporal maximizing behavior and perfect-foresight assumptions, this paper employs behavioral assumptions charactering traditional IS-LM models and naïve (or adaptive) expectations to analyze the recent Japanese economy.

The recent Japanese economy is also characterized by very-low nominal interest rates and low inflation or even deflation, as Figure-1 shows. This kind of situation is treated as a special case as a "liquidity trap" in the IS-LM models. But, one cannot analyze the dynamics of interest rates and inflation in a traditional IS-LM model since it does not allow price levels or inflation rates to change. The assumption of constant prices (or inflation rates) entails the other analytical flaw that one cannot distinguish real interest rates from nominal interest rates.

In order to consider inflation or deflation in an IS-LM framework, we need an equation that relates the rate of inflation to the level of unemployment or GDP. As "the single most important macroeconomic relationship is the Phillips curve" (Akerlof 2002, p. 418), nesting the Phillips curve into a simple IS-LM model, we will examine the dynamics of the expected inflation rates and nominal interest rates.

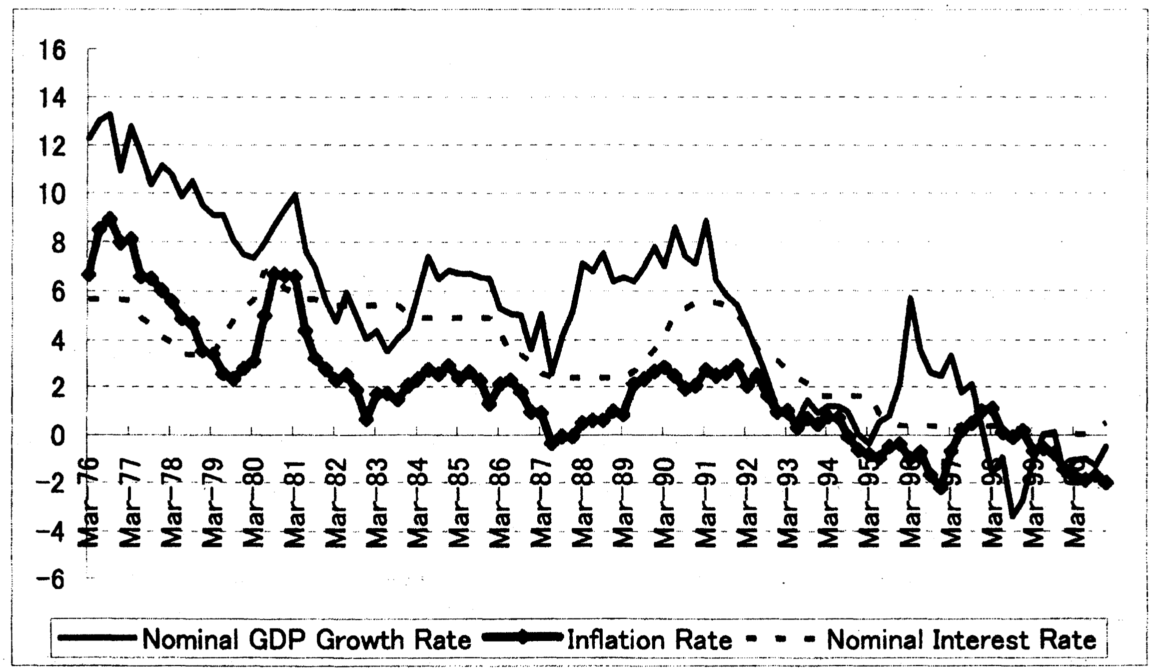

Figure-1 Recent Movements of GDP, Inflation Rates, and Nominal Interest Rates in Japan

There is the growing literature that focuses on the issues we address in this paper based upon rigorous and explicit microfoundations. For example, Benhabib, Schmitt-Grohe and Uribe (2002), and Buiter and Panigirtzoglou (2003), capturing the interactions between forward-looking prices and the agents' intertemporal maximizing behavior, discuss the possibility of liquidity trap and the economic policies to avoid or to escape from the trap. Although their approaches are totally different from ours, we do not deny the importance of the analyses, which give many very important economic insights. The main aim of this paper is to examine whether a simple IS-LM model can explain important macroeconomic phenomena, and to show that it is still useful for policy debates.

This paper is structured as follows. Section 2 sets up the basic model, derives the stationary states, and analyzes the dynamic properties. Section 3 extends the model to examine the economic policy implications. Section 4 concludes the paper.

2. The Basic Model

2-1. The Setup

The model to be presented here is nothing more than a collection of economic behavioral equations provided in such an undergraduate textbook as Mankiw (2002). Let us begin with an IS-LM model as follows:[1]

$$Y = C(Y-T) + I(r) + G = C(Y-T) + I(i-\pi^e) + G, \tag{1}$$

$$M/P = L(i, Y), \tag{2}$$

where Y is income, T is taxes, r is real interest rate, G is government purchases, i is nominal interest rate, M is money supply, and P is price level. Eq. (1) is an IS equation while (2) is an LM equation. According to the textbook, if P and π^e, in addition to the policy varibles, are given, one can find the equilibrium income and nominal or real interest rate.

Solving equation (1) for Y gives us

$$Y = F(i-\pi^e, A),$$

where A is the vector of exogenous variables including such policy variables as T, G and M. To simply the analysis, we log-linearize the above equation as below:

$$y = e_a a - e_r r = e_a a - e_r (i - \pi^e) \tag{3}$$

[1] The model is based on Adachi (1993).

where $y = \ln Y$ and $a = \ln A$, e_a and e_r are positive constants.

For the LM equation, it is assumed to take the following form:

$$M/P = k(i)Y,$$

where $k(i)$ is the Marshallian k with $k'(i) < 0$. Taking the log of both sides of the above equation and differentiating it with respect to time, we obtain

$$\mu - \pi = (k'(i)/k(i))\dot{i} + \dot{y} \tag{4}$$

where $\mu \equiv \dot{M}/M$, $\pi \equiv \dot{P}/P$. We assume a liquidity trap, i.e., there is some non-nagative rate such that, if the nominal intrest rate approches it, then the liquidity preference becomes infinite. Since the nominal interest rate cannot be nagative, let us assume the following:

$$\lim_{i \to 0}(-k'(i)/k(i)) = \infty, \quad \text{and} \quad \frac{d}{di}(-k'(i)/k(i)) > 0 \quad \text{around} \quad i = 0. \tag{5}$$

We employ two important behaviorial eguations in macroeconomics in determining the rate of inflation π^e: Phillips curve and Okun's law. The price-price Phillips curve relates the gap between π^e and π to the unemployment rate $u(t)$:

$$\pi(t) = \pi^e(t) - f(u(t)) \qquad \text{or} \qquad \pi(t) - \pi^e(t) = -f(u(t)), \quad f'(u(t)) > 0.$$

Linearizing the above around the NAIRU or the natural rate of unemployment u_n, we have

$$\pi(t) - \pi^e(t) = -\overline{\alpha}(u(t) - u_n), \quad \overline{\alpha} = f'(u_n) > 0. \tag{6}$$

On the other hand, the Okun's law shows the important empirial relationship between a change in unemployment rate and a change in income, which is expressed in descrete time:

$$u_{t+1} - u_t = -\overline{\beta}\left(\frac{Y_{t+1} - Y_t}{Y_t} - \frac{Y_{p.t+1} - Y_{p.t}}{Y_{p.t}}\right),$$

and in continuous time

$$\dot{u}(t) = -\overline{\beta}(\dot{Y}(t)/Y(t) - \dot{Y}_P(t)/Y_P(t)),$$

where $\overline{\beta} > 0$, and Y_P is the potential rate of output. Integaration of the above gives

$$u(t) = -\overline{\beta}(y(t) - y_p(t)), \tag{7}$$

where $y_p(t) \equiv \ln Y_P(t)$. From (6) and (7), we have

$$\pi(t)-\pi^e(t)=\alpha[y(t)-y_n(t)], \tag{8}$$

where $\alpha = \overline{\alpha}\overline{\beta} > 0$, and $y_n = y_p(t) - (1/\overline{\beta})u_n$ is the natural rate of output. From now on, we assume that y_n is constant because our analysis is limited to the short-run or medium-run.

To close the model, we need the equation that determines the expected rate of inflation. Here, we assume a naïve or adaptive expectation:

$$\dot{\pi}^e = \gamma(\pi - \pi^e) \tag{9}$$

where γ is a positive constant.

The system consists of four equations-(3), (4), (8) and (9)- and four unknowns- y, i, π, and π^e. From an easy manupulation, one can obtain the simplied system of two differential equations:

$$\dot{i} = \phi(i)\{\alpha(\gamma e_r + 1)[e_a a - e_r(i-\pi^e) - y_n] - (\mu - \pi^e)\} \tag{10}$$

$$\dot{\pi}^e = \alpha\gamma[e_a a - e_r(i-\pi^e) - y_n] \tag{11}$$

where $\phi(i) = [e_r - (k'(i)/k(i))]^{-1} > 0$ with $\lim_{i\to 0}\phi(i) = 0$ and $0 \le \phi(i) < 1/e_r$. Equation (11) is linear while equation (10) is non-linear. But, we should note that $\phi(i)$ is an only non-linear term, which comes from the money demand function. In other words, the LM fuction plays a key role in our model.[2]

2-2. Stationary States and Stability

As Figure-2 shows, the system has two stationary states: one with inflation E_I and the other with deflation E_D. At E_I the nominal interest rate i and the expected rate of inflation π^e, which is equal to the actual rate of inflation π, is determined by

$$\mu = \pi^{e*} \qquad \text{and} \qquad e_a a - e_r(i-\pi^e) = y_n,$$

At E_D, on the other hand, the following relationships holds:

$$i = 0 \qquad \text{and} \qquad \pi^{e*} = -r_n < \mu,$$

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[2] In contrast to our model, Romer (2000) develops the Keynesian economics without the LM function.

where $r_n = (e_a a - y_n)/e_r$ is the real interest rate consistent with the natural rate of output y_n.

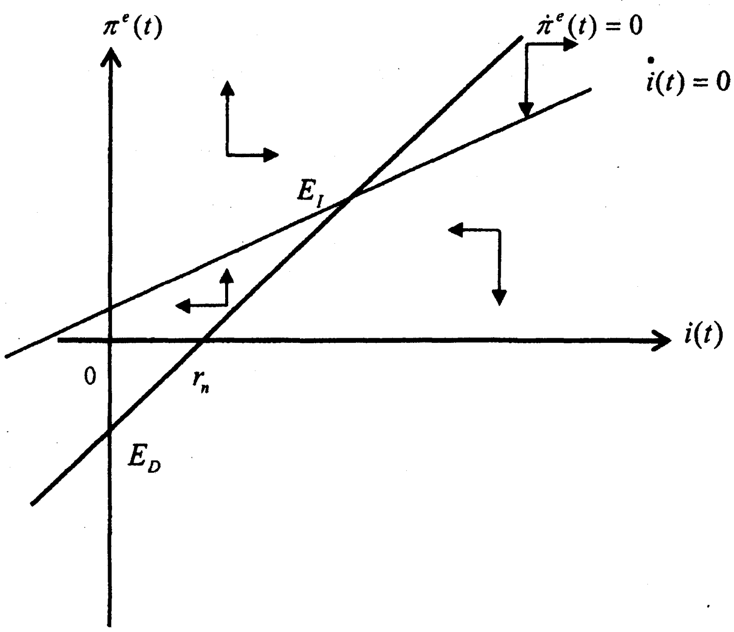

Figure-2: Two Stationary States

Linearizing the system in the neighborhood of E_I, we have the following Jacobian matrix:

$$J_I \equiv \begin{bmatrix} -\phi(i_I)\alpha e_r(\gamma e_r + 1) & \phi(i_I)[\alpha e_r(\gamma e_r + 1) + 1] \\ -\alpha\gamma e_r & \alpha\gamma e_r \end{bmatrix},$$

and hence its trace and determinant are

$$trJ_I = \alpha e_r[\gamma - \phi(i_I)(\gamma e_r + 1)] \qquad \text{and} \qquad \det J_I = \phi(i_I)\alpha\gamma e_r > 0.$$

Since the determinant is always positive, in order for the system to be asymptotically stable around E_I, the trace must be negative, which is equivalent to

$$\gamma < k(i_I)/k'(i_I),$$

where i_I is the interest rate at E_I. Therefore, the stationary state with the high nominal interest rate is asymptotically stable (a) if the adjustment speed γ of expected inflation rate (π^e) with respect to actual inflation rate (π) is sufficiently low, or (b) if the elasticity of money demand with respect to the nominal interest rate ($i_I k'(i_I)/k(i_I)$) is sufficiently small, or both. One can also establish the following proposition.

Proposition 1.

Hopf-Bifurcation occurs at $\gamma = \gamma_0$, *and hence there exist some non-constant periodic solutions of the system at some parameter values,* $\gamma \in (0, \infty)$ *are sufficiently close to* γ_0.

Proof:

The characteristic equation of the Jacobian, $\lambda^2 - (traceJ_l)\lambda + (\det J_l) = 0$, has a pair of pure imaginary roots when $traceJ_l = 0$ and $\det J_l > 0$. The latter $\det J_l > 0$ always holds. Suppose that γ_0 is the critical value of parameter γ such that $\gamma_0 = -k(i_l)/k'(i_l)$. Then, $traceJ = 0$ when $\gamma = \gamma_0$, so that the equation has a pair of pure imaginary roots. In addition, if the characteristic roots $\lambda(\gamma)$ are imaginary, then the real part: $\operatorname{Re}\lambda(\gamma)$ = $traceJ/2$. Hence, $\left.\dfrac{d(\operatorname{Re}\lambda(\gamma)}{d\gamma}\right|_{\gamma=\gamma_0} = \dfrac{\alpha e_r(1-\phi(i_l)e_r)}{2}$.

Since $trJ_l = \alpha e_r[\gamma - \phi(i_l)(\gamma e_r + 1)] = 0$ when $\gamma = \gamma_0$, $d(\operatorname{Re}\lambda(\gamma))/d\gamma|_{\gamma=\gamma_0} = \alpha e_r \phi(i_l)/2 > 0$, which means that all conditions of the Hopf-Bifurcation theorem are satisfied. (Q.E.D.)

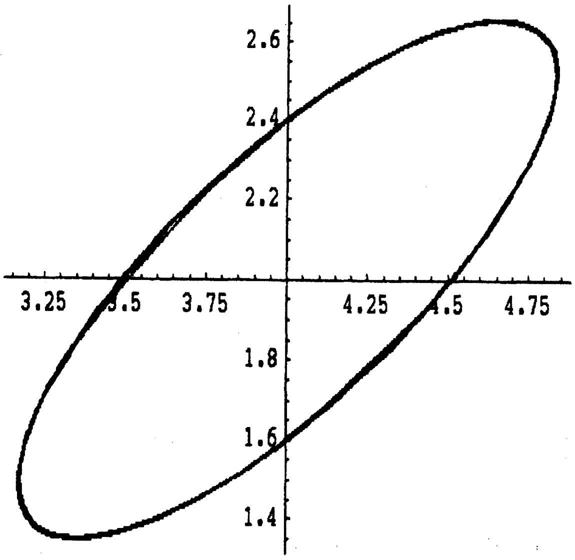

Figure-3(a) The Limit Cycle when $\gamma = 0.123$

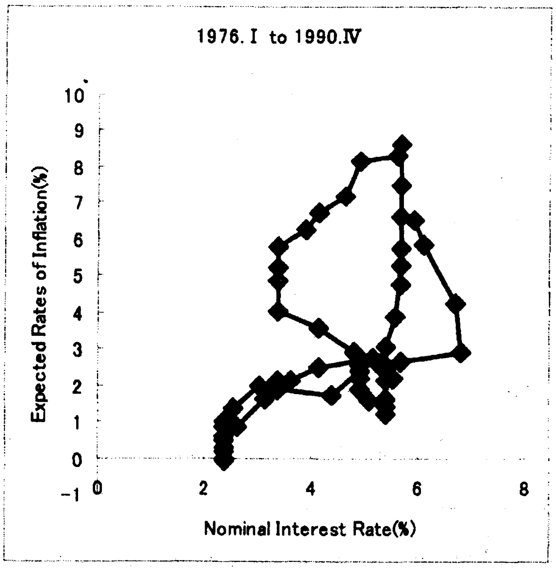

Figure-3 (b) Interest Rates and Expected Inflation Rates between 1976 and 1990

Let specify the function $k(i)$ so that the elasticity of money demand with respect to the nominal interest rate $c = ik'(i)/k(i)$ is constant or $k(i) = ai^{-c}$, where a and c are positive constants. At a certain value of γ, there exist the limit cycle as Figure-3(a) shows.[3] Our observations are consistent with the movements of nominal interest rates and expected rates of inflation in the period of late 1970's to 1980's in Japan, as Figure-3(b) shows.

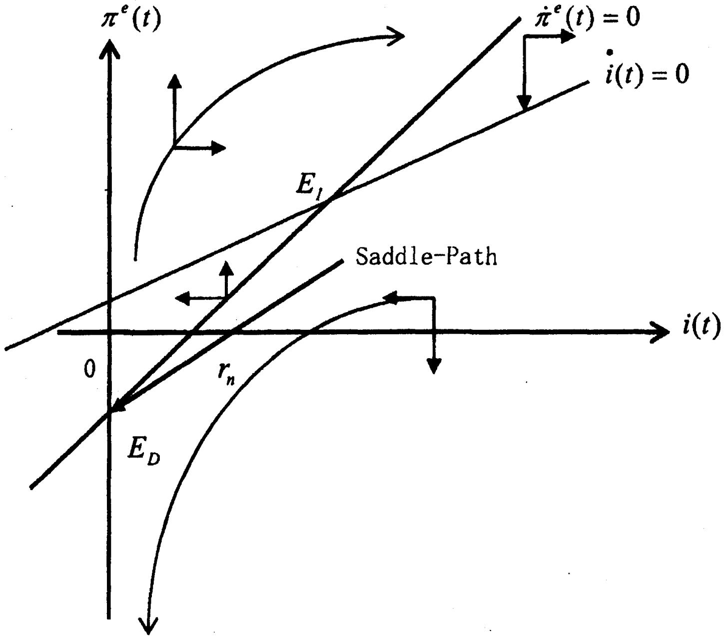

Figure-4: Saddle-Path around E_D

Linearizing the system in the neighborhood of E_D, we have the following Jacobian matrix:

$$J_D \equiv \begin{bmatrix} -\phi'(0)(\mu - \pi^e) & 0 \\ -\alpha\gamma e_r & \alpha\gamma e_r \end{bmatrix},$$

and hence its trace and determinant are:

$$trJ_D = \phi'(0)(\mu - \pi^e) + \alpha\gamma e_r \quad \text{and} \quad \det J_D = -\alpha\gamma e_r \phi'(0)(\mu - \pi^e).$$

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[3] In the following numerical simulations, all the parameters other than γ are adjusted so that the steady state is at $(i, \pi) = (2,4)$.

It is evident from (5) that $\det J_D < 0$. Therefore the associated characteristic equation has a pair of real roots which have different signs. As is shown in Figure-4, the stationary state with deflation E_D is a saddle-point. If the economy is in the region below the saddle-point path, it will fall into a deflationary spiral, which is characterized by deflation and a continued decline in income.

Proposition 2

Once the economy enters the region below the saddle-point path, it will fall into the deflationary spirals.

Proof. See Figure-4.

A Figure-5(a) shows the The results derived in the above may help understanding of the behavior of nominal interest rates and expected rates of inflation in 1990's in Japan as Figure-5(b) shows.

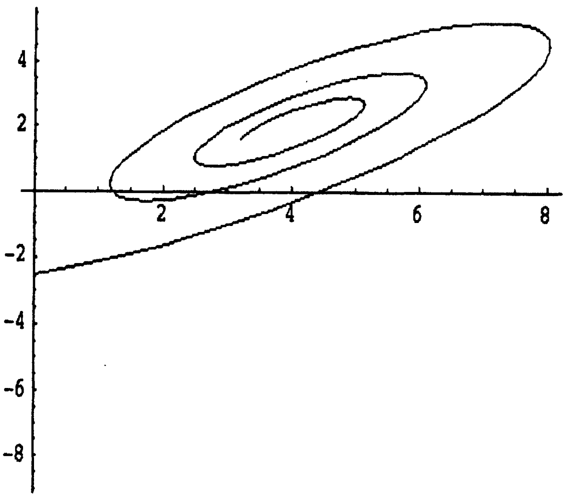


Figure-5(a) Simulation Result when $\gamma = 0.15$

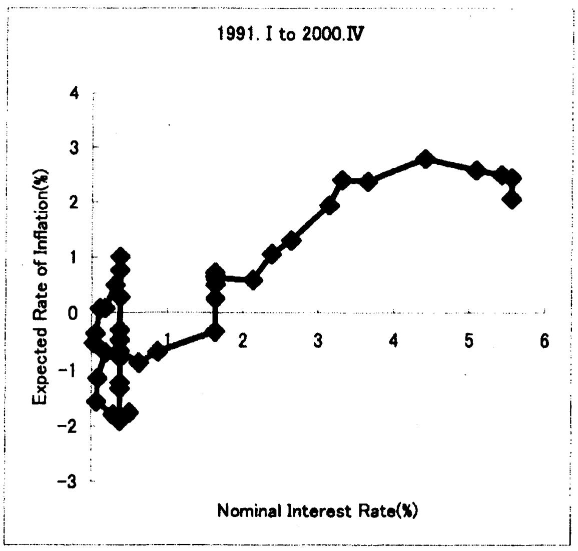
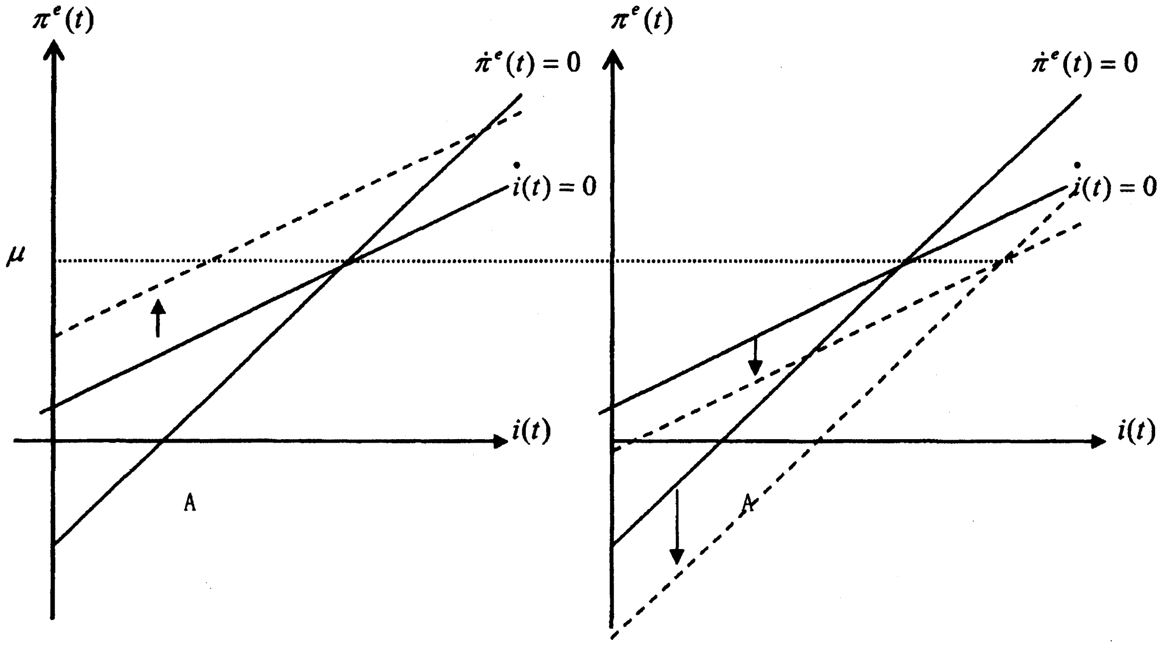


Figure-3 (b) Interest Rates and Expected Inflation Rates between 1991 and 2000

2-3. The Effects of Economic Policies

In this section, we will examine the effects of fiscal and monetary policies. If the economy is around the stationary state with inflation, i.e. E_I, and this stationary state is stable, then the policy effects are the same as found in the textbook. In contrast, the economy is around the stationary state with deflation, i.e. E_D, it will not converge to neither E_D nor E_I. Hence, one cannot appeal to the usual comparative statics method for evaluating the policy effects in this case. Instead, we will examine the possibility that the policies are able to pull the economy out of the deflationary spiral if the economy fell into the spiral without them.



(a) Expansionary Monetary Policy (b) Expansionary Fiscal Policy

Figure-6 Effects of Economic Policies

If the economy is at point A in Figure-6, it will probably go into the deflationary spirals. An expansionary monetary policy shifts up $\dot{i}(t)=0$ locus, but does not change the law of motions around point A, as Figure-6(a) shows. Hence, the policy is not effective in the sense that it cannot salvage the economy out of the deflation.

In contrast, a fiscal policy shifts down $\dot{\pi}^e(t)=0$ locus as well as $\dot{i}(t)=0$ locus. As Figure-6(b), the law of motions could be changed due to this policy. Hence, the policy is effective in that it can pull the economy out of the deflation. However, it should be noted that the economy is very unstable near-zero-interest area and it may suddenly moves into the inflation from the deflation.

3. Two Extensions

So far we have assumed a very simple naïve expectation and simple economic policies. In next subsection, we will modify the expectation formula according to Malinvaud (2000). In subsection 3-2, we will briefly discuss the effectiveness of inflation target policy using the basic model.

3-1. Normal Long-Run Expectations

Our basic model assumes the naïve (or adaptive) expectations. They are of course too simple. In fact, according to Nakayama and Ooshima (1999), 50 to 60 % of Japanese firms have adaptive expectations over inflation and 30 to 40 % of households's inflation expectations are adaptive. Here, according to Malinvaud (2000), θ portion of economic agents form adaptive expectations over inflation while $1-\theta$ have long-run normal expectations, which are consistent with the long-run equilibrium. This is expressed in discrete-time as follows:

$$\pi^e_{t+1} = \theta\gamma(\pi_t - \pi^e_t) + (1-\theta)(\pi^{e*} - \pi^e_t) + \pi^e_t ,$$

and in continuous-time:

$$\dot{\pi}^e(t) = \theta\gamma(\pi(t) - \pi^e(t)) + (1-\theta)(\pi^{e*} - \pi^e(t)),$$

where π^{e*} is the long-run normal rate of inflation rate, which is equal to the growth rate of money supply μ.

Realizing that $\pi^{e*} = \mu$, we have the following system of equations:

$$\dot{i} = \phi(i)\{\alpha(\gamma e_r + 1)[e_a a - e_r(i-\pi^e) - y_n] - (\mu - \pi^e)\}. \tag{12}$$

$$\dot{\pi}^e = \theta\alpha\gamma[e_a a - e_r(i-\pi^e) - y_n] + (1-\theta)(\mu - \pi^e), \tag{13}$$

The dynamic properties of the system of (12) and (13) are essentially the same as those of the system of (10) and (11). Hence one can come to the following proposition.

Proposition 3

Hopf-Bifurcation occurs at $\gamma = \gamma_0'$, *and hence there exist some non-constant periodic solutions of the system at some parameter values,* $\gamma \in (0,\infty)$ *are sufficiently close to* γ_0'.

Proof: Basically the same as the proof for Proposition 1.

The monetary policy could be effective under the composition of long-run normal and adaptive expectations although it is unable to pull the economy out of the deflation under the pure adaptive expectations. As Figure-7 shows, the expansionary monetary policy shifts not only $\dot{\pi}^e(t)=0$ locus up but also $\dot{i}(t)=0$ locus down. Hence, the policy can be an "effective" one in the sense that it is able to pull the economy out of the deflation.

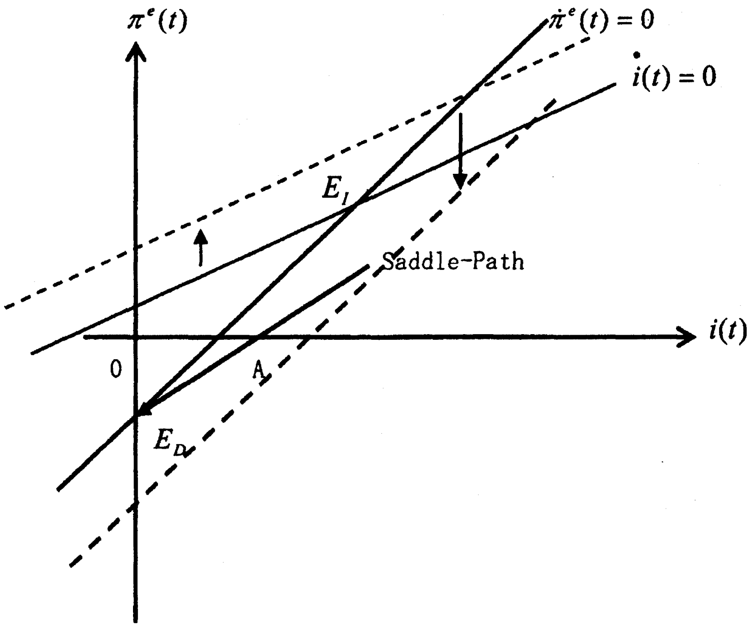


Figure-7: Expansionary Monetary Policy under Composite Expectations

3-2. Inflation Target Policy

Here we will introduce a simple inflation target policy: the monetary authority increases the growth of money supply if the actual inflation is less than the target level π^T, and vice versa. Namely,

$$\dot{\mu}=\beta(\pi^T-\pi)$$

where β is a positive constant. Substituting (9) and (11) into the above, we have

$$\dot{\mu}=\beta\{(\pi^T-\pi^e)-\alpha[e_a a-e_r(i-\pi^e)-y_n]\}. \tag{14}$$

The system now consists of equations (10), (11) and (14). At the stationary state, the following equality holds

$$\pi^T=\pi^e=\mu.$$

Hence, this target inflation policy works if the stationary state is stable. But, the question is the stability. Linearizing the system around this preferable stationary state, we have the following Jacobian Matrix

$$J_T \equiv \begin{bmatrix} -\phi(i_T)\alpha e_r(\gamma e_r+1) & \phi(i_T)[\alpha e_r(\gamma e_r+1)+1] & -\phi(i_T) \\ -\alpha\gamma e_r & \alpha\gamma e_r & 0 \\ \alpha\beta e_r & -\beta(\alpha e_r+1) & 0 \end{bmatrix},$$

where i_T is the interest rate at E_I

Defining

$$B_1 \equiv \phi(i_T)\alpha e_r(\gamma e_r+1) - \alpha\gamma e_r = -\alpha e_r[\gamma - \phi(i_T)(\gamma e_r+1)],$$

$$B_2 \equiv \phi(i_T)\alpha\gamma e_r + \phi(i_T)\alpha\beta e_r = \phi(i_T)\alpha e_r(\beta+\gamma) > 0,$$

$$B_3 \equiv -\det J_T = \phi(i_T)\alpha\beta\gamma e_r(\alpha e_r+1) - \phi(i_T)\alpha^e\beta\gamma e_r{}^2 = \phi(i_T)\alpha\gamma e_r > 0,$$

in order for the stationary state to be asymptotically stable, the following conditions must be satisfied

$$B_1 > 0 \text{ and } B_1B_2 - B_3 > 0,$$

which is equivalent to

$$B_1 > \gamma/(\beta+\gamma).$$

If $i_T = i_I$, then $B_1 = -trJ_I$. The stability condition is the basic model is $trJ_I < 0$. Since the above condition is rewritten as $trJ_I < -\gamma/(\beta+\gamma) < 0$, it is tougher than $trJ_I < 0$. Therefore, the inflation target policy introduced here tends to make the system unstable rather than stable.

4. Concluding Remarks

Krugman (1998, 2000) supposes that the Japanese economy has been in the liquidity trap and proposes an "inflation policy" for it to recover. Although the liquidity trap is usually considered in an IS-LM framework, his analysis is based on not the IS-LM but on a simple intertemporal optimization model. Since, in addition to his name, his proposal was considered as the result derived from the rigorous microfoundation, it soon attracted a great deal of attentions in the policy debate. In his model, rational expectations play a key role. Even though the deflation prevails at current period, people believe that the inflation will happen in the future thanks to an increase in money supply (or growth of money supply). This expectation of inflation decreases the real interest rate, and hence pulls the economy out of the recession.

If people have rational expectations, then his reasoning may work. If it is not the case, however, we believe that the IS-LM model is still useful to understand how the economy behaves in the short-run or in the medium-run. In contrast, in the model presented in this paper, naïve or adaptive expectations play an important role. As Krugman suggests, an increase in growth of money supply could create inflation. But, it can be the case in our model only if the adjustment speed of expected inflation rate with respect to actual inflation rate is sufficiently low. If it is high, the economy will not recover from the deflationary spiral.

The IS-LM model summarizes the interactions between the money (or bond) market and the goods market in a simple but understandable way. Hence, even those who are not familiar with intertemporal maximization models can understand it very easily. If this paper could provide some intuitions about the current Japanese economy and policy implications, our first objective was achieved.

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