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Homogeneous Law Invariant Coherent Multiperiod Value Measures and their Limits

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space. We denote $L^p(\Omega, \mathcal{F}, P)$ by L^p, $1 \leq p \leq \infty$.

Definition 1 *We say that a map $\phi : L^\infty \to \mathbf{R}$ is a coherent value measure, if the following are satisfied.*
(1) *If $X \geq 0$, then $\phi(X) \geq 0$.*
(2) *Superadditivity : $\phi(X_1 + X_2) \geq \phi(X_1) + \phi(X_2)$.*
(3) *Positive homogeneity : for $\lambda > 0$ we have $\phi(\lambda X) = \lambda\phi(X)$.*
(4) *For every constant c we have $\phi(X + c) = \phi(X) + c$.*

Then Delbaen [6] essentially proved the following.

Theorem 2 *For $\phi : L^\infty \to \mathbf{R}$, the following conditions are equivalent.*
(1) *There is a (closed convex) set of probability measures $\mathcal{Q}$ such that any $Q \in \mathcal{Q}$ is absolutely continuous with respect to P and for $X \in L^\infty$*

$$\phi(X) = \inf\{E^Q[X];\ Q \in \mathcal{Q}\}.$$

(2) *ϕ is a coherent value measure and satisfies the Fatou property, i.e., if $\{X_n\}_{n=1}^\infty \subset L^\infty$ is uniformly bounded and converging to X in probability, then*

$$\phi(X) \geq \limsup \phi(X_n).$$

(3) *ϕ is a coherent value measure and satisfies the following property. If X_n is a uniformly bounded sequence that increases to X, then $\phi(X_n)$ tends to $\phi(X)$.*

Now we introduce the following notion.

Definition 3 *We say that a map $\phi : L^\infty \to \mathbf{R}$ is law invariant, if $\phi(X) = \phi(Y)$ whenever $X, Y \in L^\infty$ have the same probability law.*

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Let $\mathcal{L}$ denote the set of probability measures on $\mathbf{R}$, $\mathcal{L}_p$, $p \in [1, \infty)$, denote the set of probability measures ν on $\mathbf{R}$ such that $\int_{\mathbf{R}} |x|^p \nu(dx) < \infty$, and $\mathcal{L}_\infty$ denote the set of probability measures ν on $\mathbf{R}$ such that $\nu(\mathbf{R} \setminus [-M, M]) = 0$ for some $M > 0$. Also, $\mathcal{M}_{[0,1]}$ be the set of probability measure on $[0, 1]$.

For $\nu \in \mathcal{L}$, let F_ν be the distribution functions of ν, i.e., $F_\nu(z) = \nu((\infty, z])$, $z \in \mathbf{R}$. Let us define $Z : [0, 1) \times \mathcal{L} \to \mathbf{R}$ by

$$Z(x, \nu) = \inf\{z; F_\nu(z) > x\}, \qquad x \in [0, 1),\ \nu \in \mathcal{L}.$$

Then $Z(\cdot, \nu) : [0, 1) \to \mathbf{R}$ is non-decreasing and right continuous, and the probability law of $Z(\cdot, \nu)$ under Lebesgue measure on $[0, 1)$ is ν (c.f.[9]). For any random variables X, we denote by μ_X the probability law of X.

For each $\alpha \in (0, 1]$, let $\eta_\alpha : \mathcal{L}_1 \to \mathbf{R}$ be given by

$$\eta_\alpha(\nu) = \alpha^{-1} \int_0^\alpha Z(x, \nu) dx, \qquad \nu \in \mathcal{L}_1.$$

Also, we define $\eta_0 : \mathcal{L}_\infty \to \mathbf{R}$ by

$$\eta_0(\nu) = \inf\{x \in \mathbf{R}; \nu((-\infty, x]) > 0\} \qquad X \in \mathcal{L}_\infty.$$

Then we have the following (cf. [8], also see Section).

Theorem 4 *Assume that $(\Omega, \mathcal{F}, P)$ is a standard probability space and P is non-atomic. Let $\phi : L^\infty \to \mathbf{R}$. Then the following conditions are equivalent.*
(1) *There is a (compact convex) subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that*

$$\phi(X) = \inf\{\int_0^1 \eta_\alpha(\mu_X) m(d\alpha);\ m \in \mathcal{M}_0\}, \qquad X \in L^\infty.$$

(2) *ϕ is a law invariant coherent value measure with the Fatou property.*

Definition 5 *We say that a map $\eta : \mathcal{L}_\infty \to \mathbf{R}$ is a mild value measure* (MVM), *if there is a subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that*

$$\eta(\nu) = \inf\{\int_0^1 \eta_\alpha(\nu) m(d\alpha);\ m \in \mathcal{M}_0\}, \qquad \nu \in \mathcal{L}_\infty.$$

For any MVM η, we define a subset $\mathcal{M}(\eta)$ of $\mathcal{M}_{[0,1]}$ by

$$\mathcal{M}(\eta) = \{m \in \mathcal{M};\ \eta(\nu) \leq \int_0^1 \eta_\alpha(\nu) m(d\alpha) \text{ for all } \nu \in \mathcal{L}_\infty\}.$$

For any $\nu \in \mathcal{L}_1$, we see that $\eta_\alpha(\nu) \leq \eta_1(\nu)$, $\alpha \in [0, 1]$. So any MVM η can be extended to a map from $\mathcal{L}_1$ to $[-\infty, \infty)$ by

$$\eta(\nu) = \inf\{\int_0^1 \eta_\alpha(\nu) m(d\alpha);\ m \in \mathcal{M}(\eta)\}, \qquad \nu \in \mathcal{L}_1.$$

We denote this map by the same symbol η.

Definition 6 *Let η be an MVM and $(\Omega, \mathcal{F}, P)$ be a probability space.*
(1) *For any integrable random variable X and any sub-σ-algebra $\mathcal{G}$, we define a $\mathcal{G}$-measurable random variable $\eta(X|\mathcal{G})$ by*

$$\eta(X|\mathcal{G}) = \eta(P(X \in dx|\mathcal{G})),$$

where $P(X \in dx|\mathcal{G})$ is a regular conditional probability law of X given a sub-σ-algebra $\mathcal{G}$. We call $\eta(X|\mathcal{G})$ a conditional value measure.
(2) *For any integrable random variable X and any filtration $\{\mathcal{F}_k\}_{k=0}^n$, we define an adapted process $\{Z_k\}_{k=0}^n$ inductively by*

$$Z_n = \eta(X|\mathcal{F}_n),$$
$$Z_{k-1} = \eta(Z_k|\mathcal{F}_{k-1}), \qquad k = n, n-1, \dots, 1.$$

We denote an $\mathcal{F}_0$-measurable random variable Z_0 by $\eta(X|\{\mathcal{F}_k\}_{k=0}^n)$, and call it a homogeneous filtered value measure.
(3) *For any filtration $\{\mathcal{F}_k\}_{k=0}^n$ and any integrable adapted process $\{X_k\}_{k=0}^n$, we define an adapted process $\{Y_k\}_{k=0}^n$ inductively by*

$$Y_n = X_n,$$

$$Y_{k-1} = X_{k-1} \wedge \eta(Y_k|\mathcal{F}_{k-1}), \qquad k = n, n-1, \dots, 1.$$

We denote an $\mathcal{F}_0$-measurable random variable Y_0 by $\eta(\{X_k\}_{k=0}^n|\{\mathcal{F}_k\}_{k=0}^n)$, and call it a homogeneous filtered value measure of an adapted process $\{X_k\}_{k=0}^n$.

In this paper, we consider two kinds of limit theorem for homogeneous filtered value measures. Let us introduce the following notion. For any MVM η and $p \in [1, \infty)$, let

$$\Delta_p(\eta) = \sup\{\int_0^1 (\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha}) m(d\alpha);\ m \in \mathcal{M}(\eta)\}.$$

1.1 Brownian-Poisson Filtration

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{B(t); t \in [0, \infty)\}$ be a d-dimensional Brownian motion and $\{N_i(t); t \in [0, \infty)\}$, $i = 1, \dots, \ell$, be Poisson processes with an intensity λ_i. We assume that they are independent. Let $\lambda = \sum_{i=1}^{\ell} \lambda_i$, and let $\mathcal{F}_t = \sigma\{B(s), N_i(s); s \leq t, i = 1, \dots, \ell\}$, $t \geq 0$.

Let η_n, $n = 1, 2, \dots$, be MVM's. We assume the following.

(A-1) There is a constant $C > 0$ such that $\Delta_2(\eta_n) \leq C 2^{-n/2}$, $n = 1, 2, \dots$.

Let $F_0(y; \alpha, \beta)$, $y \in \mathbf{R}^{\ell}$, $0 \leq \alpha \leq \beta \leq 1$, be given by

$$F_0(y; \alpha, \beta) = \inf\{\int_0^{\gamma} Z(x, \lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}) dx;\ \alpha \leq \gamma \leq \beta\}$$

$$= \inf\{\gamma \eta_{\gamma}(\lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i});\ \alpha \leq \gamma \leq \beta\},$$

and let $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \to \mathbf{R}$, $n = 1, 2, \dots$, be given by

$$b_n(x,y) = \inf\{|x|2^{n/2}(\int_0^1 \eta_\alpha(\mu_0)m(d\alpha))$$

$$+\lambda(\int_0^1 m(d\alpha)\alpha^{-1}F_0(y; 0 \vee (1-(2^n\lambda^{-1}(1-\alpha)), 1 \wedge 2^n\lambda^{-1}\alpha)); m \in \mathcal{M}(\eta_n)\}.$$

Here μ_0 is a standard normal distribution.

Then $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \to \mathbf{R}$ is concave,

$$b_n(sx, sy) = sb_n(x,y), \qquad x \in \mathbf{R}^d, \quad y \in \mathbf{R}^\ell, \quad s \geq 0,$$

and

$$b_n(x, y_1, \dots, y_\ell) \leq b_n(x', y_1', \dots, y_\ell'),$$

if $|x| \geq |x'|$, $y_1 \leq y_1'$, $\dots$, $y_\ell \leq y_\ell'$.

Let us assume the following furthermore..

(A-2) There is a continuous function $b : \mathbf{R}^d \times \mathbf{R}^\ell \to \mathbf{R}$ such that $b_n \to b$, $n \to \infty$, uniformly on compacts in $\mathbf{R}^d \times \mathbf{R}^\ell$.

Let K be a compact convex set in $\mathbf{R}^d \times \mathbf{R}^\ell$ given by

$$K = \{(z,w) \in \mathbf{R}^d \times [0,\infty)^\ell;\ b(x,y) \leq x \cdot z + \sum_{i=1}^{\ell} \lambda_i y_i w_i \text{ for all } (x,y) \in \mathbf{R}^d \times \mathbf{R}^\ell\}.$$

Also, let $\mathcal{K}$ be a set of martingales $\rho(t)$ such that there are predictable processes $\varphi : [0,\infty) \times \Omega \to \mathbf{R}^d$, $\psi_i : [0,\infty) \times \Omega \to [0,\infty)$, $i = 1, \dots, \ell$, for which

$$P((\varphi(t), \psi_1(t), \dots, \psi_\ell(t)) \in K \text{ for any } t \in [0,T]) = 1$$

and

$$\rho(t) = \prod_{i=1}^{\ell}(\prod_{s \in (0,t], \Delta N_i(s) \neq 0} \psi_i(s)) \exp(\int_0^t \varphi(s)dB(s) - \frac{1}{2}\int_0^t |\phi(s)|^2 ds - \sum_{i=1}^{\ell} \lambda_i \int_0^t (\psi_i(s)-1)ds),$$

$t \geq 0$.

Then we have the following.

Theorem 7 *Under the assumption* (A-1) *and* (A-2), *we have the following.*

For any $X \in L^2(\Omega, \mathcal{F}_T, P)$, $T > 0$,

$$\lim_{n\to\infty} \eta_n(X|\{\mathcal{F}_{2^{-n}k}\}_{k=0}^{2^{2n}}) = \inf\{E[\rho(T)X]; \rho \in \mathcal{K}\}.$$

We prove this theorem in Section 5 via a nonlinear partial differential equation.

1.2 Collective Risk

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $K \geq 1$, $p \in (1,\infty)$, $p_k \in \mathbf{R}$, $\lambda_k > 0$, and $\nu_k \in \mathcal{L}_p$, $k = 1,\dots,K$. Let $Z_i^{(k)}$, $\tau_i^{(k)}$, $k = 1,\dots,K$, $i = 1,2,\dots$, be independent random variables such that the distribution of $Z_i^{(k)}$, is ν_k, and $P(\tau_i^{(k)} > t) = \exp(-\lambda_k t)$, $t \geq 0$, for $k = 1,\dots,K$, $i = 1,2,\dots$. Let $N_i^{(k)}(t) = 1_{\{\tau_i^{(k)} \leq t\}}$, and $X_i^{(k)}(t) = Z_i^{(k)} N_i^{(k)}(t) + p_k(\tau_i^{(k)} \wedge t)$ for $t \geq 0$, $k = 1,\dots,K$, $i = 1,2,\dots$.

Let $\mathcal{F}_t = \sigma\{X_i^{(k)}(s);\ s \in [0,t], k = 1,\dots,K, i = 1,2,\dots\}$, $t \geq 0$. Also, let

$$X(t; m_1,\dots,m_K) = \sum_{k=1}^{K}\sum_{i=1}^{m_k} X_i^{(k)}(t)$$

for any $t \geq 0$, and any $m_1,\dots,m_K \in \mathbf{Z}_{\geq 0}$. Here $\mathbf{Z}_{\geq 0}$ denotes the set of non-negative integers.

Theorem 8 *Let η be MVM. Assume that $\Delta_p(\eta) < \infty$. Let $\Phi : [0,\infty)^K \times \mathbf{R}^K \to \mathbf{R}$ be given by*

$$\Phi(x,\xi) = \eta(\sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^{K}(\exp(-\lambda_k x_k)\frac{(\lambda_k x_k)^{\ell_k}}{\ell_k!}))(\nu_1 - \xi_1)^{*\ell_1} * \cdots * (\nu_K - \xi_K)^{\ell_K}) + \sum_{k=1}^{K} p_k x_k,$$

for $x \in [0,\infty)^K$, $\xi \in \mathbf{R}^K$. Here $*$ stands for the convolution and $\nu + a$ denotes a probability measure on $\mathbf{R}$ given by the following for any probability measure ν on $\mathbf{R}$ and $a \in \mathbf{R}$.

$$(\nu + a)(A) = \nu(\{x \in \mathbf{R};\ x - a \in A\}) \text{ for any Borel set } A \text{ in } \mathbf{R}.$$

Asuume that there is a C^1 fucntion $u : [0,\infty) \times [0,\infty)^K \to \mathbf{R}$ such that $u(0,x) = 0$, $x \in [0,\infty)^K$, and satisfies the following Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}u(t,x) = \Phi(x, \frac{\partial}{\partial x^1}u(t,x),\dots,\frac{\partial}{\partial x^K}u(t,x)), \qquad (t,x) \in [0,\infty) \times [0,\infty)^K.$$

Then we have the following.

$$\sup\{|h\eta(X(t;m_1,\dots,m_K)|\{\mathcal{F}_{jh}\}_{j=0}^{[h^{-2}]}) - u(t, m_1 h,\dots,m_K h)|;$$
$$t, m_1 h,\dots,m_K h \in [0,R], m_1,\dots,m_K \in \mathbf{Z}_{\geq 0}\} \to 0,$$

as $h \downarrow 0$, for any $R > 0$.

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