Homogeneous Law Invariant Coherent Multiperiod Value Measures and their Limits

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space. We denote $L^p(\Omega, \mathcal{F}, P)$ by $L^p$, $1 \leq p \leq \infty$.

Definition 1 We say that a map $\phi : L^\infty \to \mathbb{R}$ is a coherent value measure, if the following are satisfied.

1. If $X \geq 0$, then $\phi(X) \geq 0$.
3. Superadditivity: $\phi(X_1 + X_2) \geq \phi(X_1) + \phi(X_2)$.
4. Positive homogeneity: for $\lambda > 0$ we have $\phi(\lambda X) = \lambda \phi(X)$.
5. For every constant $c$ we have $\phi(X + c) = \phi(X) + c$.

Then Delbaen [6] essentially proved the following.

Theorem 2 For $\phi : L^\infty \to \mathbb{R}$, the following conditions are equivalent.

1. There is a (closed convex) set of probability measures $Q$ such that any $Q \in Q$ is absolutely continuous with respect to $P$ and for $X \in L^\infty$

$$\phi(X) = \inf \{ E^Q[X]; Q \in Q \}.$$ 

2. $\phi$ is a coherent value measure and satisfies the Fatou property, i.e., if $\{X_n\}_{n=1}^\infty \subset L^\infty$ is uniformly bounded and converging to $X$ in probability, then

$$\phi(X) \geq \lim \sup \phi(X_n).$$

3. $\phi$ is a coherent value measure and satisfies the following property. If $X_n$ is a uniformly bounded sequence that increases to $X$, then $\phi(X_n)$ tends to $\phi(X)$.

Now we introduce the following notion.

Definition 3 We say that a map $\phi : L^\infty \to \mathbb{R}$ is law invariant, if $\phi(X) = \phi(Y)$ whenever $X, Y \in L^\infty$ have the same probability law.

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Let $\mathcal{L}$ denote the set of probability measures on $\mathbb{R}$, $\mathcal{L}_{p}$, $p \in [1, \infty)$, denote the set of probability measures $\nu$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x|^p \nu(dx) < \infty$, and $\mathcal{L}_{\infty}$ denote the set of probability measures $\nu$ on $\mathbb{R}$ such that $\nu([M, M]) = 0$ for some $M > 0$. Also, $\mathcal{M}_{[0,1]}$ be the set of probability measure on $[0, 1]$.

For $\nu \in \mathcal{L}$, let $F_{\nu}$ be the distribution functions of $\nu$, i.e., $F_{\nu}(z) = \nu((\infty, z])$, $z \in \mathbb{R}$. Let us define $Z : [0, 1) \times \mathcal{L} \rightarrow \mathbb{R}$ by

$$Z(x, \nu) = \inf\{z; F_{\nu}(z) > x\}, \quad x \in [0, 1), \ \nu \in \mathcal{L}.$$

Then $Z(\cdot, \nu) : [0, 1) \rightarrow \mathbb{R}$ is non-decreasing and right continuous, and the probability law of $Z(\cdot, \nu)$ under Lebesgue measure on $[0, 1)$ is $\nu$ (c.f.[9]). For any random variables $X$, we denote by $\mu_X$ the probability law of $X$.

For each $\alpha \in (0, 1]$, let $\eta_\alpha : \mathcal{L}_1 \rightarrow \mathbb{R}$ be given by

$$\eta_\alpha(\nu) = \alpha^{-1} \int_0^1 Z(x, \nu) dx, \quad \nu \in \mathcal{L}_1.$$  

Also, we define $\eta_0 : \mathcal{L}_\infty \rightarrow \mathbb{R}$ by

$$\eta_0(\nu) = \inf\{x \in \mathbb{R}; \nu((-\infty, x]) > 0\} \quad X \in \mathcal{L}_\infty.$$

Then we have the following (cf. [8], also see Section).

**Theorem 4** Assume that $(\Omega, \mathcal{F}, P)$ is a standard probability space and $P$ is non-atomic. Let $\phi : L^\infty \rightarrow \mathbb{R}$. Then the following conditions are equivalent.

1. There is a (compact convex) subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that

$$\phi(X) = \inf\{\int_0^1 \eta_\alpha(\mu_X)m(d\alpha); m \in \mathcal{M}_0\}, \quad X \in \mathcal{L}^\infty.$$  

2. $\phi$ is a law invariant coherent value measure with the Fatou property.

**Definition 5** We say that a map $\eta : \mathcal{L}_\infty \rightarrow \mathbb{R}$ is a mild value measure (MVM), if there is a subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that

$$\eta(\nu) = \inf\{\int_0^1 \eta_\alpha(\nu)m(d\alpha); \ m \in \mathcal{M}_0\}, \quad \nu \in \mathcal{L}_\infty.$$  

For any MVM $\eta$, we define a subset $\mathcal{M}(\eta)$ of $\mathcal{M}_{[0,1]}$ by

$$\mathcal{M}(\eta) = \{m \in \mathcal{M}; \ \eta(\nu) \leq \int_0^1 \eta_\alpha(\nu)m(d\alpha) \text{ for all } \nu \in \mathcal{L}_\infty\}.$$  

For any $\nu \in \mathcal{L}_1$, we see that $\eta_\alpha(\nu) \leq \eta_1(\nu), \ \alpha \in [0, 1]$. So any MVM $\eta$ can be extended to a map from $\mathcal{L}_1$ to $[-\infty, \infty)$ by

$$\eta(\nu) = \inf\{\int_0^1 \eta_\alpha(\nu)m(d\alpha); \ m \in \mathcal{M}(\eta)\}, \quad \nu \in \mathcal{L}_1.$$  

We denote this map by the same symbol $\eta$. 
Definition 6 Let $\eta$ be an MVM and $(\Omega, \mathcal{F}, P)$ be a probability space.

1. For any integrable random variable $X$ and any sub-$\sigma$-algebra $\mathcal{G}$, we define a $\mathcal{G}$-measurable random variable $\eta(X|\mathcal{G})$ by

$$\eta(X|\mathcal{G}) = \eta(P(X \in dx|\mathcal{G})),$$

where $P(X \in dx|\mathcal{G})$ is a regular conditional probability law of $X$ given a sub-$\sigma$-algebra $\mathcal{G}$. We call $\eta(X|\mathcal{G})$ a conditional value measure.

2. For any integrable random variable $X$ and any filtration $\{F_k\}_{k=0}^{n}$, we define an adapted process $\{Z_k\}_{k=0}^{n}$ inductively by

$$Z_n = \eta(X|F_n),$$

$$Z_{k-1} = \eta(Z_k|F_{k-1}), \quad k = n, n-1, \ldots, 1.$$

We denote an $F_0$-measurable random variable $Z_0$ by $\eta(X|\{F_k\}_{k=0}^{n})$, and call it a homogeneous filtered value measure.

3. For any filtration $\{F_k\}_{k=0}^{n}$ and any integrable adapted process $\{X_k\}_{k=0}^{n}$, we define an adapted process $\{Y_k\}_{k=0}^{n}$ inductively by

$$Y_n = X_n,$$

$$Y_{k-1} = X_{k-1} \wedge \eta(Y_k|F_{k-1}), \quad k = n, n-1, \ldots, 1.$$

We denote an $F_0$-measurable random variable $Y_0$ by $\eta(\{X_k\}_{k=0}^{n}|\{F_k\}_{k=0}^{n})$, and call it a homogeneous filtered value measure of an adapted process $\{X_k\}_{k=0}^{n}$.

In this paper, we consider two kinds of limit theorem for homogeneous filtered value measures. Let us introduce the following notion. For any MVM $\eta$ and $p \in [1, \infty)$, let

$$\Delta_p(\eta) = \sup\{\int_0^1 (\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha}) m(d\alpha); m \in \mathcal{M}(\eta)\}.$$

1.1 Brownian-Poisson Filtration

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{B(t); t \in [0, \infty)\}$ be a $d$-dimensional Brownian motion and $\{N_i(t); t \in [0, \infty)\}, i = 1, \ldots, \ell$, be Poisson processes with an intensity $\lambda_i$. We assume that they are independent. Let $\lambda = \sum_{i=1}^{\ell} \lambda_i$, and let $\mathcal{F}_t = \sigma\{B(s), N_i(s); s \leq t, i = 1, \ldots, \ell\}, t \geq 0$.

Let $\eta_n, n = 1, 2, \ldots, \ell$ be MVM's. We assume the following.

(A-1) There is a constant $C > 0$ such that $\Delta_2(\eta_n) \leq C2^{-n/2}$, $n = 1, 2, \ldots$.

Let $F_0(y; \alpha, \beta), y \in \mathbb{R}^\ell, 0 \leq \alpha \leq \beta \leq 1$, be given by

$$F_0(y; \alpha, \beta) = \inf\{\int_0^\gamma Z(x, \lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}) dx; \alpha \leq \gamma \leq \beta\}$$

$$= \inf\{\gamma \eta_{\gamma}(\lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}); \alpha \leq \gamma \leq \beta\},$$
and let $b_n : \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}, n = 1, 2, \ldots$ be given by

$$b_n(x, y) = \inf \{ |x| 2^{n/2} \left( \int_0^1 \eta_n(\mu_0) m(d\alpha) \right) + \lambda \left( \int_0^1 m(d\alpha) \alpha^{-1} F_0(y; 0 \vee (1 - (2^n \lambda^{-1} (1 - \alpha)), 1 \wedge 2^n \lambda^{-1} \alpha) ; m \in \mathcal{M}(\eta_n) \right) \}.$$

Here $\mu_0$ is a standard normal distribution.

Then $b_n : \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}$ is concave,

$$b_n(sx, sy) = sb_n(x, y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^l, \quad s \geq 0,$$

if $|x| \geq |x'|, y_1 \leq y'_1, \ldots, y_\ell \leq y'_\ell$.

Let us assume the following furthermore.

(A-2) There is a continuous function $b : \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}$ such that $b_n \to b, n \to \infty$, uniformly on compacts in $\mathbb{R}^d \times \mathbb{R}^l$.

Let $K$ be a compact convex set in $\mathbb{R}^d \times \mathbb{R}^l$ given by

$$K = \{ (z, w) \in \mathbb{R}^d \times [0, \infty)^l ; b(x, y) \leq x \cdot z + \sum_{i=1}^l \lambda_i y_i w_i \text{ for all } (x, y) \in \mathbb{R}^d \times \mathbb{R}^l \}.$$

Also, let $\mathcal{K}$ be a set of martingales $\rho(t)$ such that there are predictable processes $\varphi : [0, \infty) \times \Omega \to \mathbb{R}^d, \psi_i : [0, \infty) \times \Omega \to [0, \infty), i = 1, \ldots, \ell$, for which

$$P((\varphi(t), \psi_1(t), \ldots, \psi_\ell(t)) \in K \text{ for any } t \in [0, T]) = 1$$

and

$$\rho(t) = \prod_{i=1}^t \left( \prod_{s \in [0, t], \Delta N_i(s) \neq 0} \psi_i(s) \exp \left( \int_0^t \varphi(s) dB(s) - \frac{1}{2} \int_0^t |\varphi(s)|^2 ds - \sum_{i=1}^\ell \lambda_i \int_0^t (\psi_i(s) - 1) ds \right) \right),$$

t \geq 0.

Then we have the following.

Theorem 7 Under the assumption (A-1) and (A-2), we have the following.

For any $X \in L^2(\Omega, \mathcal{F}_T, P), T > 0$,

$$\lim_{n \to \infty} \eta_n(X|\mathcal{F}_{2^{-n}k})_{k=0}^{2^n} = \inf \{ E[\rho(T)X] ; \rho \in \mathcal{K} \}.$$

We prove this theorem in Section 5 via a nonlinear partial differential equation.
1.2 Collective Risk

Let $\Omega, \mathcal{F}, P$ be a probability space. Let $K \geq 1$, $p \in (1, \infty)$, $p_k \in \mathbb{R}$, $\lambda_k > 0$, and $\nu_k \in \mathcal{L}_p$, $k = 1, \ldots, K$. Let $Z_t^{(k)}$, $\tau_t^{(k)}$, $k = 1, \ldots, K$, $i = 1, 2, \ldots$, be independent random variables such that the distribution of $Z_t^{(k)}$ is $\nu_k$, and $P(\tau_t^{(k)} > t) = \exp(-\lambda_k t)$, $t \geq 0$, for $k = 1, \ldots, K$, $i = 1, 2, \ldots$. Let

$$N_t^{(k)}(t) = 1_{\{\tau_t^{(k)} \leq t\}},$$

and $X_t^{(k)}(t) = Z_t^{(k)} N_t^{(k)}(t) + p_k (\tau_t^{(k)} \wedge t)$ for $t \geq 0$, $k = 1, \ldots, K$, $i = 1, 2, \ldots$.

Let $\mathcal{F}_t = \sigma\{X_t^{(k)}(s); s \in [0, t], k = 1, \ldots, K, i = 1, 2, \ldots\}$, $t \geq 0$. Also, let

$$X(t; m_1, \ldots, m_K) = \sum_{k=1}^{K} \sum_{i=1}^{m_k} X_t^{(k)}(t)$$

for any $t \geq 0$, and any $m_1, \ldots, m_K \in \mathbb{Z}_{\geq 0}$. Here $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers.

Theorem 8 Let $\eta$ be $MVM$. Assume that $\Delta_p(\eta) < \infty$. Let $\Phi : [0, \infty)^K \times \mathbb{R}^K \to \mathbb{R}$ be given by

$$\Phi(x, \xi) = \eta(\sum_{\ell_1, \ldots, \ell_k=0}^{\infty} \prod_{k=1}^{K} (\exp(-\lambda_k x_k) (\lambda_k x_k)^{\ell_k}/\ell_k!)((\nu_1 - \xi_1)^{\ell_1} \ast \cdots \ast (\nu_K - \xi_K)^{\ell_K}) + \sum_{k=1}^{K} p_k x_k,$$

for $x \in [0, \infty)^K$, $\xi \in \mathbb{R}^K$. Here $\ast$ stands for the convolution and $\nu + a$ denotes a probability measure on $\mathbb{R}$ given by the following for any probability measure $\nu$ on $\mathbb{R}$ and $a \in \mathbb{R}$.

$$(\nu + a)(A) = \nu(\{x \in \mathbb{R}; x - a \in A\})$$

for any Borel set $A$ in $\mathbb{R}$.

Assume that there is a $C^1$ function $u : [0, \infty) \times [0, \infty)^K \to \mathbb{R}$ such that $u(0, x) = 0$, $x \in [0, \infty)^K$, and satisfies the following Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} u(t, x) = \Phi(x, \frac{\partial}{\partial x^1} u(t, x), \ldots, \frac{\partial}{\partial x^K} u(t, x)),$$

$(t, x) \in [0, \infty) \times [0, \infty)^K$.

Then we have the following.

$$\sup\{|h \eta(X(t; m_1, \ldots, m_K); \mathcal{F}_{j=t})|^{(n-2)} - u(t, m_1 h, \ldots, m_K h)|; t, m_1 h, \ldots, m_K h \in [0, R], m_1, \ldots, m_K \in \mathbb{Z}_{\geq 0} \to 0,$$

as $h \downarrow 0$, for any $R > 0$.

References


