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Measurement of Thermal Conductivity of Granites at High Temperatures From 100°C to 690°C

By

Naoiti KUMAGAI and Hidebumi ITÔ

Geological and Mineralogical Institute, Kyoto University

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Abstract

In the first part of this paper is given a theoretical calculations of error caused by a lateral heat leakage through the basal faces of the test piece shaped in a hollow circular cylinder. The experimental procedure is to measure heat flows from the inner wall toward the outer of the hollow cylinder, with differences in temperature at these two walls ranging from 47°C to 337°C. Thermal conductivity is lastly calculated as function of temperature, and for Shirakawa-ishi two abrupt changes in the thermal conductivity are observed at 210°C and 573°C (quartz-inversion point), while for Gôshû-ishi an abrupt change is observed only at 210°C (220°C). In the concluding remarks our results are compared with those of Birch and Clark.

Adoption of a hollow circular cylinder

Our purpose is to measure the thermal conductivity of granites at high temperatures over the quartz-inversion point 573°C. For this purpose two methods are available, one being the parallel plate method and the other the hollow circular cylinder method. To determine the choice between these two methods requires a close investigation from the stand points of the effect of grain size of the test piece to be used, capacity of heat generator put inside it and lateral heat leakage and also of easiness of minimization of heat leakage. This lastly mentioned stand point seems to be very important. This has made us employ the hollow circular cylinder method.

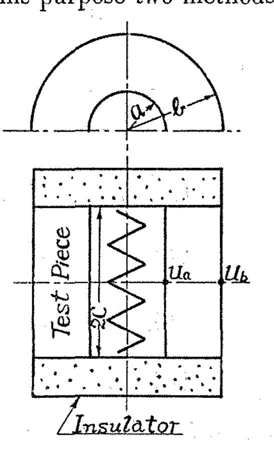

Fig. 1.

If the conductivity k is assumed to be independent of temperature, we obtain,

$$k = \frac{Q}{2\pi c} \frac{\log b/a}{u_a - u_b}, \tag{1.1}$$

where Q is half the total heat quantity generated per unit time in a heater put coaxially inside the hollow circular cylinder test piece, u_a and u_b the temperatures respectively at the inner wall and the outer of the cylinder at distances of a and b from the longitudinal axis of the cylinder and c half its height.

The equation (1.1) is valid when no heat leakage takes place through the basal faces of the cylinder. Rocks are bad conductors of heat and to insulate the basal faces from the heat leakage is difficult, even if we use the worst conductor for the insulation, and therefore we have to examine the error caused by the lateral heat leakage, as will be shown below.

In order to lessen the lateral heat leakage, $c \gg b$ is desired, and in order that the test piece of granite may be regarded as homogeneous, the thickness $b-a$ must be sufficiently large as compared with the greatest grain size of the test piece. The size of the test piece we could finally attain in our laboratory is $a \doteqdot 8$ cm, $b \doteqdot 20$ cm and $2c \doteqdot 25$ cm.

Effect of the lateral heat escape

The upper half of the apparatus is evidently symmetrical with the lower as is shown in Fig. 1. Let suffixes 1 and 2 be used for the test piece and the insulator respectively. The coordinates-axes used in the present problem should be cylindrical ones as shown in Fig. 2. At the surface of contact of the test piece with the insulator, the following conditions must be satisfied:

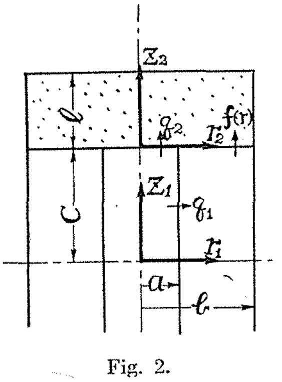

Fig. 2.

$$u_1 = u_2 \ , \quad k_1 \frac{\partial u_1}{\partial z_1} = k_2 \frac{\partial u_2}{\partial z_2} \, .$$

The latter condition indicates that the rate of normal heat flow at the boundary on the test piece side and that on the insulator side are equal, and let this rate be put equal to $f(r)$ where $r = r_1, r_2$. Further, let us assume the boundary conditions,

$$\left. \begin{aligned} &\left(\frac{\partial u_1}{\partial r_1}\right)_{r_1=a} = \frac{q_1}{k_1}, \qquad &&\left(\frac{\partial u_1}{\partial z_1}\right)_{z_1=0} = 0 \, , \\ &(u_1)_{r_1=b} = 0 \, , &&\left(\frac{\partial u_1}{\partial z_1}\right)_{z_1=c} = \frac{1}{k_1} f(r_1) \, , \end{aligned} \right\} \quad (2.1)$$

$$\left. \begin{aligned} &(u_2)_{r_2=b} = 0 \, , \qquad &&\left(\frac{\partial u_2}{\partial z_2}\right)_{z_2=0} = \frac{q_2}{k_2} \qquad && \text{for } r_2 \leqq a, \\ &(u_2)_{z_2=l} = 0 \, , && \qquad\quad = \frac{1}{k_2} f(r_2) && \text{for } a \leqq r_2 \leqq b, \end{aligned} \right\} \quad (2.2)$$

where l is the thickness of the insulator.

Now our problem is to determine $f(r)$, q_1 and q_2 from the condition $u_1 = u_2$ at the surface of contact of the test piece with the insulator. The temperature decreases as the coordinates r and z increase and k is a positive quantity, so that q_1, q_2 and $f(r)$ are negative.

q_1 may, in a strict sense, vary from place to place at the surface $r_1=a$ and $0 \leqslant z_1 \leqslant c$, and q_2 does likewise at the surface $z_2=0$ and $r_2 \leqslant a$. If the insulator were a perfect non-conductor, q_1 would be a constant and q_2 zero. The thermal conductivity k_2 of the insulator used in our experiment is, as will be mentioned later, very small as compared with k_1, so that q_1 and q_2 may be assumed to be approximately constant.

In order to obtain a very exact solution, we have to consider the effect of radiation upon the inside surface. But it is disregarded in this paper.

The differential equation of a steady state flow of heat under consideration is,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{2.3}$$

The solution of (2.3) satisfying the boundary conditions (2.1) is, as will be proved in the Appendix,

$$u_1 = \frac{q_1}{k_1} a \log \frac{r}{b} + \sum_{s=1}^{\infty} \frac{1}{\nu_s} \frac{2\chi(r,s)}{b^2\chi'^2(b,s) - a^2\chi(a,s)} \frac{\cosh \nu_s z_1}{\sinh \nu_s c} \int_a^b \frac{f(t)}{k_1} \chi(t,s)\, t dt, \tag{2.4}$$

where

$$\chi(r,s) = \frac{J_0(\nu_s r)}{J_0(\nu_s b)} - \frac{Y_0(\nu_s r)}{Y_0(\nu_s b)}, \tag{2.5}$$

$$\chi'(r,s) = \frac{\partial \chi(r,s)}{\partial(\nu_s r)}, \tag{2.6}$$

ν_s ($\nu_1 < \nu_2 < \cdots\cdots < \nu_s < \cdots\cdots\cdots$) are the roots of $\chi'(a,s)=0$, and, putting $\nu_s r = x$, $J_0(x)$ is the Bessel function of zeroth degree and $Y_0(x)$ the Neumann's cylindrical function (the cylindrical harmonic of the second kind) of zeroth degree.

For the boundary condition (2.2) the solution is,

$$u_2 = -2a \sum_{s=1}^{\infty} \frac{q_2}{k_2} \frac{1}{\lambda_s^2} \frac{\sinh \frac{\lambda_s}{b}(l-z_2)}{\cosh \frac{\lambda_s}{b} l} \frac{J_0\left(\frac{\lambda_s}{b} r\right) J_1\left(\frac{\lambda_s}{b} a\right)}{[J_1(\lambda_s)]^2}$$

$$-\frac{2}{b} \sum_{s=1}^{\infty} \frac{1}{\lambda_s} \frac{\sinh \frac{\lambda_s}{b}(l-z_2)}{\cosh \frac{\lambda_s}{b} l} \frac{J_0\left(\frac{\lambda_s}{b} r\right)}{[J_1(\lambda_s)]^2} \int_a^b \frac{1}{k_2} f(t) J_0\left(\frac{\lambda_s}{b} t\right) t dt, \tag{2.7}$$

where λ_s are the roots of $J_0(x) = 0$.

In order to satisfy the condition at the surface of contact, $u_1(z_1=c) = u_2(z_2=0)$, let us assume that the function $f(r)$ will be given by a finite series such as,

$$f(r) = \sum_{n=1}^{m} C_n \chi(r,n), \tag{2.8}$$

where $\chi(r,n)$ is the cylindrical function defined by (2.5) and C_n a constant which will be determined later. Then (2.4) and (2.7) become respectively

$$u_1=\frac{q_1}{k_1}\,a\,\log\frac{r}{b}+\sum_{n=1}^{m}\frac{1}{\nu_n}\,\frac{\cosh\nu_n z_1}{\sinh\nu_n c}\,\frac{1}{k_1}\,C_n\chi(r,n)\,, \tag{2.9}$$

and

$$u_2=-2a\frac{q_2}{k_2}\sum_{s=1}^{\infty}\frac{J_0\left(\frac{\lambda_s}{b}r\right)J_1\left(\frac{\lambda_s}{b}a\right)}{[J_1(\lambda_s)]^2}\,\frac{1}{\lambda_s^2}\,\frac{\sinh\frac{\lambda_s}{b}(l-z_2)}{\cosh\frac{\lambda_s}{b}l}$$

$$-2a\frac{1}{k_2}\sum_{n=1}^{m}C_n\chi(a,n)\sum_{s=1}^{\infty}\frac{1}{b^2\nu_n^2-\lambda_s^2}\,\frac{\sinh\frac{\lambda_s}{b}(l-z_2)}{\cosh\frac{\lambda_s}{b}l}$$

$$\frac{J_0\left(\frac{\lambda_s}{b}r\right)J_1\left(\frac{\lambda_s}{b}a\right)}{[J_1(\lambda_s)]^2}\,. \tag{2.10}$$

Then, the condition at the surface of contact, $u_1=u_2$, takes the form,

$$\frac{1}{k_1}\sum_{n=1}^{m}\frac{C_n}{a\nu_n}\,\frac{\cosh\nu_n c}{\sinh\nu_n c}\,\chi(r,n)$$

$$+\frac{2}{k_2}\sum_{n=1}^{m}C_n\,\chi(a,n)\sum_{s=1}^{\infty}\frac{1}{b^2\nu_n^2-\lambda_s^2}\,\frac{\sinh\frac{\lambda_s}{b}l}{\cosh\frac{\lambda_s}{b}l}\,\frac{J_0\left(\frac{\lambda_s}{b}r\right)J_1\left(\frac{\lambda_s}{b}a\right)}{[J_1(\lambda_s)]^2}$$

$$+2\frac{q_2}{k_2}\sum_{s=1}^{\infty}\frac{J_0\left(\frac{\lambda_s}{b}r\right)J_1\left(\frac{\lambda_s}{b}a\right)}{[J_1(\lambda_s)]^2}\,\frac{1}{\lambda_s^2}\,\frac{\sinh\frac{\lambda_s}{b}l}{\cosh\frac{\lambda_s}{b}l}=-\frac{q_1}{k_1}\log\frac{r}{b}\,. \tag{2.11}$$

If q_1 is assumed to be given, this equation involves m+1 unknown constants, C_n (n= 1, 2, ...m) and q_2. It is likely that the quantity of heat passing through a point at r=a perpendicularly to the base of the insulator is continuous, and therefore let us asume $f(a)=q_2$, that is,

$$\sum_{n=1}^{m}C_n\,\chi(a,n)-q_2=0\,. \tag{2 12}$$

From (2.8) and (2.5), naturally we get f $(b)=0$.

Solving the m+1 simaltaneous equations consisting of m equations (2.11) satisfied at $r=r_i$ with $a<r<b$ and $i=1, 2, \cdots\cdots$, m and (2.12), we can find the m different values of C_n and the value of q_2.

Taking $a=3$ cm, $b=12$ cm, $c=10$ cm and $l=12$ cm, let us in the next calculate the effect of the lateral escape of heat. The roots of $\chi'(a,s)=0$ have been obtained graphically, giving $a\nu_1=0.6670$, $a\nu_2=1.6450$, $a\nu_3=2.6643$, $a\nu_4=3.7007$, $a\nu_5=4.7405$,........... The roots of $J_0(x)=0$ are well known. In (2.8) let us choose m=3. In order to see the order of magnitude of C_n, we assume $f(r)\propto\log(r/b)$, and then through some labour of calculations we get $C_n\propto\chi(a,s)/(a\nu_n)^2$. Therefore we obtain $C_1 : C_4=\chi(a,1) : 0.032\ \chi(a,4)$. From this result, approximation of $f(r)$ by $m=3$ does not seem to involve great error. With $m=3$ we have to solve the simultaneous linear equations having 4 unknowns.

By taking $r_1=4$ cm, $r_2=6$ cm and $r_3=9$ cm in (2.11) and by using (2.12), we get the following 4 simultaneous equations:

$$\left.\begin{aligned}
&\left(-\frac{1}{k_1}9.677-\frac{2}{k_2}4.268_0\right)C_1+\left(\frac{1}{k_1}2.353+\frac{2}{k_2}1.060_9\right)C_2\\
&\qquad+\left(\frac{1}{k_1}1.265+\frac{2}{k_2}0.566_5\right)C_3+\frac{2}{k_2}0.1428\,q_2=\frac{1}{k_1}1.098_6q_1\\
&\left(-\frac{1}{k_1}8.549-\frac{2}{k_2}3.787_8\right)C_1+\left(\frac{1}{k_1}0.233+\frac{2}{k_2}0.134_0\right)C_2\\
&\qquad+\left(-\frac{1}{k_1}1.122-\frac{1}{k_2}0.582_6\right)C_3+\frac{2}{k_2}0.069_7q_2=\frac{1}{k_1}0.693_1\,q_1\\
&\left(-\frac{1}{k_1}4.337-\frac{2}{k_2}2.043_9\right)C_1+\left(-\frac{1}{k_1}1.645-\frac{2}{k_2}0.830_4\right)C_2\\
&\qquad+\left(\frac{1}{k_1}0.517+\frac{2}{k_2}0.258_8\right)C_3+\frac{2}{k_2}0.024_4q_2=\frac{1}{k_1}0.287_7q_1\\
&\qquad-6.45_4\,C_1+4.46_7\,C_2+5.11_8\,C_3-q_2=0,
\end{aligned}\right\}\quad(2.13)$$

For the material of the insulator, the insulation fire brick manufactured by Isoraito Kôgyô K. K., Nanao, Ishikawa Prefecture, is used. Coefficient of thermal conductivity reported by this maker is 0.147 Kcal/m. h. °C, so that $k_2=0.0017$ joule/cm. sec. °C. If we employ the average value of the thermal conductivity of granites already known, we are to take $k_1=0.023$ joule/cm. sec. °C.[1] These values make $k_1/k_2=13.5$. It is evident that the larger the ratio k_1/k_2, the smaller the escaping heat. Therefore let us employ two values 10 and 5 for this ratio, which will determine from (2.13) the ratio $C_1:C_2:C_3:q_2:q_1$ as shown in Table I.

Table I

k_1/k_2	C_1	:	C_2	:	C_3	:	q_2	:	q_1
10	-7.13	:	5.34	:	3.52	:	87.89	:	1000.00
5	-13.16	:	9.95	:	6.75	:	163.90	:	1000.00

From this table are computed the graphs of Figs. 3 and 4.

If we denote by $u_1(a,0)$ the temperature at $r_1=a$ and $z_1=0$, the temperature at $r_1=b$ and $z_1=0$ being taken as zero always, the conductivity k can be given from (2.9) by,

$$k_1=\frac{q_1a}{u_1(a,o)}\left\{\log\frac{a}{b}+\sum_{n=1}^{m}\frac{C_n}{q_1}\frac{1}{a\nu_n}\frac{1}{\sinh\nu_nc}\,\chi\,(a,n)\right\}. \qquad (2.14)$$

The constants q_1 and q_2 are simply related with Q which is half the rate of total heat generation of a heater put inside the inner space of the test piece, by an equation $q_1 = -Q / (2\pi ac + \pi a^2 q_2/q_1)$ where q_1 and q_2 are negative as already mentioned. By neglecting small quantities, $(C_n/q_1)^2$ and $(q_2/q_1)^2$ and higher terms, we finally obtain from (2.14),

$$k_1 = \frac{Q}{u_1(a,o)}\frac{\log b/a}{2\pi c} - \frac{Q}{u_1(a,o)}\frac{1}{2\pi c}\left\{\frac{a}{2c}\frac{q_2}{q_1}\log\frac{b}{a} + \sum_{n=1}^{m}\frac{C_n}{q_1}\frac{1}{a\nu_n}\frac{\chi(a,n)}{\sinh \nu_n c}\right\}. \quad (2.15)$$

The first term on the right side of (2.15) is the conductivity to be found when the length of the cylinder $2c$ is very long as compared with its outer radius b. The second term is, therefore, a correction to be applied to the first term when the length is not large. The value of the second term with it's minus sign exclusive has been proved to be positive for $m=3$.

If the first term is denoted by k'_1, the ratio $(k'_1-k_1)/k_1$ (>0) has been computed for $k_1/k_2=10$ and 5, giving,

$$(k'_1-k_1)/k_1 = \begin{cases} 2.48\% & \text{for } k_1/k_2=10 \\ 4.67\% & \text{for } k_1/k_2=5. \end{cases}$$

In our experiment, we cool the outer surface of the test piece with water and that of the insulator is exposed in air, so that the lateral heat escape through the insulator is evidently less than that of the case in which the outer surface of the insulator is likewise cooled with water, and therefore the above mentioned errors become much smaller for the case actually employed. Therefore, we finally disregard the second term on the right side of (2.15) and employ the first term as giving the conductivity k, that is,

$$k_1 = \frac{Q}{u_1(v,o)}\frac{\log b/a}{2\pi c} \quad (2.16)$$

of which the maximum error will be a few percentage for $5 < k_1/k_2 < 10$ which is of our case as will be seen later.

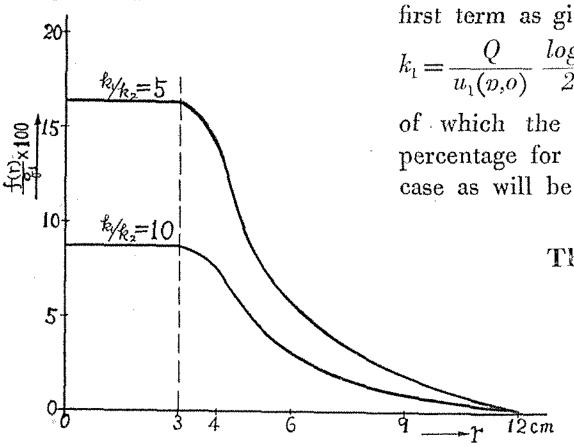

Fig. 3. Variation with r of percentage relative heat leakage $f(r)/q_1$

The apparatus

Two kinds of granite have been chosen for the test piece: one is Shirakawa-ishi, or Hiei-Granite from between Mt. Hiei and Mt. Daimonji in Kyoto, and the other Gôshû-ishi, or Hira-Granite from Mt. Hira in Shiga Prefecture. The latter granite is pegmatitic.

Form of our test piece has been made into a hollow circular cylinder.

Finishing of its whole surface has been done by a stonemason. The finished piece are shown in Fig. 5. The form of the cylinders has been carefully measured and the results are shown in Table II.

The sketch of the apparatus is shown in Fig. 6. Junctions of thermo-couples are fixed in tiny hollows made at the extremities of three diameters separated by 60° angle and lying on the horizontal plain bisecting the test piece; alumel-chromel at the inside and cupper-constantan out-side. Each wire of thermo-couple is contained inside an insulation tube of an outer diameter of about 1.5 mm. Good contact of the thermo-couples to the walls of the test piece is secured by pressing thin layers of plaster of Paris kneaded with water glass toward the walls.

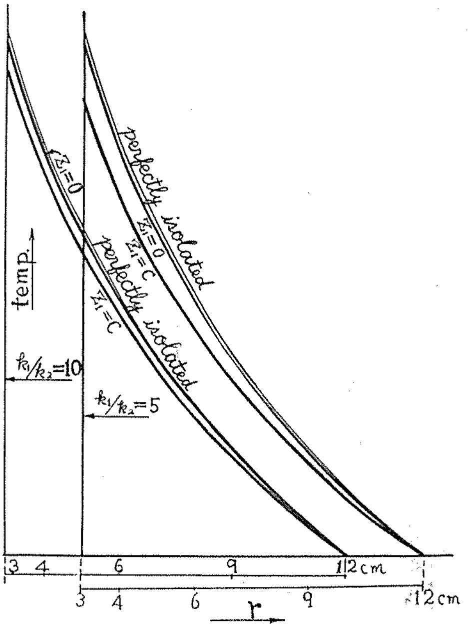

Fig. 4. Temperature u_1 computed by (2.9)

Chemically pure plaster of Paris is used for the inner wall where the temperature is higher and the ordinary one for the outerwall where the temperature is lower. For temperatures higher

Table II. Mean dimensions of test piece (in mm)

	Outer dia.	Inner dia.	Height
Shirakawa-ishi	201.95 ± 0.27	98.41 ± 0.95	194.5
Gôshû-ishi	197.30 ± 0.39	85.45 ± 0.48	180.6

than about 200°C, plaster of Paris ($CaSO_4\frac{1}{2}H_2O$) becomes dead burnt gypsum ($CaSO_4$) which is chemically stable against temperature change above this temperature. In order to lessen an error produced by minute irregularity of the inner and outer cylindrical surfaces of the test piece with which the stuffing material, the plaster of Paris, is in contact, it is desirable that the conductivity of the test piece and that of the stuffing material are equal as possible. The conductivity of plaster of Paris is 0.011 joule/cm. sec. °C[1] which is of the same order of magnitude with that of granite, but the thermal conductivity of dead burnt gypsum is regrettably unknown. We have nothing to do, therefore, but to assume the thermal conductivity of dead burnt gypsum to be approximately equal to that of plaster of Paris. This assumption may give rise to unknown

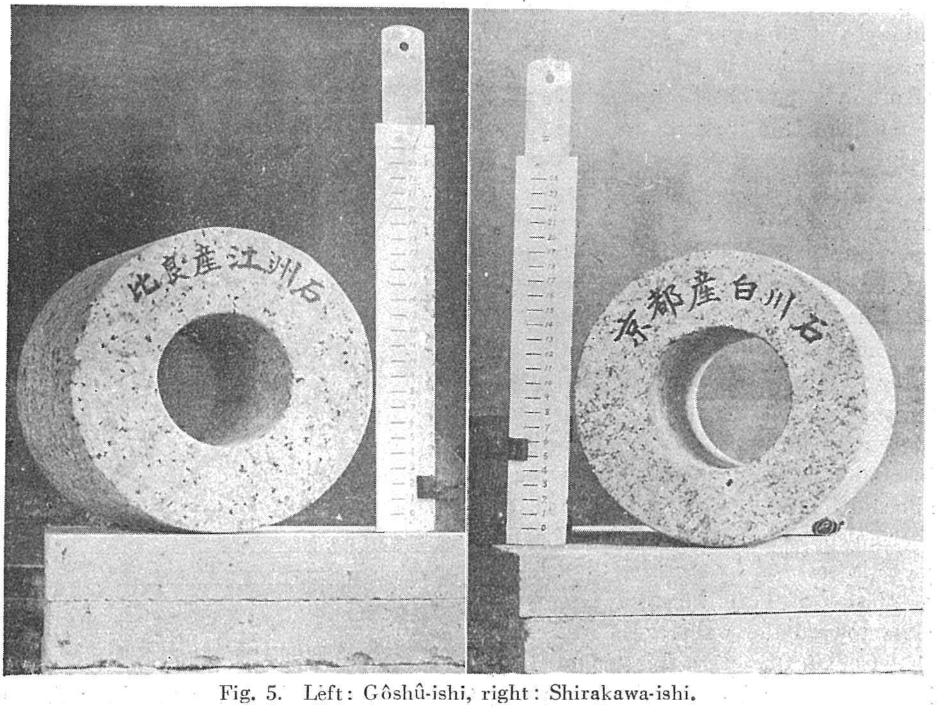

Fig. 5. Left: Gôshû-ishi, right: Shirakawa-ishi.

errors in the values of conductivity reported in this paper.

Representation of thermal conductivity as function of temperature

First let us assume that the bases of the cylinder are perfectly insulated. This assumption comes from the neglect of the second term on the right side of (2.15) of which the effect is small as already mentioned. Let us now denote a temperature by a letter v expressed by an arbitrary scale. Then we obtain

$$2\pi r\ k\ (v)\ \frac{dv}{dr} = -\ q,$$

and we get

$$\frac{q}{2\pi}\ log\left(\frac{b}{a}\right) = \int_{v_b}^{v_a} k\ (v)\ dv, \qquad (4.1)$$

where q is the quantity of heat applied per unit time and per uint length along the longitudinal axis, and v_a and v_b the temperatures respectively at the inner wall and the outer of the test piece.

As it is most general to represent the unknown function $k\ (v)$ in power series of v, we get

$$k(v) = a_0 + a_1 v + a_2 v^2, \qquad (4.2)$$

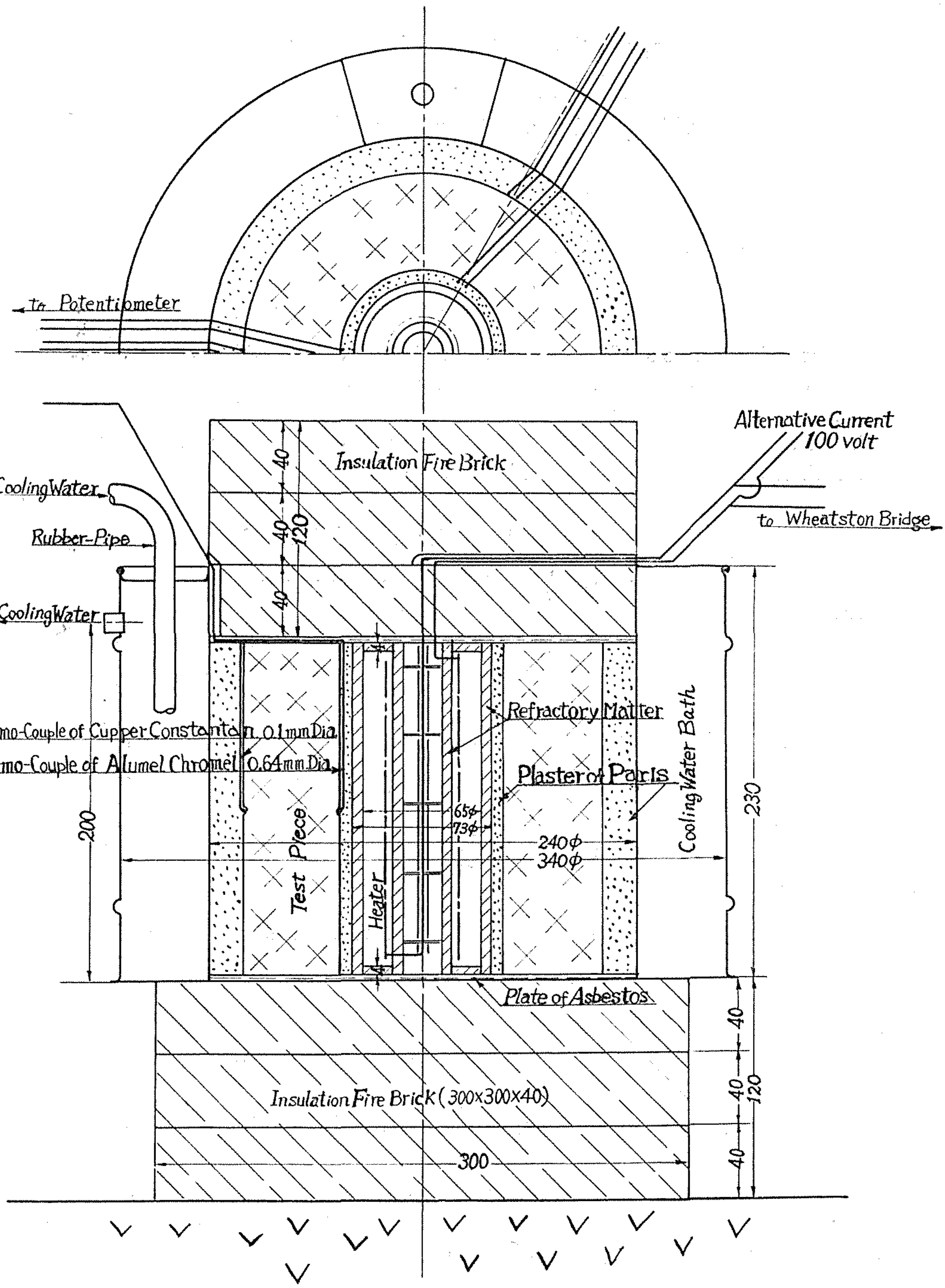

Fig. 6. Apparatus

in which a_0, a_1 and a_2 are the constants to be determined. For simplicity put

$$q\ log\ (b/a)/\,2\pi = \xi, \tag{4.3}$$

which is a measurable quantity, and the mean conductivity, say $\bar{k}$, is given from (4.1) and (4.3) by

$$\bar{k} = \xi\ /\ (v_a - v_b). \tag{4.4}$$

From (4.3), (4.2) and (4.1), we get

$$\xi = a_0(v_a - v_b) + \frac{a_1}{2}(v_a^2 - v_b^2) + \frac{a_2}{3}(v_a^3 - v_b^3). \tag{4.5}$$

If we are given n sets of observations of (4.5) with $n > 3$, the constants a_0, a_1 and a_2 can be determined by the method of least squares.

At the quartz-inversion point 573°C, we may anticipate $k(v)$ as discontinuous. In such a case two different sets of constants will occur, one for the temperatures lower than 573°C and the other higher than it.

Experimental results on Gôshû-ishi

Table III. Measurements for Gôshû-ishi

No. of measurement i	Temp. inside u_a°C	Temp. outside u_b°C	Quantity of heat 2Q joule	Mean conductivity $\bar{k}$ joule/cm. sec. °C
1	283	169	258.99	0.0168
2	308	180	285.69	164
3	379	217	311.01	141
4	430	244	358.93	142
5	495	272	447.79	148
6	557	302	482.74	140
7	608	322	534.07	138
8	690	353	616.21	135

For the sake of calculation, let the temperature v be defined by

$$100v = u - 400, \tag{5.1}$$

where u is the temperature in °C.

At first let us make use of all the measurements shown in Table III, and the least squares solutions for (4.5) give,

$$\begin{matrix} a_0 = 1.402, & a_1 = -0.0799, & a_2 = 0.0220, & \text{and } \varepsilon = 0.1098, \\ \pm 34 & \pm 201 & \pm 209 & \end{matrix} \tag{5.2}$$

where the errors annexed to a_0, a_1 and a_2 are the mean errors and ε the mean error for a single measurement of ξ. Then transforming the variable from v to u, we get from (5.2), (5.1) and (4.2),

$$k(u) = 1.402 \cdot 10^{-2} - 0.0799\ (u\text{-}400) \cdot 10^{-4} + 0.0220\ (u\text{-}400)^2 \cdot 10^{-6}, \quad \text{for } 169°\text{C} < u < 690°\text{C}. \tag{5.3}$$

From a close inspection of Table III, it is noticed that when we pass from i=2 to 3, $\bar{k}$ suddenly decreases, after we pass i=3 it shows a gradual increase until we reach i=5 and after i=5 it does, on the contrary, a gradual decrease. Then, if we omit the two data of i=1 and 2, the least squares solutions applied for the remaining 6 data give,

$$a_0 = 1.440, \quad a_1 = -0.0223, \quad a_2 = -0.0291, \text{ and } \varepsilon = 0.0776. \tag{5.4}$$
$$\pm 27 \qquad \pm 251 \qquad \pm 235$$

The similar transformation of v into u gives,

$$k(u) = 1.440 \cdot 10^{-2} - 0.0223\ (u\text{-}400) \cdot 10^{-4} - 0.0291\ (u\text{-}400)^2 \cdot 10^{-6}, \quad \text{for } 217°\text{C} < u < 690°\text{C}. \tag{5.5}$$

The value of ε given in (5.4) is less than that in (5.2), and therefore the result (5.5) will be more likely than (5.3), so that let us assume that $k(u)$ given by (5.5) will be the thermal conductivity of Gôshû-ishi for 217°C$<u<$690°C.

For the purpose of obtaining the thermal conductivity $k(u)$ for temperatures below 217°C, number of the available data are regrettably only two and the conductivity has to be expressed by a linear equation. If we denote by v_s (seperation temperature) the lower limit of the temperature range for which (5.5) is valid, the following two kinds of conductivity come into play on the right side of (4.1) :

$$k(v) = b_0 + b_1 v \qquad \text{for } v < v_s, \tag{5.6}$$

and

$$k(v) = a_0 + a_1 v + a_2 v^2 \qquad \text{for } v_s < v, \tag{5.7}$$

where the values of a_0, a_1 and a_2 are given by (5.4) and b_0 and b_1 are the constants to be determined. Putting these into (4.1), we get

$$\xi - a_0(v_a - v_s) - \frac{a_1}{2}\left(v_a^2 - v_s^2\right) - \frac{a_2}{3}\left(v_a^3 - v_s^3\right) = b_0\left(v_s - v_b\right) + \frac{b_1}{2}\left(v_s^2 - v_b^2\right), \tag{5.8}$$

where $v_b < v_s < v_a$. 217°C (u_b for i=3) may not be a plausible temperature corresponding to the separation temperature v_s. The separation temperature may lie in the neighbourhood of 217°C which is the lower limit of the temperature range for which (5.5) is determined. Let us, therefore, assume two kinds of separation temperature, one being 210°C and the other 220°C, which give rise to two kinds of v_s, -1.9 and -1.8. Values of b_0 and b_1 will be then determined from (5.8) by using the two data of i=1 and 2. The transformtion of v into u will be similarly done. The results with $v_s = -1.9$ are

$$k(u) = 11.93 \cdot 10^{-2} + 4.65(u\text{-}400) \cdot 10^{-4}, \quad \text{for } 169°\text{C} < u < 210°\text{C}. \tag{5.9}$$

And the results with $v_s = -1.8$ are,

$$k(u) = 9.63 \cdot 10^{-2} + 3.67(u-400) \cdot 10^{-4}, \quad \text{for } 169°\text{C} < u < 220°\text{C}. \qquad (5\ 10)$$

The two coefficients on the right sides of (5.9) and (5.10) involve no errors because of only the two data.

Graphs showing the equations (5.3), (5.5), (5.9) and (5.10) are represented in Fig. 7. Difference between the values of v_s, −1.9 and −1.8, produces appreciable change in the conductivity below 217°C. This indicates that from our experimental data the separation temperature has an important influence for the determination of $k(u)$ below 217°C. But regrettably we have no basis by which we can reasonably determine the separation temperature. The two kinds of $k(u)$ below 217°C obtained above are to be seen as showing only a general tendency. It is worth noting that the conductivity abruptly changes when we pass through the neighbourhood of 217°C and also the conductivity below this temperature very steeply increases with temperature. And we have to remark that it can not be justified from our present data whether the conductivity is as really discontinuous at 210°C or 220°C as shown in Fig. 7.

Beforehand we have had for granites an expectation of occurrnce of a discontinuity or an abrupt change in the conductivity at the quartz-inversion point 573°C. In the next section, it will be shown that the conductivity of Shirakawa-ishi shows an abrupt increase when the temperature passes 573°C toward higher temperatures. For Gôshû-ishi, however, the values of $\bar{k}$ (Table III) show a steady decrease, even when u_a passes 573°C toward higher temperatures. Therefore,

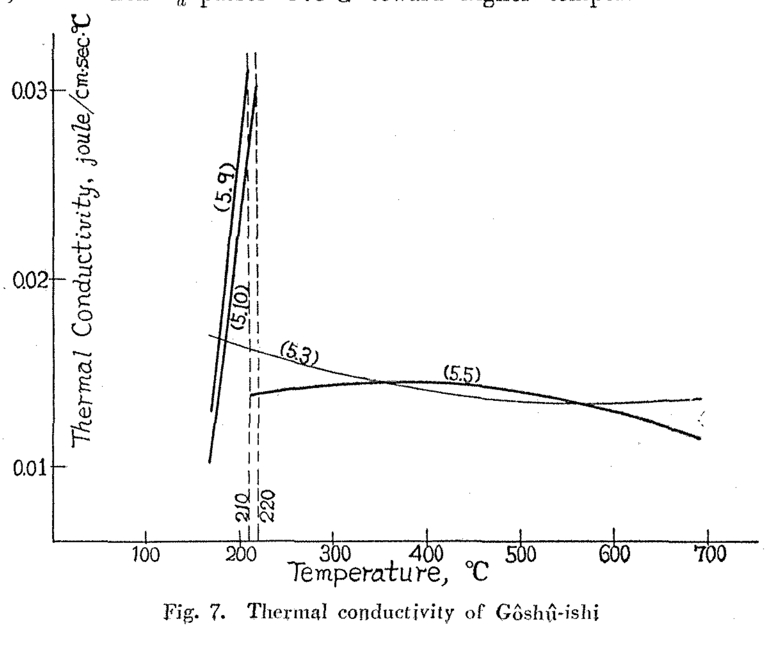

Fig. 7. Thermal conductivity of Gôshû-ishi

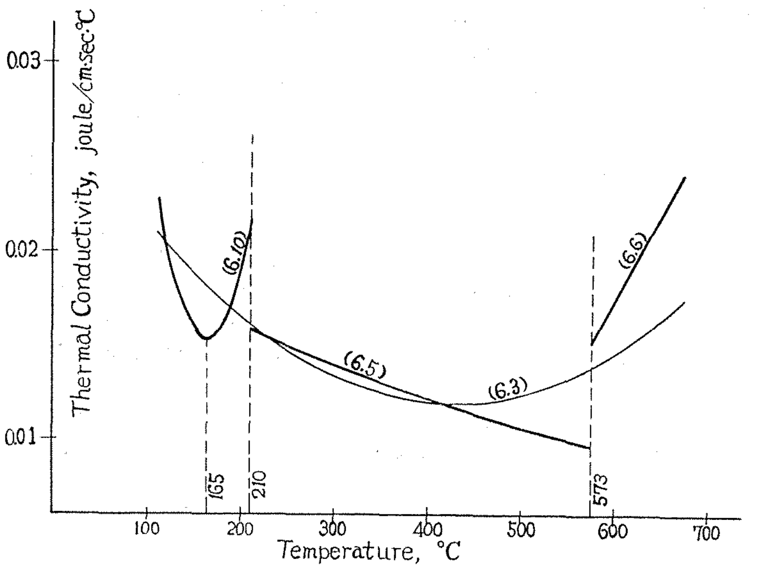

Fig. 8. Thermsl conductivity of Shirawa-ishi.

it is likely that the thermal conductivity of Gôshû-ishi is not discontinuous or does not show an abrupt change at 573°C.

Experimental results on Shirakawa-ishi

Table IV. Measurements for Shirakawa-ishi

No. of measurement i	Temp. inside u_a°C	Temp. outside u_b°C	Quantity of heat $2Q$ joule	Mean conductivity $\bar{k}$ joule/cm. sec. °C
1	159	112	148.07	0.0185
2	195	133	170.92	162
3	239	152	250.73	170
4	284	173	297.03	157
5	377	213	391.91	140
6	420	226	448.45	136
7	489	257	497.56	126
8	560	282	559.43	118
9	628	306	686.76	125
10	668	335	768.41	130

In this case the favourable transformation of temperature is

$$100\, v = u - 300. \tag{6.1}$$

As well as in the case of Gôshû-ishi, let us firstly make use of all the measuremnts shown in Table IV, for which the least squares solutions are,

$$a_0=1.328,\ a_1=-0.2294,\ a_2=0.0916, \text{ and } \varepsilon=0.1070, \qquad (6.2)$$
$$\pm 30 \qquad \pm 338 \qquad \pm 588$$

and $k(u)=1.328\cdot10^{-2}-0.2294\,(u-300)\cdot10^{-4}+0.0916\,(u-300)^2\cdot10^{-6},$
for $112^\circ\text{C} < u < 668^\circ\text{C}$. (6.3)

Observing the values of $\bar{k}$ (Table IV) carefully, we find that 1) when the higher temperature u_a exceeds 560°C, which is just below the quartz-inversion point 573°C, $\bar{k}$ clearly shows an increase, while $\bar{k}$ does a steady decrease as u_a increases up to 560°C, and 2) if $\bar{k}$ for i=3 (u_a=239°C, u_b=152°C) is put aside, all the values of $\bar{k}$ up to i=8 steadily decrease. The former fact suggests that the thermal conductivity of Shirakawa-ishi will abrudtly change when we pass the quartz-inversion point. The latter does that the thermal conductivtity will also abruptly change when we pass a certain temperature falling between 239°C and 152°C. For this temperature let us assume 210°C or 220°C we have employed for Gôshû-ishi. Let the measurements given in Table IV be divided, therefore, into three groups by 210°C and 573°C. The first group consists of the measurements of i=1,2,3 and 4, of which the lower temperature u_b does not exceed 210°C, the third group i=9 and 10, of which the higher temperature u_a exceeds 573°C, and the second group the rest of the measuremnts, i=5, 6, 7 and 8.

The results of the calculation of $k(u)$ for the second group are,

$$a_0=1.396,\ a_1=0.2027,\ a_2=0.0164, \text{ and } \varepsilon=0.0127, \qquad (6.4)$$
$$\pm 5 \qquad \pm 269 \qquad \pm 163$$

and $k(u)=1.396\cdot10^{-2}-0.2027\,(u-300)\cdot10^{-4}+0.0164\,(u-300)^2\cdot10^{-6},$
for $213^\circ\text{C} < u < 573^\circ\text{C}$ (the second group). (6.5)

$k(u)$ for the third group has been calculated in the similar way we have used for Gôshû-ishi for the temperatures lower than 210°C. The result is

$k(u)=-1.073\cdot10^{-2}+0.936\,(u-300)\cdot10^{-4},$
for $573^\circ\text{C} < u < 668^\circ\text{C}$ (the third group). (6.6)

For the first group, the higher temperatres u_a for i=3 and 4 exceed the separation temperature 210°C, and therefore it will be necessary to obtain two hypothetical data of which u_a is taken as equal to 210°C and u_b to 152°C for i=3 and to 173°C for i=4. For doing this, let this separation temperature be denoted by v_s, the conductivity for $v < v_s$ by $k_\alpha(v)$ and that for $v > v_s$ by $k_\beta(v)$ which is identical with (6.5) if u is transformed to v. Then we have

$$\xi=\int_{v_b}^{v_s} k_\alpha(v)\,dv+\int_{v_s}^{v_a} k_\beta(v)\,dv. \qquad (6.7)$$

If we give to v_a the values corresponding to 239°C (i=3) and 284°C (i=4), two values of the second integral on the right side of (6.7) can be evaluated. Since the values of ξ on the left side of (6.7) can be found from (4.3), two values of the first integral can be also done, which are the values of ξ of the two hypothetieal data mentioned above. Thus we obtain Table V.

Table V. Measurements modified by 210°C for Shirakawa-ishi

i	u_a°C	u_b°C	ξ joule/cm. sec	$\bar{k}$ joule/cm. sec. °C
1	159	112	0.8711	0.0185
2	195	133	1.0055	162
3	210	152	1.0230	176
4	210	173	0.6309	170

Expressing $k_\alpha(v)$ by,

$$k_\alpha(v) = c_0 + c_1 v + c_2 v^2, \tag{6.8}$$

the solutions become

$$c_0 = 7.246,\ c_1 = 8.437,\ c_2 = 3.116, \text{ and } \varepsilon = 0.0487, \tag{6.9}$$
$$\pm 3.923 \quad \pm 5.849 \quad \pm 2.113$$

and $$k(u) = 7.246 \cdot 10^{-2} + 8.437\,(u-300) . 10^{-4} + 3.116\,(u-300)^2 \cdot 10^{-6},$$
$$\text{for } 112^\circ\text{C} < u < 210^\circ\text{C (the first group).} \tag{6.10}$$

The errors annexed to the values of c_0, c_1 and c_2 in (6.9) are relatively large. The origin of these great errors will lie in that both the separation temperature assumed at 210°C and the approximation of $k_\alpha(v)$ by a second order equation (6.8) may not be satisfactory.

Graphs of the equations (6.3), (6.5), (6.6) and (6.10) are shown in Fig. 8. For Shirakawa-ishi, there are two characteristic temperatures 210°C and 573°C (quartz-inversion point) at which the thermal conductivity shows abrupt change. Below 210°C the conductivity has a sharp minimum at about 165°C. Between 210°C and 573°C the conductivity decreases as the temperature increases. Above the quartz-inversion point, the conductivity increases steeply with temperature. It is regrettable that we could not obtain more than two data above 573°C owing to melting of the heating wire.

Concluding remarks

I) Change of thermal conductivity with temperature of Gôshû-ishi expressed by (5.9) (or (5.10)) and (5.5), which have been obtained by the separation of the data at 210°C (or 220°C), will explain the observed data more satisfactorily than that expressed by (5.3) which has been obtained by the method in which the separation is not employed. For showing this, standard deviations σ have been computed by these two kinds of equations. The results are: By the set of the equations (5.9) (or (5.10)) and (5.5) $\sigma = 0.0476$ which is clearly less than $\sigma = 0.0868$ obtained by the equation (5.3). Similarly, the equations (6.10), (6.5) and (6.6) which have been obtained for Shirakawa-ishi by the separations at 210°C and 573°C will also explain the observed data more satisfactorily than the equation (6.3), as a standard deviation calculated by the former set of equations is $\sigma=0.0159$ which is also clearly much less than $\sigma=0.0895$ obtained by (6.3).

II) As for the results obtained for the two kinds of granites, Gôshû-ishi (pegmatitic) and Shirakawa-ishi, there are two questions which follow: 1) why does not Gôshû-ishi show an abrupt change in conductivity at 573°C (quartz-inversion point) while Shirakawa-ishi does?, and 2) why do the conductivities of both of the two granites change abruptly at about 210°C? These two questions seem to be important, and prior to finding answers to them we have to make other experiments to investigate into the above two facts. One experiment will be that in which we have to observe such suitably selected physical quantities other than thermal conductivity that change with temperature. The other will be that in which ranges of temperatures for which the mean conductivities $\bar{k}$ are to be obtained are made as small as possible than those employed in our present experiments.

III) Lastly we have to make an important reference to the works of Birch and Clark.[2] They measured by an ingenious device change of thermal conductivity with temperature between 0°C and 400°C of 18 igneous rocks and seven sedimentary and metamorphic rocks and etc. The specimen they used was shaped into a flat circular disk of 6.35 mm thickness and 38.1 mm diameter and the difference in temperature at the two faces of the specimen was about 5°C. Such a smallness of the temperature difference ensures their result safely to be the thermal conductivity as function of the temperature, that is $k(u)$. Therefore, we may compare our results of $k(u)$ to those of Birch and Clark. But here it must be remembered that our results are deduced from the observed mean conductivity $\bar{k}$ for large temperature ranges. On close inspection of their result for anorthosite and diabasic rocks, nine in number, shown in their Fig. 5, it may be worth noting that thermal conductivities of diabase (Vinal Haven) and gabbro (Wisconsin 2) show an increase, though not very conspicuous, in the shape of a cusp at about 200°C, and similar cuspidal increase can also be observed for diabase (Mt. Holyoke) and albitite (Sylmar, Pa.), while Birch and Clark represent the thermal conductivities of these rocks by smooth curves. Excepting gabbro (French Creek), the thermal conductivities of diabase (Maryland), gabbro (Wisconsin 1), anorthosite (Stillwater) and anorthosite (Transvaal) are not reported at temperatures above 200°C, but they increase steadily with temperature toward 200°C. It is also noteworthy that diabase (Maryland) shows a minimun thermal conductivity at about 150°C. These facts above noticed are consistent qualitatively with our results obtained for Gôshû-ishi and Shirakawa-ishi. On the other hand, the thermal conductivities of Rockport granites 1 and 2 represented in their Fig. 4 show steady decrease with increasing temperature up to nearly 300°C and that of Barre granite does similar decrease up to a little less than 200°C. On the contrary, our two granites do not show such a decrease in a temperature range from 112°C to 220°C.

Appendix

In the following is given the derivation of the equation (2.4) which is the solution of (2.3) satisfying the boundary conditions (2.1). For doing this, the suffix 1 in (2.4) is dropped for simplicity's sake. If we pnt $u=v+w$, it is required to determine v and w which satisfy the following equations:

$$\left(\frac{\partial v}{\partial r}\right)_{r=a}=\frac{q}{k}\,,\ (v)_{r=b}=0\,,\ \left(\frac{\partial v}{\partial z}\right)_{z=0}=0\,,\ \left(\frac{\partial v}{\partial z}\right)_{z=c}=0\,, \tag{8.1}$$

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}=0\,, \tag{8.2}$$

and

$$\left(\frac{\partial w}{\partial r}\right)_{r=a}=0\,,\ (w)_{r=b}=0\,,\ \left(\frac{\partial w}{\partial z}\right)_{z=0}=0\,,\ \left(\frac{\partial w}{\partial z}\right)_{z=c}=\frac{1}{k}f(r)\,, \qquad (8.3),(8.4),(8.5),(8.6)$$

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}=0\,. \tag{8.7}$$

The solution of (8.2) subject to the boundary conditions (8.1) is given by

$$v=\frac{q}{k}\,a\log\frac{r}{b}\,. \tag{8.8}$$

For finding w let us put $w=Z(z)\cdot R(r)$. Then from (8.7) we obtain,

$$\frac{d^2Z}{dz^2}-\nu^2 Z=0,\ \frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\nu^2 R=0, \qquad (8.9),(8.10)$$

where ν is a constant. The solution of (8.10) is

$$R(r)=AJ_0(r\nu)+BY_0(r\nu)\,, \tag{8.11}$$

where A and B are arbitrary constants. In order that this equation $R(r)$ satisfies (8.3) and (8.4), ν must be a root of the equation

$$\begin{vmatrix} J_1(a\nu) & Y_1(a\nu) \\ J_0(b\nu) & Y_0(b\nu) \end{vmatrix}=0\,, \tag{8.12}$$

which equation can be replaced by $\chi'(a,s)=0$ as will be easily seen through simple calculation (Cf. (2.5) and (2.6)). Putting $a\nu=x$ and $b/a=\rho$, (8.12) becomes

$$J_1(x)\cdot Y_0(\rho x)-J_0(\rho x)\cdot Y_1(x)=0\,. \tag{8.13}$$

Using the asymptotic values of $J_n(x)$ and $Y_n(x)$ for large, positive values of x, $J_n(x)=\sqrt{\frac{2}{\pi x}}\cos\left[x-\frac{\pi}{4}(2n+1)\right]$ and $Y_n(x)=\sqrt{\frac{2}{\pi x}}\sin\left[x-\frac{\pi}{4}(2n+1)\right]$, the left side of (8.13) is approximately equal to $\frac{2}{\pi x}\frac{1}{\rho}\cos(\rho-1)x$ for large values of

x, and therefore we know that (8.12) has an infinitly large number of positive roots. Let these roots be denoted by $\nu_1 < \nu_2 \cdots\cdots < \nu_s < \cdots\cdots$. Determining the arbitrary constants A and B by (8.3) and (8.4), we obtain

$$R\,(r) = C_s\left(\frac{J_0(\nu_s r)}{J_0(\nu_s b)} - \frac{Y_0(\nu_s r)}{Y_0(\nu_s b)}\right) = C_s \chi(r,s)\;, \tag{8.14}$$

where C_s is a constant. On the other hand, the solution of (8.9) satisfying (8.5) is

$$Z\,(z) = C_s'\,\frac{\cosh \nu_s z}{\sinh \nu_s c}\,, \tag{8.15}$$

where C_s' is another constant. Therefore the general solution of w is

$$w = \sum_{s=1}^{\infty} K_s \frac{\cosh \nu_s z}{\sinh \nu_s c}\ \chi\,(r,s)\;, \tag{8.16}$$

where $K_s = C_s\,C_s'$. From (8.6) we get

$$\frac{1}{k} f\,(r) = \sum_{s=1}^{\infty} K_s \nu_s \chi\,(r,s)\;, \tag{8.17}$$

Multiplying $r\,\chi\,(r,m)$ with $m=1,\ 2,\ \cdots\cdots s, \cdots\cdots$ to both sides of (8.17) and then integrating from $r=a$ to $r=b$, we obtain

$$K_s = \frac{1}{\nu_s}\,\frac{2}{b^2\chi'^2(b,s) - a^2\chi^2(a,s)}\int_a^b \frac{f(t)}{k}\,\chi\,(t,s)\ tdt\;, \tag{8.18}$$

and hence

$$w = \sum_{s=1}^{\infty}\frac{1}{\nu_s}\,\frac{2\,\chi\,(r,s)}{b^2\chi'^2(b,s) - a^2\chi^2(a,s)}\,\frac{\cosh \nu_s z}{\sinh \nu_s c}\int_a^b \frac{f(t)}{k}\,\chi(t,s)\ tdt\;. \tag{8.19}$$

One can easily prove that the equations (8.8) and (8.19) are respectively the solution of (8.2) satisfying (8.1) and that of (8.7) satisfying (8.3), (8.4), (8.5) and (8.6), and therefore that $v+w=u$ is the required solution.

References

1) International Critical Table, Vol. II, p. 315 (1927)
2) Birch, FRANCIS and Harry CLARK, Am. Jour. Sci. Vol. 238, p. 529 (1940)