数理解析研究所講究録 1391 巻 2004 年 192-211

Optimal Portfolio of Low Liquid Assets

Koichi Matsumoto
R&D Promotion Department
Mizuho-DL Financial Technology Co., Ltd.
Ote Center Building 12F, 1-3, Otemachi 1-chome,
Chiyoda-ku, Tokyo 100-0004, Japan
E-mail : koichi-matsumoto@fintec.co.jp

Abstract

When an asset is completely liquid, an investor can realize his desirable strategy. But when the asset is not sufficiently liquid, the investor cannot trade the asset continuously and his strategy is restricted. He has to consider the risk of the failure of the trade.

In this paper a risky asset is traded at random times. We solve an optimal portfolio problem. And a procedure of an asymptotic expansion of the optimal strategy is proposed. Further we discuss the convergence of the value function when the liquidity of the asset increases.

1 Introduction

As various assets are traded in the market, the liquidity risk becomes more important. There are many studies related with the liquidity risk. For example, a transaction cost has a close relation to the liquidity. Leland[6], Boyle and Vorst[1], Kusuoka[5] analyze a replication strategy of the derivatives, using the transaction cost. Also Subramanian and Jarrow[4] consider a liquidation strategy, using the price impact and the execution delays. Further they modify the standard VaR computation.

In this paper we represent the liquidity by the success rate of the trade and consider an optimal portfolio problem between a risky asset and the saving account in a finite period. We consider the investor who has a log-utility function or a power utility function. The investor can trade an asset at the random times distributed exponentially. When the risky asset is completely liquid, the optimal portfolio problem is solved by Merton[9]. The Merton's optimal strategy is to keep the risky asset ratio constant. In our setting the investor cannot realize the Merton's optimal strategy because he has to trade the risky asset continuously for the Merton's optimal strategy. We consider how the optimal strategy and the value function change when the risky asset becomes less liquid. The following results are shown.

1. The optimal strategy exists and it converges to the Merton's optimal strategy as the liquidity increases.

2. A procedure of an asymptotic expansion of the optimal strategy is given concretely.

3. When the liquidity becomes lower, the utility becomes lower. The value function converges to the Merton's value function as the liquidity increases.

This paper is organized as follows. The next section provides the setting of the market and explains our problem. In Section 3, we discuss the problems when the investor has a log-utility function. In Section 4 we consider the investor whose utility function is a power function.

The author wishes to express my deepest thanks to Prof. Sigeo Kusuoka for the encouragement in this study.

2 Setup

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t; 0 \leq t \leq T\})$ be a filtered probability space satisfying the usual condition. Under P, $\{B(t); 0 \leq t \leq T, B(0) = 0\}$ is a $\{\mathcal{F}_t\}$-Brownian motion and $\{P(t); 0 \leq t \leq T, P(0) = 0\}$ is a $\{\mathcal{F}_t\}$-Poisson process with intensity λ. We denote by $\beta(t)$ the saving account and by $S(t)$ the price of the risky asset. They are assumed to be governed by $\beta(0) = 1$, $S(0) = S_0$,

$$\begin{aligned} d\beta(t) &= r\beta(t)dt, \\ dS(t) &= \mu S(t)dt + \sigma S(t)dB(t) \end{aligned}$$

where S_0, r, μ and σ are positive constants and $r < \mu$. The investor invests a part of wealth in the risky asset and the rest in the safety asset. Let the amount invested in the risky asset be $W_1(t)$ and the amount invested in the safety asset be $W_0(t)$. The investor tries to trade the risky asset worth of $V(t)$ at t but the trade succeeds only at the jump times of the Poisson process. We fix constants w_1, w_0. For any predictable locally bounded process V, we consider the following stochastic differential equations

$$W_0(t) = w_0 + \int_0^t W_0(s-)\frac{d\beta(s)}{\beta(s)} - \int_0^t V(s)dP(s), \tag{2.1}$$

$$W_1(t) = w_1 + \int_0^t W_1(s-)\frac{dS(s)}{S(s)} + \int_0^t V(s)dP(s). \tag{2.2}$$

Then these stochastic differential equations have a unique solution $W_0(t)$, $W_1(t)$ by Theorem 14.6 of Elliot[3].

We say V is an admissible strategy if V satisfies

$$-W_1(t-) \leq V(t) \leq W_0(t-)$$

for $0 \leq t \leq T$. This means that the investor cannot make a short sale of the low liquid risky asset and must not invest more risky asset than his total asset. If V is admissible, then $W_0(t)$ and $W_1(t)$ are nonnegative.

We denote by $W(t)$ the total asset and by $X(t)$ the fraction of the wealth invested in the risky asset, i.e.,

$$\begin{aligned} W(t) &= W_0(t) + W_1(t), \\ X(t) &= \frac{W_1(t)}{W_0(t) + W_1(t)}. \end{aligned}$$

Let $v(t)$ be given by

$$v(t) = \frac{V(t) + W_1(t-)}{W_0(t-) + W_1(t-)}.$$

By the Ito formula, we have

$$W(t) = w_0 + w_1 + \int_0^t W(s-)((\mu - r)X(s-) + r)ds + \int_0^t W(s-)X(s-)\sigma dB(s), \tag{2.3}$$

$$\begin{aligned} X(t) = {} & \frac{w_1}{w_0 + w_1} + \int_0^t X(s-)(1 - X(s-))(\mu - r - \sigma^2 X(s-))ds \\ & + \int_0^t X(s-)(1 - X(s-))\sigma dB(s) + \int_0^t (v(s) - X(s-))dP(s). \end{aligned} \tag{2.4}$$

We define a set of processes by

$$\mathcal{V}[t, T] = \{v | v \text{ is predictable and } 0 \leq v(s) \leq 1 \text{ for } t \leq s \leq T\}.$$

If V is admissible, then $v \in \mathcal{V}[0,T]$.

For any $v \in \mathcal{V}[0,T]$, (2.3) and (2.4) have a unique solution $X(t)$, $W(t)$ by Theorem 14.6 of Elliot[3]. We can show $0 \le X(t) \le 1$. Then

$$\begin{aligned} W_0(t) &= W(t)(1-X(t)), \\ W_1(t) &= W(t)X(t), \\ V(t) &= (v(t)-X(t-))W(t-) \end{aligned}$$

is a solution of (2.1), (2.2) and V is admissble. Therefore there is a one-to-one correspondence between $W_0(t)$, $W_1(t)$, $V(t)$ and $W(t)$, $X(t)$, $v(t)$. Further, V is admissible if and only if $v \in \mathcal{V}[0,T]$. Therefore we consider $W(t)$, $X(t)$, $v(t)$ and we call v a strategy instead of V. When we emphasize that a process depends on v, we denote $W(t)$, $X(t)$ by $W(t;v)$, $X(t;v)$.

For the utility function of investor $U : \boldsymbol{R} \to \boldsymbol{R}$, our problem is to find an optimal strategy v^λ which maximizes $E[U(W(T;v))]$ among $v \in \mathcal{V}[0,T]$ and to analyze the value function given by

$$V^\lambda(t,x,w) = \sup_{v\in\mathcal{V}[t,T]} E[U(W(T;v))|\mathcal{F}_t]|_{(X(t),W(t))=(x,w)}.$$

For preparation, we define $L_n(t) = \int_0^t s^n e^{-s} ds$ for $n \in \boldsymbol{N} \cup \{0\}$. Note that $0 \le L_n(\lambda t) \le n!$ and

$$\frac{L_n(\lambda t)}{L_m(\lambda t)} \le \frac{n!}{m!}, \tag{2.5}$$

for $0 \le m \le n$, since $L_n(\lambda t) = n!\left(1 - e^{-\lambda t}\sum_{i=0}^n (\lambda t)^i/i!\right)$.

3 Log-Utility function

In this section we consider the investor who has a log-utility function. We will prove three theorems. For preparatory steps of these proofs, some lemmas are necessary. Since we aim for concise presentation, we sketch these proofs. Please refer to Matsumoto[7] for the details.

Let $U(W) = \log W$ and

$$x_0 = \frac{\mu - r}{\sigma^2}.$$

We assume that $0 < x_0 < 1$. The Merton's optimal strategy $v^\infty(t)$ and the Merton's value function $V^\infty(t,x,w)$ are given by

$$\begin{aligned} v^\infty(t) &= x_0, \\ V^\infty(t,x,w) &= \log w + \left(r + \frac{(\mu-r)^2}{2\sigma^2}\right)(T-t). \end{aligned}$$

For the details, see Merton[10], Duffie[2], etc.

3.1 Optimal Strategy and Value Function

In this subsection we prove the existence of optimal strategy exists and consider its convergence as the liquidity increases.

Let

$$A^\lambda(t,x) = \int_0^t e^{-\lambda s} K(s,x) ds + \lambda \int_0^t \sup_{0\le y\le 1}\left(\int_0^s e^{-\lambda u} K(u,y) du\right) ds \tag{3.1}$$

where

$$K(t,y) = E[f(Y^y(t))],$$
$$f(y) = (\mu - r)y + r - \frac{1}{2}\sigma^2 y^2,$$
$$Y^y(t) = \frac{yS(t)/S_0}{yS(t)/S_0 + (1-y)\beta(t)}.$$

By the Ito formula we get

$$dY^y(t) = Y^y(t)(1-Y^y(t))(\mu - r - \sigma^2 Y^y(t))dt + Y^y(t)(1-Y^y(t))\sigma dB(t). \quad (3.2)$$

Also if $0 \le y \le 1$, then $0 \le Y^y(t) \le 1$ and specially if $y = 0$ or 1, then $Y^y(t) = y$.

Theorem 3.1 *The optimal strategy exists and the value function is given by*

$$V^\lambda(t,x,w) = \log w + A^\lambda(T-t,x). \quad (3.3)$$

Specially if λ is sufficiently large, an optimal strategy is unique and satisfies

$$|v^\lambda(t) - x_0| \le C\frac{1}{\lambda}, \quad 0 \le t \le T \quad (3.4)$$

for some constant C.

For the preparation of the proof of this theorem, we will show some lemmas.

Lemma 3.1 *The optimal strategy exists. The value function is given by*

$$V^\lambda(t,x,w) = \log w + A^\lambda(T-t,x).$$

Proof. Because $A^\lambda(T-t,x)$ is continuous with respect to x, there exists $\hat{v}(t)$ which satisfies

$$A^\lambda(T-t,\hat{v}(t)) = \sup_{0\le x\le 1} A^\lambda(T-t,x).$$

$\hat{v}$ is a deterministic process.

It can be shown that $\log W(t;v) + A^\lambda(t,X(t;v),W(t;v))$ is a supermartingale for all $v \in \mathcal{V}[0,T]$ and specially $\log W(t;\hat{v}) + A^\lambda(t,X(t;\hat{v}),W(t;\hat{v}))$ is a martingale. Then we have

$$\begin{aligned} E[\log(W(T;v))|\mathcal{F}_t]|_{(X(t),W(t))=(x,w)} &\le \log w + A^\lambda(T-t,x) \\ &= E[\log(W(T;\hat{v}))|\mathcal{F}_t]|_{(X(t),W(t))=(x,w)}. \end{aligned}$$

Therefore $\hat{v}$ is an optimal strategy and the result follows. □

Let

$$g^\lambda(t,x) = \int_0^t e^{-\lambda s}K(s,x)ds = \frac{1}{\lambda}\int_0^{\lambda t} e^{-u}K(\frac{u}{\lambda},x)du,$$
$$K_{i,j} = \frac{\partial^{i+j}K(0,x_0)}{\partial t^i \partial x^j},$$
$$B_n = \sup\left\{\left|\frac{\partial^n K(t,x)}{\partial t^i \partial x^{n-i}}\right| \,\middle|\, 0 \le i \le n,\, 0 \le t \le T,\, 0 \le x \le 1\right\},$$

for $i,j,n \in \boldsymbol{N} \cup \{0\}$. Because the second term of (3.1) does not depend on x, $A^\lambda(t,\cdot)$ has an absolute maximum at the same point as $g^\lambda(t,\cdot)$. It can be shown that $K_{0,1} = 0$, $K_{0,2} = -\sigma^2$, $K_{0,k} = 0 (k \ge 3)$, $K_{1,0} = -\frac{1}{2}\sigma^4 x_0^2(1-x_0)^2$, $K_{1,1} = -\sigma^4 x_0(1-x_0)(1-2x_0)$.

Lemma 3.2 *Suppose that*

$$\lambda \geq \max\left(\frac{H_1+2H_2}{H_3}, 1, \frac{4(B_3+B_4/3)}{|K_{0,2}|}\right)$$

where

$$\begin{aligned} H_1 &= \left|\frac{K_{1,1}}{K_{0,2}}\right| = \sigma^2 x_0(1-x_0)|1-2x_0|, \\ H_2 &= \frac{B_3}{2|K_{0,2}|}(H_1^2+2H_1+2), \\ H_3 &= \min\left(\frac{|K_{0,2}|}{4(B_3+B_4/2)}, 1-x_0, x_0\right). \end{aligned}$$

Then there uniquely exists $h^\lambda(t)$ which satisfies

$$\begin{aligned} \frac{\partial g^\lambda(t, x_0+h^\lambda(t))}{\partial x} &= 0, \\ |h^\lambda(t)| &\leq H_3. \end{aligned}$$

Further $h^\lambda(t)$ satisfies

$$|h^\lambda(t)| \leq (H_1+2H_2)\frac{1}{\lambda}.$$

Proof. By the Taylor's theorem, we get

$$\lambda\frac{\partial g^\lambda(t,x_0+h)}{\partial x} = \sum_{n=0}^{N-1}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\lambda^k}h^{n-k}K_{k,1+n-k}L_k(\lambda t) + S_N(\lambda,t,h,1) \tag{3.5}$$

where

$$\begin{aligned} S_N(\lambda,t,h,l) &= \int_0^{\lambda t} e^{-u}R_N(\frac{u}{\lambda},h,l)du, \\ R_N(t,h,l) &= \sum_{k=0}^{N}\binom{N}{k}t^k h^{N-k}\int_0^1 \frac{(1-s)^{N-1}}{(N-1)!}\frac{\partial^{l+N}K(st,x_0+sh)}{\partial t^k \partial x^{l+N-k}}ds. \end{aligned}$$

Since

$$|R_N(t,h,l)| \leq \sum_{k=0}^{N}\binom{N}{k}t^k|h|^{N-k}B_{N+l}\int_0^1\frac{(1-s)^{N-1}}{(N-1)!}ds = \sum_{k=0}^{N}\binom{N}{k}t^k|h|^{N-k}B_{N+l}\frac{1}{N!},$$

we get

$$|S_N(\lambda,t,h,l)| \leq L_0(\lambda t)\frac{B_{N+l}}{N!}\sum_{k=0}^{N}\binom{N}{k}\frac{1}{\lambda^k}|h|^{N-k}k!. \tag{3.6}$$

Substituting 2 for N in (3.5), we get

$$\frac{\lambda}{K_{0,2}L_0(\lambda t)}\frac{\partial g^\lambda(t,x_0+h)}{\partial x} = h + \frac{1}{\lambda}\frac{K_{1,1}L_1(\lambda t)}{K_{0,2}L_0(\lambda t)} + \frac{S_2(\lambda,t,h,1)}{K_{0,2}L_0(\lambda t)}.$$

Therefore h satisfies $\partial g^\lambda(t,x_0+h)/\partial x = 0$ if only if

$$h + \frac{1}{\lambda}\frac{K_{1,1}L_1(\lambda t)}{K_{0,2}L_0(\lambda t)} + \frac{S_2(\lambda,t,h,1)}{K_{0,2}L_0(\lambda t)} = 0. \tag{3.7}$$

By (3.6) we get

$$|S_2(\lambda,t,h,1)| \leq L_0(\lambda t)\frac{B_3}{2}\left(|h|^2 + 2\frac{1}{\lambda}|h| + 2\frac{1}{\lambda^2}\right).$$

Also we have

$$\left|\frac{\partial S_2(\lambda,t,h,1)}{\partial h}\right| \le L_0(\lambda t)\left(\frac{B_4}{6}|h|^2+\left(B_3+\frac{B_4}{3}\frac{1}{\lambda}\right)\left(|h|+\frac{1}{\lambda}\right)\right) \tag{3.8}$$

since

$$\left|\frac{\partial}{\partial h}R_2(\frac{u}{\lambda},h,1)\right| \le B_3\left(|h|+\frac{u}{\lambda}\right)+\frac{B_4}{6}\left(|h|^2+2|h|\frac{u}{\lambda}+\frac{u^2}{\lambda^2}\right).$$

We solve (3.7) by the successive approximation. Let

$$h_1(t) = -\frac{1}{\lambda}\frac{K_{1,1}L_1(\lambda t)}{K_{0,2}L_0(\lambda t)},\quad h_n(t) = -\frac{1}{\lambda}\frac{K_{1,1}L_1(\lambda t)}{K_{0,2}L_0(\lambda t)} - \frac{S_2(\lambda,t,h_{n-1}(t),1)}{K_{0,2}L_0(\lambda t)},\quad n\ge 2.$$

By (2.5) we get

$$|h_1(t)| \le \frac{1}{\lambda}\left|\frac{K_{1,1}}{K_{0,2}}\right| = \frac{H_1}{\lambda} \le \frac{H_1}{H_1+2H_2}H_3 \le H_3$$

and then

$$|h_2(t)-h_1(t)| = \left|\frac{S_2(\lambda,t,h_1(t),1)}{K_{0,2}L_0(\lambda t)}\right| \le \frac{B_3}{2|K_{0,2}|}\left(\frac{H_1^2}{\lambda^2}+2\frac{H_1}{\lambda^2}+2\frac{1}{\lambda^2}\right) = \frac{H_2}{\lambda^2}.$$

Since $\lambda \ge 1$,

$$|h_2(t)| = |h_2(t)-h_1(t)|+|h_1(t)| \le \frac{H_2}{\lambda^2}+\frac{H_1}{\lambda} \le \frac{H_1+2H_2}{\lambda} \le H_3.$$

We show that for all $n \ge 2$ the following inequalities hold.

$$|h_n(t)-h_{n-1}(t)| \le \frac{1}{2^{n-2}}|h_2(t)-h_1(t)|,$$
$$|h_n(t)| \le \frac{H_1+2H_2}{\lambda} \le H_3.$$

The inequalities hold for $n=2$. Suppose that inequalities hold for all $k=2,\dots,n$. By this assumption and (3.8), we get

$$|h_{n+1}(t)-h_n(t)| \le \left(\frac{(B_3+B_4/2)}{|K_{0,2}|}H_3+\frac{(B_3+B_4/3)}{|K_{0,2}|}\frac{1}{\lambda}\right)|h_n(t)-h_{n-1}(t)| \le \frac{|h_2(t)-h_1(t)|}{2^{n-1}}$$

and

$$|h_{n+1}(t)| \le \sum_{k=2}^{n+1}\frac{1}{2^{k-2}}|h_2(t)-h_1(t)|+|h_1(t)| \le \frac{2H_2}{\lambda^2}+H_1\frac{1}{\lambda} \le \frac{H_1+2H_2}{\lambda} \le H_3.$$

Therefore the inequalities hold for all $n \ge 2$. Then $h_n(t)$ has the limit $h(t)$ satisfying

$$\frac{\partial g^\lambda(t,x_0+h(t))}{\partial x} = 0,$$
$$|h(t)| \le \frac{H_1+2H_2}{\lambda} \le H_3.$$

Suppose that $\tilde{h}(t)$ is a second solution which satisfies $\frac{\partial g^\lambda(t,x_0+\tilde{h}(t))}{\partial x} = 0$, $|\tilde{h}(t)| \le H_3$. In the similar way to the above argument we can show

$$|h(t)-\tilde{h}(t)| = \left|\frac{S_2(\lambda,t,h(t),1)}{K_{0,2}L_0(\lambda t)} - \frac{S_2(\lambda,t,\tilde{h}(t),1)}{K_{0,2}L_0(\lambda t)}\right| \le \frac{1}{2}|h(t)-\tilde{h}(t)|.$$

Therefore $h(t)$ is a unique solution and the results follows. □

By the definition of $g^\lambda(t,x)$ the following lemma can be shown.

Lemma 3.3 *For all $0 \le t \le T$ and $0 \le x \le 1$*

$$\left| \frac{\lambda g^\lambda(t,x)}{L_0(\lambda t)} - f(x) \right| \le B_1 \frac{1}{\lambda},$$

that is, $\lambda g^\lambda(t,x)/L_0(\lambda t)$ converges to $f(x)$ uniformly when λ tends to ∞.

By Lemma 3.1 we have already shown the first half of Theorem 3.1. We prove the latter half.

Proof of Theorem 3.1. Let

$$\epsilon_1 = \frac{1}{2}\sigma^2 H_3^2 = \sup_{|x-x_0|\le H_3} f(x_0) - f(x)$$

If $\lambda \ge H_4 = 2B_1/\epsilon_1$, by Lemma 3.3 we obtain

$$\left| \frac{\lambda g^\lambda(t,x)}{L_0(\lambda t)} - f(x) \right| \le \frac{\epsilon_1}{2}.$$

Suppose that

$$\lambda \ge \max\left(\frac{H_1 + 2H_2}{H_3}, 1, \frac{4(B_3 + B_4/3)}{|K_{0,2}|}, H_4 \right).$$

If $|x - x_0| > H_3$, then $f(x_0) - f(x) > \epsilon_1$ and then

$$\frac{\lambda g^\lambda(t,x_0)}{L_0(\lambda t)} - \frac{\lambda g^\lambda(t,x)}{L_0(\lambda t)} > \left(f(x_0) - \frac{1}{2}\epsilon_1\right) - \left(f(x) + \frac{1}{2}\epsilon_1\right) > (f(x_0) - f(x)) - \epsilon_1 > 0.$$

Therefore $\lambda g^\lambda(t,x)/L_0(\lambda t)$ has a maximum in $|x - x_0| \le H_3$. By Lemma 3.2, $x_0 + h^\lambda(t)$ is a unique extreme point of $g^\lambda(t,x)$ in $|x - x_0| \le H_3$. The result follows. □

Remark 3.1 *By the above proof, the optimal strategy can be represented by*

$$v^\lambda(T-t) = x_0 + h^\lambda(t),$$

when λ is sufficiently large.

3.2 Asymptotic Expansion of the Optimal Strategy

In this subsection we show the asymptotic expansion of the optimal strategy.

When λ is sufficiently large, we have

$$\left| \sum_{n=0}^{N-1} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\lambda^k} h^\lambda(t)^{n-k} K_{k,1+n-k} L_k(\lambda t) \right| \le \frac{\tilde{C}_{N-1} L_0(\lambda t)}{\lambda^N} \tag{3.9}$$

for some constant $\tilde{C}_{N-1}$ by Theorem 3.1, (3.5) and (3.6).

Let

$$G(z,\psi) = \psi + \sum_{i=1}^{\infty} \sum_{k=1}^{i} \gamma_{i,k} z^k \psi^{i-k}$$

where $\gamma_{i,k}$, $1 \le k \le i$ are constants. We seek a formal power series of

$$\Psi(z) = \sum_{j=1}^{\infty} \phi_j^* z^j, \quad \text{such that} \quad G(z, \Psi(z)) = 0.$$

The solution of this problem is given by solving the equations in terms of the coefficients of

$$\Psi(z) + \sum_{i=1}^{\infty} \sum_{k=1}^{i} \gamma_{i,k} z^k \Psi(z)^{i-k} = 0.$$

These equations are of the form

$$\phi_1^* = -\gamma_{1,1}, \quad \phi_j^* = -P_j(\gamma_{i,k}, \phi_l^* : 1 \le k \le i \le j, l \le j-1) \ (j \ge 2)$$

where P_j is a polynomial with positive integer coefficients. Therefore we can solve recursively for the coefficients $\phi_j^*, j \ge 1$ and they are uniquely determined.

We define $h_j^*(t)$, by replacing $\gamma_{i,k}$ in ϕ_j^* by

$$\begin{pmatrix} i \\ k \end{pmatrix} \frac{1}{i!} \frac{K_{k,i-k+1}}{K_{0,2}} \frac{L_k(t)}{L_0(t)}.$$

By the procedure for making h_1^*, the following proposition can be proved.

Proposition 3.1 *Let $h_1^* : [0,\infty) \to \boldsymbol{R}$ be given by*

$$h_1^*(t) = -\frac{K_{1,1}L_1(t)}{K_{0,2}L_0(t)} = -\sigma^2 x_0(1-x_0)(1-2x_0)\frac{L_1(t)}{L_0(t)}, \quad t > 0$$

and $h_1^*(0) = h_1^*(0+)$. Then h_1^* is bounded and continuous. If λ is sufficiently large, there exists a constant C_1 such that

$$\left| h^\lambda(t) - h_1^*(\lambda t)\frac{1}{\lambda} \right| \le C_1 \frac{1}{\lambda^2}, \quad 0 \le t \le T, \tag{3.10}$$

$$\left| \frac{\partial}{\partial x} A^\lambda(t, x_0 + h_1^*(\lambda t)\frac{1}{\lambda}) \right| \le C_1 \frac{1}{\lambda^3}, \quad 0 \le t \le T, \tag{3.11}$$

$$\left| A^\lambda(t, x_0 + h^\lambda(t)) - A^\lambda(t, x_0 + h_1^*(\lambda t)\frac{1}{\lambda}) \right| \le C_1 \frac{1}{\lambda^5}, \quad 0 \le t \le T. \tag{3.12}$$

By the mathematical induction, we prove the following theorem.

Theorem 3.2 *There exist bounded continuous functions $h_i^* : [0,\infty) \to \boldsymbol{R}$, $i \ge 1$ such that, for all $n \in \boldsymbol{N}$, there exist $C_n > 0$ and $\lambda_n > 0$ satisfying*

$$\left| v^\lambda(T-t) - \left(x_0 + \sum_{i=1}^n \frac{h_i^*(\lambda t)}{\lambda^i} \right) \right| \le C_n \frac{1}{\lambda^{n+1}}, \quad 0 \le t \le T, \lambda \ge \lambda_n,$$

$$\left| \frac{\partial}{\partial x} V^\lambda(T-t, x_0 + \sum_{i=1}^n \frac{h_i^*(\lambda t)}{\lambda^i}, w) \right| \le C_n \frac{1}{\lambda^{n+2}}, \quad 0 \le t \le T, \lambda \ge \lambda_n,$$

$$\left| V^\lambda(T-t, v^\lambda(T-t), w) - V^\lambda(T-t, x_0 + \sum_{i=1}^n \frac{h_i^*(\lambda t)}{\lambda^i}, w) \right| \le C_n \frac{1}{\lambda^{2n+3}}, \quad 0 \le t \le T, \lambda \ge \lambda_n.$$

Proof. For $n = 1$ it reduces to Proposition 3.1. Suppose that the assertion holds for $n \le N$. By (2.5), we have

$$\left| \begin{pmatrix} i \\ k \end{pmatrix} \frac{1}{i!} \frac{K_{k,i-k+1}}{K_{0,2}} \frac{L_k(t)}{L_0(t)} \right| \le \left| \frac{K_{k,i-k+1}}{K_{0,2}} \right| \frac{1}{(i-k)!} \le \left| \frac{K_{k,i-k+1}}{K_{0,2}} \right|$$

for $1 \le k \le i$. Since $h_n^*(t)$ is a polynomial of

$$\begin{pmatrix} i \\ k \end{pmatrix} \frac{1}{i!} \frac{K_{k,i-k+1}}{K_{0,2}} \frac{L_k(t)}{L_0(t)}, \quad 1 \le k \le i \le n,$$

h_n^* is bounded and continuous. By the definition of h_n^*,

$$\left| \sum_{j=0}^{N+1} \frac{1}{j!} \sum_{k=0}^{j} \begin{pmatrix} j \\ k \end{pmatrix} \frac{1}{\lambda^k} \left(\sum_{i=1}^{N+1} \frac{h_i^*(\lambda t)}{\lambda^i} \right)^{j-k} K_{k,1+j-k} L_k(\lambda t) \right| \le \frac{\hat{C}_{N+1} L_0(\lambda t)}{\lambda^{N+2}} \tag{3.13}$$

for some constant $\hat{C}_{N+1}$. By (3.9), the induction hypothesis and (3.13), we get

$$\left|\left(h^\lambda(t) - \sum_{i=1}^{N+1}\frac{h_i^*(\lambda t)}{\lambda^i}\right)K_{0,2}L_0(\lambda t)\right| \le \left|\sum_{j=0}^{N+1}\frac{1}{j!}\sum_{k=0}^{j}\begin{pmatrix} j \\ k \end{pmatrix}\frac{1}{\lambda^k}h^\lambda(t)^{j-k}K_{k,1+j-k}L_k(\lambda t)\right|$$

$$+\left|\sum_{j=2}^{N+1}\frac{1}{j!}\sum_{k=1}^{j}\begin{pmatrix} j \\ k \end{pmatrix}\frac{1}{\lambda^k}\left(h^\lambda(t)^{j-k} - \left(\sum_{i=1}^{N+1}\frac{h_i^*(\lambda t)}{\lambda^i}\right)^{j-k}\right)K_{k,1+j-k}L_k(\lambda t)\right|$$

$$+\left|\sum_{j=0}^{N+1}\frac{1}{j!}\sum_{k=0}^{j}\begin{pmatrix} j \\ k \end{pmatrix}\frac{1}{\lambda^k}\left(\sum_{i=1}^{N+1}\frac{h_i^*(\lambda t)}{\lambda^i}\right)^{j-k}K_{k,1+j-k}L_k(\lambda t)\right| \le \frac{\hat{C}}{\lambda^{N+2}}L_0(\lambda t)$$

for some constant $\hat{C}$. Therefore we obtain

$$\left|h^\lambda(t) - \sum_{i=1}^{N+1}\frac{h_i^*(\lambda t)}{\lambda^i}\right| \le \frac{\hat{C}}{|K_{0,2}|}\frac{1}{\lambda^{N+2}}.$$

When $|h^\lambda(t) - h(t)| \le C_{N+1}/\lambda^{N+2}$,

$$\left|\lambda\frac{\partial g^\lambda(t, x_0 + h(t))}{\partial x}\right| = \left|\lambda\frac{\partial g^\lambda(t, x_0 + h(t))}{\partial x} - \lambda\frac{\partial g^\lambda(t, x_0 + h^\lambda(t))}{\partial x}\right|$$

$$\le \left|\sum_{j=0}^{N+1}\frac{1}{j!}\sum_{k=0}^{j}\begin{pmatrix} j \\ k \end{pmatrix}\frac{h^\lambda(t)^{j-k} - h(t)^{j-k}}{\lambda^k}K_{k,1+j-k}L_k(\lambda t)\right| + \frac{C'_{N+1}L_0(\lambda t)}{\lambda^{N+2}} \le \frac{C''_{N+1}L_0(\lambda t)}{\lambda^{N+2}}$$

for some constants C'_{N+1} and C''_{N+1}. Therefore we obtain

$$\left|\frac{\partial}{\partial x}A^\lambda(t, x_0 + \sum_{i=1}^{N+1}\frac{h_i^*(\lambda t)}{\lambda^i})\right| \le \frac{(\hat{C}' + \hat{C}_{N+1})L_0(\lambda t)}{\lambda^{N+3}},$$

$$\left|A^\lambda(t, x_0 + h^\lambda(t)) - A^\lambda(t, x_0 + \sum_{i=1}^{N+1}\frac{h_i^*(\lambda t)}{\lambda^i})\right| \le \frac{\hat{C}C''_{N+1}L_0(\lambda t)}{|K_{0,2}|\lambda^{2(N+1)+3}}.$$

By the above argument, the assertion for $n = N+1$ holds and then the result follows. □

3.3 Limit of the Value Function

We have shown that the optimal strategy converges to the Merton's strategy when λ tends to ∞. In this subsection we discuss the limit of the value function.

The following lemma can be proved similarly to Theorem 3.2.

Lemma 3.4 *When λ is sufficiently large, there exists a constant C_0 such that*

$$\left|\frac{\partial A^\lambda(t, x_0)}{\partial x}\right| \le C_0\frac{1}{\lambda^2}, \quad 0 \le t \le T,$$

$$|A^\lambda(t, x_0 + h^\lambda(t)) - A^\lambda(t, x_0)| \le C_0\frac{1}{\lambda^3}, \quad 0 \le t \le T.$$

By Lemma 3.4 we get

$$\left|A^\lambda(T-t, x) - \left(K(0, x_0)(T-t) + \frac{1}{\lambda}\left(K(0, x) - K(0, x_0) + \frac{\partial K(0, x_0)}{\partial t}(T-t)\right)\right)\right|$$

$$\le \left|-\frac{1}{\lambda}e^{-\lambda(T-t)}(K(T-t, x) - K(T-t, x_0)) + \frac{1}{\lambda}\int_0^{T-t}e^{-\lambda s}\left(\frac{\partial K(s, x)}{\partial t} - 2\frac{\partial K(s, x_0)}{\partial t}\right)ds\right.$$

$$\left.+\int_0^{T-t}\int_0^{s}\frac{1}{\lambda}e^{-\lambda u}\frac{\partial^2 K(u, x_0)}{\partial t^2}duds\right| + \hat{C}\frac{1}{\lambda^2}$$

for some constant $\hat{C}$. Therefore we get the following consequence.

Theorem 3.3 *For* $0 \le t \le T$,

$$V^\lambda(t,x,w) \to V^\infty(t,x,w) \tag{3.14}$$

and

$$\lambda(V^\infty(t,x,w) - V^\lambda(t,x,w)) \to \frac{1}{2}\sigma^2(x-x_0)^2 + \frac{1}{2}\sigma^4 x_0^2(1-x_0)^2(T-t) \tag{3.15}$$

as $\lambda \to \infty$ *uniformly in* $0 \le x \le 1$.

4 Power Utility Function

In this section the investor has a power utility function. Since we aim for concise presentation, we sketch the proofs. Please refer to Matsumoto[8] for the details.

Let $U(W) = W^\alpha$ for fixed $0 < \alpha < 1$ and

$$x_\alpha = \frac{\mu - r}{(1-\alpha)\sigma^2}.$$

We assume that $0 < x_\alpha < 1$. The Merton's optimal strategy $v^\infty(t)$ and the Merton's value function $V^\infty(t,x,w)$ are given by

$$\begin{aligned} v^\infty(t) &= x_\alpha, \\ V^\infty(t,x,w) &= w^\alpha \exp\left(\alpha\left(r + \frac{(\mu-r)^2}{2(1-\alpha)\sigma^2}\right)(T-t)\right). \end{aligned}$$

For the details, see Merton[10], Duffie[2], etc.

4.1 Value Function and Optimal Strategy

In this subsection, we show the existence and the uniqueness of the optimal strategy.

Let $A^\lambda(t,x)$ be a solution of

$$A^\lambda(t,x) = \int_0^t D(t-s,x)e^{-\lambda(t-s)}\lambda\tilde{A}^\lambda(s)ds + D(t,x)e^{-\lambda t} \tag{4.1}$$

where

$$\begin{aligned} &\tilde{A}^\lambda(t) = \sup_{0\le x\le 1} A^\lambda(t,x), \\ &D(t,y) = E\left[e^{\int_0^t f(Y^y(s))ds}\right], \\ &f(y) = \alpha(\mu-r)y + \alpha r - \frac{1}{2}\alpha(1-\alpha)\sigma^2 y^2 \end{aligned}$$

and $Y^y(t)$ is a solution of

$$Y^y(t) = y + \int_0^t Y^y(s)(1-Y^y(s))(\mu - r - \sigma^2(1-\alpha)Y^y(s))ds + \int_0^t Y^y(s)(1-Y^y(s))\sigma dB(s).$$

Note that $Y^y(t)$ has a unique solution by Theorem 14.6 of Elliot[3]. If $0 \le y \le 1$, then $0 \le Y^y(t) \le 1$. Specially if $y = 0$ or 1, then $Y^y(t) = y$. By the definition, $D(0,x) = 1$ and $\partial D(0,x)/\partial t = f(x)$. Since $f(x) \le f(x_\alpha)$, we have

$$\begin{aligned} 0 < e^{\min(f(0),f(1))t} \le D(t,x) \le e^{f(x_\alpha)t}, \\ \min(f(0),f(1)) \le \frac{\partial D(0,x)}{\partial t} \le f(x_\alpha). \end{aligned} \tag{4.2}$$

By the successive approximation, the following lemma can be shown.

Lemma 4.1 *There exists a unique solution $\tilde{A}^\lambda(t)$ of*

$$\tilde{A}^\lambda(t) = \sup_{0\le x\le 1} \int_0^t D(t-s,x)e^{-\lambda(t-s)}\lambda\tilde{A}^\lambda(s)ds + D(t,x)e^{-\lambda t}. \tag{4.3}$$

Further $\tilde{A}^\lambda(t)$ satisfies

$$0 \le \tilde{A}^\lambda(t) \le e^{f(x_\alpha)t}. \tag{4.4}$$

By Lemma 4.1, (4.1) has a unique solution.

It can be shown that $W(t;v)^\alpha A^\lambda(T-t, X(t;v))$ is a supermartingale for $v \in \mathcal{V}[0,T]$. The following lemma can be proved similarly to Lemma 3.1.

Lemma 4.2 *The optimal strategy exists and the value function is given by*

$$V^\lambda(t,x,w) = w^\alpha A^\lambda(T-t,x).$$

By Lemmas 4.1 and 4.2, we have $w^\alpha \tilde{A}^\lambda(t) \ge w^\alpha e^{\alpha r t}$ and then

$$\tilde{A}^\lambda(t) \ge e^{\alpha r t} \ge 1. \tag{4.5}$$

Let

$$B_n = \sup\left\{ \left|\frac{\partial^n D(t,x)}{\partial t^i \partial x^{n-i}}\right| \middle| 0\le i\le n,\ 0\le t\le T,\ 0\le x\le 1\right\},$$

$$M_n(\lambda,t) = \int_0^{\lambda t} s^n e^{-s}(\tilde{A}^\lambda(t-\frac{s}{\lambda})-1)ds$$

for $i,j,n \ge 0$. Also $g^\lambda(t,x)$ is defined by

$$g^\lambda(t,x) = \frac{\lambda(A^\lambda(t,x) - M_0(\lambda,t) - 1)}{M_1(\lambda,t) + L_0(\lambda t)}. \tag{4.6}$$

$g^\lambda(t,x)$ has an absolute maximum at the same point as $A^\lambda(t,x)$. By (4.5) and Lemma 4.1,

$$0 \le M_n(\lambda,t) \le (e^{f(x_\alpha)t}-1)L_n(\lambda t). \tag{4.7}$$

Lemma 4.3 *Suppose that*

$$\lambda \ge \max\left(2H_2, \frac{2H_1}{H_3}\right)$$

where

$$\begin{aligned} H_1 &= \frac{1}{\alpha(1-\alpha)\sigma^2}B_3 e^{f(x_\alpha)T}, \\ H_2 &= \frac{1}{\alpha(1-\alpha)\sigma^2}B_4 e^{f(x_\alpha)T}, \\ H_3 &= \min\left(1-x_\alpha, x_\alpha\right). \end{aligned}$$

Then there uniquely exists $h^\lambda(t)$ which satisfies

$$\frac{\partial g^\lambda(t, x_\alpha + h^\lambda(t))}{\partial x} = 0,$$

$$h^\lambda(t) \le H_3.$$

Further $h^\lambda(t)$ satisfies

$$h^\lambda(t) \le 2H_1\frac{1}{\lambda}.$$

Proof. By (4.1)

$$A^\lambda(t,x) = \int_0^{\lambda t} D(\frac{u}{\lambda},x)e^{-u}\left(\tilde{A}^\lambda(t-\frac{u}{\lambda})-1\right)du + \frac{1}{\lambda}\int_0^{\lambda t}\frac{\partial}{\partial t}D(\frac{u}{\lambda},x)e^{-u}du + 1. \tag{4.8}$$

By Taylor's theorem, we have

$$\begin{aligned} A^\lambda(t,x) &= M_0(\lambda,t) + \frac{1}{\lambda}(M_1(\lambda,t)+L_0(\lambda t))f(x) \\ &+\frac{1}{\lambda^2}\int_0^{\lambda t} u^2 e^{-u}\left(\tilde{A}^\lambda(t-\frac{u}{\lambda})-1\right)\int_0^1 (1-s)\frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu \\ &+\frac{1}{\lambda^2}\int_0^{\lambda t} ue^{-u}\int_0^1 \frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu + 1. \end{aligned}$$

By (4.6) we have

$$g^\lambda(t,x) = f(x) + \frac{1}{\lambda}(g_1^\lambda(t,x)+g_2^\lambda(t,x)) \tag{4.9}$$

where

$$\begin{aligned} g_1^\lambda(t,x) &= \int_0^{\lambda t} u^2 e^{-u}\left(\tilde{A}^\lambda(t-\frac{u}{\lambda})-1\right)\int_0^1 (1-s)\frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu \Big/ (M_1(\lambda,t)+L_0(\lambda t)), \\ g_2^\lambda(t,x) &= \int_0^{\lambda t} ue^{-u}\int_0^1 \frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu \Big/ (M_1(\lambda,t)+L_0(\lambda t)). \end{aligned}$$

Differentiating $g^\lambda(t,x)$ with respect to x and substituting $x = x_\alpha + h$,

$$\frac{\partial g^\lambda(t,x_\alpha+h)}{\partial x} = -h\alpha(1-\alpha)\sigma^2 + \frac{1}{\lambda}\left(\frac{\partial g_1^\lambda(t,x_\alpha+h)}{\partial x} + \frac{\partial g_2^\lambda(t,x_\alpha+h)}{\partial x}\right).$$

By (4.7) and (2.5), we have

$$\begin{aligned} \left|\frac{\partial g_1^\lambda(t,x)}{\partial x}\right| &\le \frac{B_3}{2}\frac{M_2(\lambda,t)}{M_1(\lambda,t)+L_0(\lambda t)} \le \frac{B_3}{2}\frac{(e^{f(x_\alpha)t}-1)L_2(\lambda t)}{L_0(\lambda t)} \le B_3(e^{f(x_\alpha)t}-1), \\ \left|\frac{\partial g_2^\lambda(t,x)}{\partial x}\right| &\le B_3\frac{L_1(\lambda t)}{M_1(\lambda,t)+L_0(\lambda t)} \le B_3. \end{aligned}$$

Similarly we get

$$\left|\frac{\partial^2 g_1^\lambda(t,x)}{\partial x^2} + \frac{\partial^2 g_2^\lambda(t,x)}{\partial x^2}\right| \le B_4 e^{f(x_\alpha)t}. \tag{4.10}$$

We solve

$$\frac{\partial g^\lambda(t,x_\alpha+h)}{\partial x} = 0$$

by the successive approximation. Let

$$h_1(t) = 0,\ \ h_n(t) = \frac{1}{\alpha(1-\alpha)\sigma^2}\left(\frac{\partial g_1^\lambda(t,x_\alpha+h_{n-1}(t))}{\partial x} + \frac{\partial g_2^\lambda(t,x_\alpha+h_{n-1}(t))}{\partial x}\right)\frac{1}{\lambda},\ \ n\ge 2.$$

Then

$$|h_2(t)-h_1(t)| = \left|\frac{1}{\alpha(1-\alpha)\sigma^2}\left(\frac{\partial g_1^\lambda(t,x_\alpha)}{\partial x} + \frac{\partial g_2^\lambda(t,x_\alpha)}{\partial x}\right)\frac{1}{\lambda}\right| \le \frac{B_3 e^{f(x_\alpha)t}}{\alpha(1-\alpha)\sigma^2}\frac{1}{\lambda} \le H_1\frac{1}{\lambda}.$$

We show that for all $n \ge 2$ the following inequality holds.

$$|h_n(t)-h_{n-1}(t)| \le \frac{1}{2^{n-2}}|h_2(t)|.$$

The inequality holds for $n=2$. Suppose that the inequality holds for all $k=2,\dots,n$. By this assumption and (4.10), we get

$$|h_{n+1}(t)-h_n(t)| \le \frac{B_4 e^{f(x_\alpha)t}}{\alpha(1-\alpha)\sigma^2}\frac{1}{\lambda}|h_n(t)-h_{n-1}(t)| \le \frac{H_2}{\lambda}|h_n(t)-h_{n-1}(t)| \le \frac{1}{2^{n-1}}|h_2(t)|.$$

Therefore the inequality holds for all $n \ge 2$. Since

$$|h_n(t)| \le \sum_{k=2}^{n}|h_k(t)-h_{k-1}(t)|+|h_1(t)| \le 2|h_2(t)| \le 2H_1\frac{1}{\lambda},$$

we obtain for all $n \ge 2$,

$$|h_n(t)| \le 2H_1\frac{1}{\lambda} \le H_3.$$

Then $h_n(t)$ has the limit $h(t)$ satisfying

$$\frac{\partial g^\lambda(t,x_\alpha+h(t))}{\partial x}=0,$$
$$|h(t)| \le 2H_1\frac{1}{\lambda} \le H_3.$$

Suppose that $\tilde{h}(t)$ is a second solution which satisfies

$$\frac{\partial g^\lambda(t,x_\alpha+\tilde{h}(t))}{\partial x}=0,$$
$$|\tilde{h}(t)| \le H_3.$$

By (4.10) we obtain

$$|h(t)-\tilde{h}(t)| \le \frac{B_4 e^{f(x_\alpha)t}}{\alpha(1-\alpha)\sigma^2}\frac{1}{\lambda}|h(t)-\tilde{h}(t)| \le \frac{H_2}{\lambda}|h(t)-\tilde{h}(t)| \le \frac{1}{2}|h(t)-\tilde{h}(t)|.$$

Therefore $h(t)$ is a unique solution and the results follow. □

Similarly to (4.10) we have $\left|g_1^\lambda(t,x)+g_2^\lambda(t,x)\right| \le B_2 e^{f(x_\alpha)t}$. By (4.9) and this inequality, the following lemma can be proved.

Lemma 4.4 *For all* $0 \le t \le T$ *and* $0 \le x \le 1$

$$\left|g^\lambda(t,x)-f(x)\right| \le B_2 e^{f(x_\alpha)T}\frac{1}{\lambda},$$

that is, $g^\lambda(t,x)$ *converges to* $f(x)$ *uniformly when* λ *tends to* ∞.

By Lemma 4.2 we have shown the first half of the following theorem. The latter half can be proved by the above lemma similarly to Theorem 3.1.

Theorem 4.1 *The optimal strategy exists and the value function is given by*

$$V^\lambda(t,x,w) = w^\alpha A^\lambda(T-t,x).$$

Specially if λ *is sufficiently large, an optimal strategy is unique and satisfies*

$$|v^\lambda(t)-x_\alpha| \le C_0\frac{1}{\lambda}, \quad 0 \le t \le T$$

where C_0 *is some constant.*

Remark 4.1 *When* λ *is sufficiently large, the optimal strategy can be represented by*

$$v^\lambda(T-t)=x_\alpha+h^\lambda(t).$$

4.2 Asymptotic Expansion of the Optimal Strategy

In this subsection we propose a procedure of an asymptotic expansion of the optimal strategy.

Let

$$D_{i,j} = \frac{\partial^{i+j} D(0, x_\alpha)}{\partial t^i \partial x^j}.$$

Then $D_{0,0} = 1$, $D_{0,k} = 0 (k \geq 1)$, $D_{1,0} = \frac{\alpha}{2(1-\alpha)} \frac{(\mu - r)^2}{\sigma^2} + \alpha r$, $D_{1,1} = 0$, $D_{1,2} = -\alpha(1-\alpha)\sigma^2$, $D_{1,k} = 0 (k \geq 3)$, $D_{2,0} = \left(\frac{\alpha}{2(1-\alpha)} \frac{(\mu-r)^2}{\sigma^2} + \alpha r \right)^2 + \frac{1}{2}\alpha(\alpha-1)x_\alpha^2(1-x_\alpha)^2\sigma^4$, $D_{2,1} = \alpha(\alpha-1)x_\alpha(1-x_\alpha)(1-2x_\alpha)\sigma^4$.

Differentiating (4.8) with respect to x, we have by Taylor's theorem

$$\begin{aligned} \frac{\partial A^\lambda(t, x_\alpha + h)}{\partial x} &= \sum_{n=2}^{N} \sum_{k=1}^{n} \frac{h^{n-k}}{\lambda^k} D_{k,n-k+1} \left(\binom{n}{k} \frac{M_k(\lambda, t)}{n!} + \binom{n-1}{k-1} \frac{L_{k-1}(\lambda t)}{(n-1)!} \right) \\ &\quad + S_N(\lambda, t, h, 1) \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} S_N(\lambda, t, h, l) &= \int_0^{\lambda t} R_{N+1}(\frac{u}{\lambda}, h, 0, l) e^{-u} \left(\tilde{A}^\lambda(t - \frac{u}{\lambda}) - 1 \right) du + \frac{1}{\lambda} \int_0^{\lambda t} R_N(\frac{u}{\lambda}, h, 1, l) e^{-u} du, \\ R_N(t, h, m, l) &= \sum_{k=0}^{N} \binom{N}{k} t^k h^{N-k} \int_0^1 \frac{(1-s)^{N-1}}{(N-1)!} \frac{\partial^{m+l+N} D(st, x_\alpha + sh)}{\partial t^{m+k} \partial x^{l+N-k}} ds \end{aligned}$$

for $l, m, N \in \mathbf{N}$ and $0 \leq x_\alpha + h \leq 1$.

Since

$$|R_N(t, h, m, l)| \leq \sum_{k=0}^{N} \binom{N}{k} t^k |h|^{N-k} B_{N+m+l} \frac{1}{N!},$$

we get by (4.7) and (2.5)

$$\begin{aligned} &\left| \frac{S_N(\lambda, t, h, l)}{L_0(\lambda t)} \right| \\ &\leq \frac{B_{N+l}}{(N+1)!} \sum_{k=0}^{N+1} \binom{N+1}{k} \frac{|h|^{N+1-k} (e^{f(x_\alpha)t} - 1) k!}{\lambda^k} + \frac{B_{N+l+1}}{N!} \sum_{k=0}^{N} \binom{N}{k} \frac{|h|^{N-k} k!}{\lambda^{k+1}}. \end{aligned} \tag{4.12}$$

Suppose that $T_1, T_2, \ldots$ are independently uniformly distributed in $[0, 1]$ under P. Let

$$\begin{aligned} J^\lambda(t; h) &= e^{-\lambda t} \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!} F(n+1, t; h), \\ F(n, t; h) &= E_n[\prod_{i=1}^{n} D(tT_i, x_\alpha + h(t\hat{T}_i))] \end{aligned}$$

where $x_\alpha + h \in \mathcal{V}[0, T]$, $\hat{T}_i = 1 - \sum_{j=1}^{i-1} T_j$ and $E_n[\cdot] = E[\cdot | \sum_{i=1}^{n} T_i = 1]$. By (4.2),

$$0 < e^{\min(f(0), f(1))t} \leq F(n, t; h) \leq e^{f(x_\alpha)t}$$

and then

$$0 < e^{\min(f(0), f(1))t} \leq J^\lambda(t; h) \leq e^{f(x_\alpha)t}. \tag{4.13}$$

By the definition

$$\tilde{A}^\lambda(t) = \int_0^t D(t-s, x_\alpha + h^\lambda(t)) e^{-\lambda(t-s)} \lambda \tilde{A}^\lambda(s) ds + D(t, x_\alpha + h^\lambda(t)) e^{-\lambda t}.$$

Therefore we get for all $N \in \boldsymbol{N}$,

$$\begin{aligned}
&e^{\lambda t_0}\tilde{A}^\lambda(t_0) = D(t_0, x_\alpha + h^\lambda(t_0)) \\
&+\sum_{n=1}^{N}\lambda^n\int_0^{t_0}\cdots\int_0^{t_{n-1}}\left(\prod_{i=0}^{n-1}D(t_i - t_{i+1}, x_\alpha + h^\lambda(t_i))\right)D(t_n, x_\alpha + h^\lambda(t_n))dt_n\cdots dt_1 \\
&+\lambda^{N+1}\int_0^{t_0}\cdots\int_0^{t_N}\left(\prod_{i=0}^{N}D(t_i - t_{i+1}, x_\alpha + h^\lambda(t_i))\right)e^{\lambda t_{N+1}}\tilde{A}^\lambda(t_{N+1})dt_{N+1}\cdots dt_1.
\end{aligned}$$

The following lemma can be proved.

Lemma 4.5 *$\tilde{A}^\lambda(t)$ satisfies*

$$\tilde{A}^\lambda(t) = J^\lambda(t; h^\lambda). \tag{4.14}$$

By Taylor's theorem,

$$\log D(t, x_\alpha + h) \quad = \quad \left(D_{1,0} + \frac{1}{2}D_{1,2}h^2\right)t + Z(t, x_\alpha + h)t^2$$

where

$$Z(t,x) = \int_0^1 (1-s)\left(\frac{\partial^2 D(st,x)}{\partial t^2}D(st,x) - \left(\frac{\partial D(st,x)}{\partial t}\right)^2\right)\bigg/ D(st,x)^2 ds.$$

Let

$$Z_n = \sup\left\{\left|\frac{\partial^n Z(t,x)}{\partial t^i \partial x^{n-i}}\right| \middle| 0 \le i \le n, 0 \le t \le T, 0 \le x \le 1\right\}.$$

Lemma 4.6 *Let*

$$h_0^*(t) = 0$$

for $0 \le t \le T$. Then

$$\left|J^\lambda(t; h_0^*) - e^{f(x_\alpha)t}\right| \le C\frac{1}{\lambda}, \quad 0 \le t \le T,\ \lambda \ge \lambda_0, \tag{4.15}$$

$$\left|\tilde{A}^\lambda(t) - J^\lambda(t; h_0^*)\right| \le C\frac{1}{\lambda^2}, \quad 0 \le t \le T,\ \lambda \ge \lambda_0, \tag{4.16}$$

$$\left|\tilde{A}^\lambda(t) - e^{f(x_\alpha)t}\right| \le C\frac{1}{\lambda}, \quad 0 \le t \le T,\ \lambda \ge \lambda_0 \tag{4.17}$$

for some constants C and λ_0.

Proof. By the definition,

$$F(n,t;h) \quad = \quad e^{D_{1,0}t}E_n[\exp\left(\sum_{i=1}^{n}\frac{1}{2}D_{1,2}h(tT_i)^2 tT_i + Z(tT_i, x_\alpha + h(tT_i))t^2T_i^2\right)].$$

Since

$$|e^{D_{1,0}t} - F(n,t;h_0^*)| \quad \le \quad e^{D_{1,0}t + Z_0t^2}Z_0t^2E_n[\sum_{i=1}^{n}T_i^2] = e^{D_{1,0}t+Z_0t^2}Z_0t^2\frac{2}{n+1},$$

we get

$$|e^{D_{1,0}t} - J^\lambda(t;h_0^*)| \quad \le \quad e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}|e^{D_{1,0}t} - F(n+1,t;h_0^*)| \le 2e^{D_{1,0}t+Z_0t^2}Z_0\frac{t}{\lambda}$$

and then yields (4.15). For all $x_\alpha + h_0,\ x_\alpha + h_1 \in \mathcal{V}[0,T]$,

$$
\begin{aligned}
&|F(n,t;h_1) - F(n,t;h_0)| \qquad (4.18)\\
\le\ & e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2} E_n[\sum_{i=1}^{n} \frac{|D_{1,2}|}{2}|h_1(tT_i)^2 - h_0(tT_i)^2|tT_i + Z_1|h_1(tT_i) - h_0(tT_i)|t^2T_i^2].
\end{aligned}
$$

By Theorem 4.1 we get

$$|F(n,t;h^\lambda) - F(n,t;h_0^*)| \ \le\ e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2}\left(\frac{1}{2}|D_{1,2}|\frac{C_0^2}{\lambda^2}t + Z_1\frac{C_0}{\lambda}t^2\frac{2}{n+1}\right)$$

for some constant C_0. Therefore we get

$$|J^\lambda(t;h^\lambda) - J^\lambda(t;h_0^*)| \ \le\ e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2}\left(\frac{1}{2}|D_{1,2}|C_0 + 2Z_1\right)\frac{C_0}{\lambda^2}t.$$

By Lemma 4.5 we get (4.16). (4.17) is an immediate consequence of (4.15) and (4.16). □

Let

$$G(z,\psi) = \psi + \sum_{i=2}^{\infty}\sum_{k=2}^{i}\gamma_{i,k}z^{k-1}\psi^{i-k}$$

where $\gamma_{i,k}$, $2 \le k \le i$ are constants. We seek a formal power series of

$$\Psi(z) = \sum_{j=1}^{\infty}\phi_j^* z^j, \quad \text{such that} \quad G(z,\Psi(z)) = 0.$$

The solution of this problem is given by solving the equations in terms of the coefficients of

$$\Psi(z) + \sum_{i=2}^{\infty}\sum_{k=2}^{i}\gamma_{i,k}z^{k-1}\Psi(z)^{i-k} = 0.$$

These equations are of the form

$$\phi_1^* = -\gamma_{2,2},\ \phi_j^* = -P_j(\gamma_{i,k},\phi_l^* : 2 \le k \le i \le j+1, l \le j-1)\ (j \ge 2)$$

where P_j is a polynomial with positive integer coefficients. Therefore we can solve recursively for the coefficients $\phi_j^*, j \ge 1$ and they are uniquely determined.

We define h_j^* by replacing $\gamma_{i,k}$ in ϕ_j^* by

$$\frac{D_{k,i-k+1}}{D_{1,2}(\alpha_1+\beta_0)}\left(\begin{pmatrix} i \\ k \end{pmatrix}\frac{\alpha_k}{i!} + \begin{pmatrix} i-1 \\ k-1 \end{pmatrix}\frac{\beta_{k-1}}{(i-1)!}\right).$$

Since h_j^* depends on $\alpha_1,\dots,\alpha_{j+1},\beta_0,\dots,\beta_j$, we denote h_j^* by

$$h_j^*(\vartheta_j)$$

where $\vartheta_j = (\alpha_1,\dots,\alpha_{j+1},\beta_0,\dots,\beta_j)$. Then we have the following lemma.

Lemma 4.7 *For all $n \in \boldsymbol{N}$ if there exist $\tilde{M}_m : [0,\infty)\times[0,T] \to \boldsymbol{R}$ for $0 \le m \le n+1$ satisfying*

$$(\min(1,e^{f(1)t}) - 1)L_m(\lambda t) \le \tilde{M}_m(\lambda,t) \le (e^{f(x_\alpha)t} - 1)L_m(\lambda t), \quad 0 \le t \le T, \qquad (4.19)$$

there exists a constant C_n such that for $0 < t \le T,\ \lambda > 0$,

$$|h_n^*(\theta_n(\lambda,t))| \le C_n, \qquad (4.20)$$

$$\left|\sum_{i=2}^{n+1}\sum_{k=1}^{i}\Gamma_{i,k}(\tilde{M}_k(\lambda,t),L_{k-1}(\lambda t))\frac{1}{\lambda^k}\left(\sum_{j=1}^{n}\frac{h_j^*(\theta_j(\lambda,t))}{\lambda^j}\right)^{i-k}\right| \le \frac{C_n}{\lambda^{n+2}} \qquad (4.21)$$

where

$$\Gamma_{i,k}(\alpha_0,\beta_0) = D_{k,i-k+1}\left(\begin{pmatrix} i \\ k \end{pmatrix}\frac{\alpha_0}{i!} + \begin{pmatrix} i-1 \\ k-1 \end{pmatrix}\frac{\beta_0}{(i-1)!}\right),$$
$$\theta_n(\lambda,t) = (\tilde{M}_1(\lambda,t),\ldots,\tilde{M}_{n+1}(\lambda,t),L_0(\lambda t),\ldots,L_n(\lambda t)).$$

The following lemma can be proved by Lemma 4.7, using the mathematical induction. Refer to Matsumoto[8] for the details.

Lemma 4.8 *Suppose that there exist $M_{n,N}(\lambda,t)$ satisfying*

$$(\min(1,e^{f(1)t})-1)L_n(\lambda t) \le M_{n,N}(\lambda,t) \le (e^{f(x_\alpha)t}-1)L_n(\lambda t), \quad 0\le t\le T, \tag{4.22}$$
$$|M_n(\lambda,t) - M_{n,N}(\lambda,t)| \le C\frac{L_0(\lambda t)}{\lambda^N}, \quad 0\le t\le T,\ 1\le n\le N+1 \tag{4.23}$$

for some constant C and $N\ge 1$. Let

$$\theta_{n,N}(\lambda,t) \;=\; (M_{1,N}(\lambda,t),\ldots,M_{n+1,N}(\lambda,t),L_0(\lambda t),\ldots,L_n(\lambda t)).$$

Then for $1\le n\le N$ there exist C_n and λ_n such that

$$|h_n^*(\theta_{n,N}(\lambda,t))| \le C_n, \quad 0<t\le T,\ \lambda\ge\lambda_n, \tag{4.24}$$
$$\left|h^\lambda(t) - \sum_{i=1}^n \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i}\right| \le C_n\frac{1}{\lambda^{n+1}}, \quad 0<t\le T,\ \lambda\ge\lambda_n, \tag{4.25}$$
$$\left|\frac{\partial A^\lambda}{\partial x}(t,x_\alpha+\sum_{i=1}^n\frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i})\right| \le C_n\frac{1}{\lambda^{n+2}}, \quad 0<t\le T,\ \lambda\ge\lambda_n, \tag{4.26}$$
$$\left|\tilde{A}^\lambda(t) - A^\lambda(t,x_\alpha+\sum_{i=1}^n\frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i})\right| \le C_n\frac{1}{\lambda^{2n+3}}, \quad 0<t\le T,\ \lambda\ge\lambda_n. \tag{4.27}$$

By (4.13) and (4.18), the following lemma can be shown.

Lemma 4.9 *Suppose that λ is sufficiently large and $\tilde{h}(t)$ satisfies*

$$0\le x_\alpha+\tilde{h}(t)\le 1, \quad 0\le t\le T,$$
$$|h^\lambda(t)-\tilde{h}(t)| \le C\frac{1}{\lambda^{N+1}}, \quad 0\le t\le T$$

for some constant C. Let

$$\tilde{M}_n(\lambda,t) = \int_0^{\lambda t} u^n e^{-u}\left(J^\lambda(t-\frac{u}{\lambda};\tilde{h})-1\right)du.$$

Then

$$(\min(1,e^{f(1)t})-1)L_n(\lambda t) \le \tilde{M}_n(\lambda,t) \le (e^{f(x_\alpha)t}-1)L_n(\lambda t), \quad 0\le t\le T, \tag{4.28}$$
$$|M_n(\lambda,t) - \tilde{M}_n(\lambda,t)| \le C_n\frac{L_0(\lambda t)}{\lambda^{N+2}}, \quad 0\le t\le T \tag{4.29}$$

for some constant C_n.

Let

$$M_{n,1}(\lambda,t) \;=\; \left(e^{f(x_\alpha)t}-1\right)L_n(\lambda t).$$

Theorem 4.2 *For all $N \in N$ there exists an approximation of the optimal strategy, v_N^λ such that*

$$|v^\lambda(t) - v_N^\lambda(t)| \le C_N \frac{1}{\lambda^{N+1}}, \quad 0 \le t \le T,\ \lambda \ge \lambda_N, \tag{4.30}$$

$$\left|\frac{\partial A^\lambda(t, v_N^\lambda(T-t))}{\partial x}\right| \le C_N \frac{1}{\lambda^{N+2}}, \quad 0 \le t \le T,\ \lambda \ge \lambda_N, \tag{4.31}$$

$$|A^\lambda(t, v^\lambda(T-t)) - A^\lambda(t, v_N^\lambda(T-t))| \le C_N \frac{1}{\lambda^{2N+3}}, \quad 0 \le t \le T,\ \lambda \ge \lambda_N \tag{4.32}$$

for some constants C_N and λ_N.

Proof. We use the mathematical induction. By Lemma 4.6,

$$\begin{aligned}
&|M_n(\lambda,t) - M_{n,1}(\lambda,t)| \\
\le\ & \left|\int_0^{\lambda t} u^n e^{-u}\left(J^\lambda(t-\frac{u}{\lambda}; h^\lambda) - 1\right) du - \int_0^{\lambda t} u^n e^{-u}\left(e^{f(x_\alpha)(t-\frac{u}{\lambda})} - 1\right) du\right| \\
&+ \left|\int_0^{\lambda t} u^n e^{-u}\left(e^{f(x_\alpha)(t-\frac{u}{\lambda})} - 1\right) du - \left(e^{f(x_\alpha)t} - 1\right) L_n(\lambda t)\right| \le C_1 \frac{L_n(\lambda t) + L_{n+1}(\lambda t)}{\lambda}
\end{aligned}$$

for some constant C_1. By Lemma 4.8 the assertion holds for $N = 1$.

Suppose that the assertion holds for $N \le N_1$. Let $h_{N_1}^\lambda(t) = v_{N_1}^\lambda(T-t) - x_\alpha$ and

$$M_{n,N_1+1}(\lambda,t) = \int_0^{\lambda t} u^n e^{-u}\left(J^\lambda(t-\frac{u}{\lambda}; h_{N_1}^\lambda) - 1\right) du.$$

By Lemma 4.9

$$\begin{aligned}
&(\min(1, e^{f(1)t}) - 1)L_n(\lambda t) \le M_{n,N_1+1}(\lambda,t) \le (e^{f(x_\alpha)t} - 1)L_n(\lambda t), \quad 0 \le t \le T, \\
&|M_n(\lambda,t) - M_{n,N_1+1}(\lambda,t)| \le C_{N_1+1}\frac{1}{\lambda^{N_1+2}}, \quad 0 \le t \le T,\ \ 1 \le n \le N_1 + 2
\end{aligned}$$

for some constant C_{N_1+1}. By Lemma 4.8, the assertion holds for $N = N_1 + 1$ and then the result follows. □

Remark 4.2 *By the arguments before Lemma 4.7, we have shown how to determine h_n^* recursively. Further we have shown how to construct $M_{n,N}$ successively in the proof of Theorem 4.2. Therefore we can construct v_N^λ successively.*

We have the following corollary from the proof of Theorem 4.2.

Corollary 4.1 *Let*

$$\begin{aligned}
h_1^*(\alpha_1, \alpha_2, \beta_0, \beta_1) &= -\frac{D_{2,1}(\alpha_2/2 + \beta_1)}{D_{1,2}(\alpha_1 + \beta_0)}, \\
\theta_{1,1}(\lambda,t) &= (M_{1,1}(\lambda,t), M_{2,1}(\lambda,t), L_0(\lambda t), L_1(\lambda t)).
\end{aligned}$$

Let v_1^λ be given by

$$v_1^\lambda(T-t) = x_\alpha + \frac{h_1^*(\theta_{1,1}(\lambda,t))}{\lambda} = x_\alpha - \sigma^2 x_\alpha(1-x_\alpha)(1-2x_\alpha)\frac{\left(e^{f(x_\alpha)t} - 1\right)L_2(\lambda t)/2 + L_1(\lambda t)}{\left(e^{f(x_\alpha)t} - 1\right)L_1(\lambda t) + L_0(\lambda t)}\frac{1}{\lambda}$$

for $0 < t \le T$ and $v_1^\lambda(T) = x_\alpha$. Then v_1^λ satisfies (4.30), (4.31) and (4.32) for $N = 1$.

4.3 Limit of the Value Function

In the previous subsection, we have shown that the optimal strategy converges to x_α when λ tends to ∞. In this subsection we show the limit of the value function.

Lemma 4.10 *There exists $C_1 > 0$ such that*

$$\left|J^\lambda(t;h_0^*) - \left(e^{f(x_\alpha)t} + 2Z(0,x_\alpha)e^{f(x_\alpha)t}\frac{t}{\lambda}\right)\right| \le C_1\frac{1}{\lambda^2}, \quad 0 \le t \le T.$$

Proof. We have

$$\left|e^{D_{1,0}t}\left(1 + Z(0,x_\alpha)t^2\frac{2}{n+1}\right) - F(n,t;h_0^*)\right|$$

$$\le e^{D_{1,0}t}Z_1t^3E_n[\sum_{i=1}^n T_i^3] + \frac{1}{2}e^{D_{1,0}t}Z_0^2t^4\exp\left(Z_0t^2\right)E_n[\left(\sum_{i=1}^n T_i^2\right)^2] \le e^{D_{1,0}t}t^3\frac{6Z_1 + 14Z_0^2t\exp\left(Z_0t^2\right)}{(n+2)(n+1)}.$$

Also we have

$$\left|e^{D_{1,0}t}\left(1 + 2Z(0,x_\alpha)\frac{t}{\lambda}\right) - e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^n}{n!}e^{D_{1,0}t}\left(1 + Z(0,x_\alpha)\frac{2t^2}{n+2}\right)\right| \le 2Z(0,x_\alpha)e^{D_{1,0}t}\frac{1}{\lambda^2}.$$

Therefore we get

$$\left|J^\lambda(t;h_0^*) - e^{D_{1,0}t}\left(1 + 2Z(0,x_\alpha)\frac{t}{\lambda}\right)\right| \le \left(6Z_1t + 14Z_0^2t^2\exp\left(Z_0t^2\right) + 2Z(0,x_\alpha)\right)e^{D_{1,0}t}\frac{1}{\lambda^2}.$$

The result follows. □

By Lemma 4.6 the following lemma can be shown.

Lemma 4.11 *There exists $C_1 > 0$ such that*

$$\left|\frac{\partial J^\lambda(t;h_0^*)}{\partial t} - f(x_\alpha)e^{f(x_\alpha)t}\right| \le C_1\frac{1}{\lambda}, \quad 0 \le t \le T.$$

By Lemmas 4.6, 4.10 and 4.11 and (4.1), we have

$$\Bigg|A^\lambda(t,x) - e^{f(x_\alpha)t} - 2Z(0,x_\alpha)e^{f(x_\alpha)t}\frac{t}{\lambda}$$
$$- \int_0^t\left(\frac{\partial D(t-s,x)}{\partial t}e^{f(x_\alpha)s} - D(t-s,x)f(x_\alpha)e^{f(x_\alpha)s}\right)e^{-\lambda(t-s)}ds\Bigg| \le C'\frac{1}{\lambda^2}$$

for some constant C'. Then we get the following theorem.

Theorem 4.3 *For $0 \le t \le T$,*

$$V^\lambda(t,x,w) \to V^\infty(t,x,w)$$

and

$$\lambda(V^\infty(t,x,w) - V^\lambda(t,x,w))$$
$$\to \frac{1}{2}w^\alpha\alpha(1-\alpha)\sigma^2e^{f(x_\alpha)(T-t)}\left((x-x_\alpha)^2 + x_\alpha^2(1-x_\alpha)^2\sigma^2(T-t)\right) \ge 0 \tag{4.33}$$

as $\lambda \to \infty$ uniformly in $0 \le x \le 1$.

References

[1] Boyle, P. and T. Vorst (1992) : Option Replication in Discrete Time with Transaction Costs, *Journal of Finance*, 47, 271-293

[2] Duffie, D. (1996) : *Dynamic Asset Pricing Theory*, Princeton University Press

[3] Elliot, R. (1982) : *Stochastic Calculus and Applications*, Springer-Verlag

[4] Subramanian, A. and R. Jarrow (2001) : The Liquidity Discount, *Mathematical Finance*, 11, 447-474

[5] Kusuoka, S. (1995) : Limit Theorem on Option Replication cost with Transaction Costs, *Annals of Applied Probability*, 1, 198-221

[6] Leland, H. E. (1985) : Option Pricing and Replication with Transactions Costs, *Journal of Finance*, 40, 1283-1301

[7] Matsumoto, K. (2002) : Optimal Portfolio of the Low Liquid Asset with the Log-Utility Function, preprint, UTMS 2002-28

[8] Matsumoto, K. (2003) : Optimal Portfolio of Low Liquid Assets with a Power Utility Function, *to appear in Journal of Mathematical Sciences, The University of Tokyo*

[9] Merton, R.C. (1971) : Optimum Consumption and Portfolio Rules in a Continuous Time Model, *Journal of Economic Theory*, 3, 373-413

[10] Merton, R.C. (1990) : *Continuous-Time Finance*, Blackwell Publishers