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Cournot Oligopoly with Network Formation

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Abstract

We consider an oligopoly model where the firm's (marginal) cost depends upon the number of pairwise collaborative links it has with the other firms. This reflects the benefit resulting from information exchange between two firms or common use of basic means of technology. We analyze the firm's incentive to form a pairwise link and a structure of resulting network.

We find that the complete network and the unlinked network in which three firms form links each other (triangle) and one firm has no link (isolated firm) can be pairwise stable networks. Furthermore, there can be environment under which cycle of network structures emerge.

Keywords: Network Formation, Oligopoly, Collaboration, Pairwise Stability

JEL classification: C70; L13; L20

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1 Introduction

We consider an oligopoly model where the firm's (marginal) cost depends upon the number of pairwise collaborative links it has with the other firms. This reflects the benefit resulting from information exchange between two firms or common use of basic means of technology. We analyze the firm's incentive to form a pairwise link and a structure of resulting network.

Each firm announces the name of the firm which it would like to form collaborative link anticipating the final result of Cournot competition in the later stage which determines the distribution of profits. A link is formed if and only if a pair of firms finds it advantageous to do so. On the other hand, a link may be disconnected by single firm's decision irrespective of the profit of the partner. The final pairwise stable network is the subgame perfect equilibrium.

The basic idea governing the evolutionally process of network formation is the pairwise stability as described in their papers and in Jackson and Watts(2002).[1] In the four-firm model of oligopoly, we find that, depending on the firms' cost parameters and the effects of pairwise collaboration, two different stable networks will emerge. One is the complete network in which all pairs of firms are linked. The complete network results when the cost saving is decreasing in the number of links. The other is the unlinked network in which three firms form links each other (triangle) and one firm has no link (isolated firm). Furthermore, there can be environment under which a cycle of network structures emerge. This cycle includes, e.g., a circle network, a wine glass network which consists of triangle with one additional link between a central firm and one peripheral firm, and some other.

Most closely related to this paper is the above mentioned literature on the net-

[1]Dutta and Mutuswami(1997) also analyzes a similar problem by considering a strategic-form game of network formation.

work formation . (For a survey, see Slikker and van den Nouweland (2001).) Recently, like this paper, Goyal and Joshi (2003) analyzed network formation game in oligopoly. Compared with the work by Goyal and Joshi, our analysis lays particular emphasis on the convexity (or concavity) of cost function with respect to the number of links. This yields variety of outcomes even for cases of small number of firms.

The rest of paper is organized as follows. Section 2 introduces the model. In Section 3 we will explain some of the basic concepts of graph theory and illustrate possible network patterns . In Section 4 the concept of pairwise stability is defined. In Section 5, we will derive the equilibrium profits for each network. Section 6 examines the pairwise stable networks and the transition movement between networks. Section 7 concludes the paper.

2 The Model

We consider a market of homogeneous products with the following linear demand function:

$$p = a - bX, \quad a, b > 0, \tag{1}$$

where p and X denote the price and the amount of demand respectively.

Let $N = \{1, \cdots, n\}$ be a set of *ex ante* identical firms. The cost function of firm i depends upon its collaboration links with other firms. Each collaboration link is pairwise and induces lower costs of production. Specifically, assume that firm i's cost function is given by

$$C(x_i; d_i) = c(d_i)x_i, \tag{2}$$

where x_i denotes firm i's outputs and d_i is the number of links firm i has. (We do not consider the fixed cost of production or link formation.) We assume that the marginal cost $c(d_i)$ is strictly decreasing in the number of links d_i;

Assumption 1 *For any $m \in \mathbb{Z}$, $c(m) > c(m+1)$.*

The link by two firms can be interpreted as, e.g., the exchange of technology, sharing and standardization of parts, and so on.

Finally, we put a restriction in order to ensure the positive amount of output for every firm.

Assumption 2

$$a > c(0)$$

The market clearing condition requires

$$X = \sum_i x_i. \tag{3}$$

3 Structure of Network

The link between player i (firm i) and player j (firm j) is denoted $\{i, j\}$. Define $L^N \equiv \{\{i, j\} | ij \subset N,\ j \neq i\}$. Thus, L^N represents the set of all possible links on N, that is, L^N is the complete graph on N. A network (of links) on N is a graph (N, L) which has the set of firms as its vertices and pairwise links as the set of edges $L \subset L^N$. Let $L_i \subset L$ be the set of links in which player i is involved. Then, d_i is the cardinality of L_i.

Besides the complete network L^N, there are several interesting forms of networks. Suppose that there exists some i^0 and network is described by

$$L = \{\{i^0, j\} | j \in N,\ j \neq i^0\}.$$

This network is called a *star* network and i^0 can be referred to a central firm. The network

$$L = \{\{i, i+1\} | i \in N \backslash n\}$$

is a line network, and

$$L = \{\{i, i+1\} | i \in N\},$$

where $\{n, n+1\}$ is identified as $\{n, 1\}$, is a circle network.

Each network (N, L) is characterized by (i) the total number of edges (projects) which is same as the cardinality of L, and (ii) the degree of firm i, d_i, which shows the number of links in which firm i is involved. Since each firm's profit from a network depends upon its own d_i and rivals' d_j, we can express each firm's profit by the degree sequence $d = (d_1, \cdots, d_n)$ which characterizes the form of L. To be specific, we focus on the cases three or four players. When $n = 3$, we can classify players into three types; the type A firm has no link ($d_A = 0$), the type B firm is involved in one link ($d_B = 1$), and the type C has two projects ($d_C = 2$). The possible network patterns when $n = 3$ are summarized in Figure 1, where, for example, A^s denotes the type A player in network L^s. Note that only the line network and complete network are connected when $n = 3$.

Similarly, when $n = 4$, we can classify players into four types, that is, additional to type A, B, and C firms, there may be type D firm who has three links with other firms ($d_D = 3$). The possible network patterns when $n = 4$ are summarized in Figure 2, where L^0 to L^4 indicate unconnected networks, and L^5 to L^N are connected networks. We assume that all these networks are independent of the naming of vertices (firms).

[Figure 1 and 2 are inserted around here]

4 Pairwise Stability

For a given network L, let $\pi_i(L)$ be the profit of firm i. We consider the case in which players are free to form new links or sever existing links as long as this will increase their payoffs. The concept of pairwise stability reflects this idea, and is defined formally as follows:

Definition 1 *A network (N, L) is pairwise stable if these two conditions are satisfied.*

1. *For all $\{i,j\} \in L$, $\pi_i(L) \geq \pi_i(L \backslash \{i,j\})$ and $\pi_j(L) \geq \pi_j(L \backslash \{i,j\})$.*

2. *For all $\{i,j\} \notin L$, if $\pi_i(L) < \pi_i(L \cup \{i,j\})$, then $\pi_j(L) > \pi_j(L \cup \{i,j\})$.*

We adopted this definition from Jackson and Wolinsky (1996).

5 Cournot Equilibrium under Fixed Network

Remember that firms first form collaboration links, and then compete in the product market. Applying the backward induction, we derive the equilibrium level of $\pi_i(L)$ (and x_i) for each fixed network L in order to examine the incentive for link formation.

Given any network L, it is easily checked that the Cournot equilibrium output for firm i can be written as

$$x_i(L) = \frac{a - nc(d_i) + \sum_{j \neq i} c(d_j)}{b(n+1)}, \ i \in N, \tag{4}$$

and the equilibrium profit for firm i is $\pi_i(L) = b(x_i(L))^2$.

The equilibrium profit for each firm depends only upon its own type and rival firms' types, which are characterized only by the network form. Thus, for any network form, each type firm's profit is independent of firm's name. We denote type M firm's equilibrium profit in network L by $\pi_M(L)$, where $M = A, B, C$ when $n = 3$, and $M = A, B, C, D$ when $n = 4$. In order to analyze the incentives to form links (and change networks), we will compare levels of the equilibrium profits below.

$n = 3$ **case**

First, let us consider the case that $n = 3$. In this case, we have the following lemma which ranks the equilibrium profits:

Lemma 1 *Suppose that $n = 3$. Then, under Assumption 1 and 2, we have the following results.*

1. $\pi_C(L^{\mathrm{III}}) > \pi_C(L^N)$, $\pi_C(L^N) > \pi_B(L^{\mathrm{III}})$, $\pi_C(L^N) > \pi_A(L^{\mathrm{I}})$, $\pi_B(L^{\mathrm{II}}) > \pi_A(L^{\mathrm{I}})$, *and* $\pi_A(L^{\mathrm{I}}) > \pi_A(L^{\mathrm{II}})$ *always hold.*

2. $\pi_B(L^{\mathrm{III}}) > \pi_A(L^{\mathrm{I}})$ *and* $\pi_B(L^{\mathrm{II}}) > \pi_C(L^{\mathrm{IV}})$ *if and only if* $c(d_A)+c(d_C) > 2c(d_B)$ (**Condition I**).

3. $\pi_B(L^{\mathrm{III}}) > \pi_A(L^{\mathrm{II}})$ *if and only if* $3c(d_A) + c(d_C) > 4c(d_B)$ (**Condition II**).

4. $\pi_C(L^{\mathrm{III}}) > \pi_B(L^{\mathrm{II}})$ *if and only if* $4c(d_B) > c(d_A) + 3c(d_C)$ (**Condition III**).

Figure 3 summarizes the results of Lemma 1.

[Insert Figure 3 around here.]

Note that Condition I is the sufficient condition for Condition II. Furthermore, Condition I can be rewritten as $c(d_B) - c(d_A) > c(d_C) - c(d_B)$, which asserts that the effect of additional link decreases as the number of links increases. This closely relates to the convexity of the link formation game.

$n = 4$ **case**

Next, we examine the case $n = 4$, where we focus only on connected graphs (L^5 to L^N). (In the following section, we will discuss link formation movements and pairwise stable networks considering only unconnected graphs. However, it can be shown that the possible pairwise stable networks among all four firm graphs are L^N and L^4.) The following lemma characterizes the equilibrium profits for each network structures. Figure 4 summarizes the main results which will be used below.

Lemma 2 *Suppose that $n = 4$. Then, under Assumption 1 and 2, we have the following results.*

1. $\pi_D(L^5) > \pi_C(L^6)$, $\pi_D(L^5) > \pi_D(L^7)$, $\pi_D(L^7) > \pi_D(L^9)$, $\pi_D(L^9) > \pi_D(L^N)$, $\pi_D(L^N) > \pi_C(L^8)$, $\pi_C(L^6) > \pi_C(L^8)$, $\pi_C(L^8) > \pi_C(L^9)$, $\pi_C(L^8) > \pi_B(L^5)$, $\pi_C(L^8) > \pi_B(L^6)$, $\pi_B(L^5) > \pi_B(L^7)$, *and* $\pi_B(L^6) > \pi_B(L^7)$ *always hold.*

2. $\pi_C(L^7) > \pi_C(L^8)$ *and* $\pi_B(L^5) > \pi_B(L^6)$ *if and only if* $c(d_B) + c(d_D) > 2c(d_C)$ (**Condition 1**).

3. $\pi_C(L^6) > \pi_D(L^9)$ *if and only if* $2c(d_B) + 3c(d_D) > 5c(d_C)$ (**Condition 2**).

4. $\pi_C(L^6) > \pi_D(L^N)$ *and* $\pi_C(L^9) > \pi_B(L^5)$ *if and only if* $2c(d_B)+c(d_D) > 3c(d_C)$ (**Condition 3**).

5. $\pi_C(L^6) > \pi_D(L^7)$ *and* $\pi_C(L^7) > \pi_D(L^9)$ *if and only if* $c(d_B)+4c(d_D) > 5c(d_C)$ (**Condition 4**).

6. $\pi_C(L^9) > \pi_B(L^7)$ *and* $\pi_C(L^7) > \pi_B(L^6)$ *if and only if* $5c(d_C) > 4c(d_B)+c(d_D)$ (**Condition 5**).

7. $\pi_C(L^9) > \pi_B(L^6)$ *if and only if* $3c(d_B) + 2c(d_D) > 5c(d_C)$ (**Condition 6**).

8. $\pi_C(L^7) > \pi_D(L^N)$ *if and only if* $c(d_B) + 2c(d_D) > 3c(d_C)$ (**Condition 7**).

Moreover, note that Condition 1 is sufficient condition for Condition 3, 5, and 6, and Condition 4 is sufficient for Condition 1.

[Insert Figure 4 around here.]

6 Network Formation and Firm Behavior

We next go back to the first stage of the game, and examine firm's incentive to form collaboration links. Remember that the process of network formation is governed by the pairwise stability. As a result, a new link will be formed if a pair of players finds it advantageous to cooperate and a link will be severed if some firm thinks that the opposite is the case.

$n = 3$ case

When $n = 3$, Lemma 1 says that

1. The transition $L^{\mathrm{I}} \rightarrow L^{\mathrm{II}}$ always realizes. A pair of A^{I} players forms a link since $\pi_B(L^{\mathrm{II}}) > \pi_A(L^{\mathrm{I}})$.
2. The transition $L^{\mathrm{III}} \rightarrow L^{N}$ always realizes. A pair of B^{III} players forms a link since $\pi_C(L^{N}) > \pi_B(L^{\mathrm{III}})$.
3. The transition $L^{\mathrm{II}} \rightarrow L^{\mathrm{III}}$ realizes if both Condition II and Condition III are satisfied. A pair of B^{II} and A^{II} forms a new link since both $\pi_B(L^{\mathrm{III}}) > \pi_A(L^{\mathrm{II}})$ and $\pi_C(L^{\mathrm{III}}) > \pi_B(L^{\mathrm{II}})$ are satisfied under Condition II and Condition III. If the reverse inequality of Condition II or Condition III holds, then $L^{\mathrm{III}} \rightarrow L^{\mathrm{II}}$.

As a result, we obtain the following results.

Proposition 1

Suppose that $n = 3$. Then,

1. *If both Condition II and Condition III are satisfied* $(3c(d_A)+c(d_C) > 4c(d_B) > c(d_A) + 3c(d_C))$*, the complete network* L^{IV} *is pairwise stable.*
2. *If the reverse inequality of Condition II or that of Condition III holds* $(3c(d_A)+c(d_C) < 4c(d_B)$ *or* $4c(d_B) < c(d_A)+3c(d_C))$*, both* L^{II} *and the complete network*

L^N are pairwise stable. In this case, (i) if the initial network is L^{I} or L^{II}, then link formation processes reaches L^{II}, (ii) if the initial network is L^{III}, then link formation processes can realizes both L^{II} and L^N (multi-valued), and (iii) if the initial network is L^N, then the network does not change.

$n = 4$ **case**

When $n = 4$, Lemma 2 says that

1. The transition $L^5 \rightarrow L^7$ is always possible. A pair of B^5 players forms a link since $\pi_C(L^7) > \pi_B(L^5)$.

2. The transition $L^6 \rightarrow L^8$ is always possible. A pair of B^6 players forms a link since $\pi_C(L^8) > \pi_B(L^6)$.

3. The transition $L^8 \rightarrow L^9$ is always possible. A pair of C^8 players forms a link since $\pi_D(L^9) > \pi_C(L^8)$.

4. The transition $L^9 \rightarrow L^N$ is always possible. A pair of C^9 players forms a link since $\pi_D(L^{\mathrm{N}}) > \pi_C(L^9)$.

5. If the reverse inequality of Condition 4 and Condition 5 are satisfied, then the transition $L^6 \rightarrow L^7$ is possible. A pair of C^6 and B^6 forms a link since $\pi_D(L^7) > \pi_C(L^6)$ under the reverse inequality of Condition 4 and $\pi_C(L^7) > \pi_B(L^6)$ under Condition 5. If Condition 4 or the reverse inequality of Condition 5 holds, the transition $L^7 \rightarrow L^6$ is possible.

6. If the reverse inequality of Condition 1 holds, then the transition $L^5 \rightarrow L^6$ is possible. A pair of B^5 and A^5 forms a link, with severing a link with D^5, since $\pi_B(L^6) > \pi_B(L^5)$ under the reverse inequality of Condition 1 and $\pi_A(L^6) > \pi_A(L^5)$. Under Condition 1, the transition $L^6 \rightarrow L^5$ may occur.

7. If Condition 1 holds, then the transition $L^8 \rightarrow L^7$ is possible. Some C^8 player severs a link and form a link another firm since $\pi_C(L^7) > \pi_C(L^8)$ under Condition 1 and $\pi_D(L^7) > \pi_C(L^8)$ always. Otherwise, the transition $L^7 \rightarrow L^8$ may occur.

8. If the reverse inequality of Condition 5 holds, then the transition $L^9 \rightarrow L^7$ may occur. A pair of B^9 and C^9 players severs a link since $\pi_A(L^7) > \pi_B(L^9)$ and $\pi_B(L^7) > \pi_C(L^9)$ under Condition 1. Under Condition 5, the transition $L^7 \rightarrow L^9$ is possible.

When focusing on connected networks with four firms, Proposition 2 summarizes the transitions of network structures, where the transition dynamics is multi-valued in many cases:

Proposition 2

Suppose that $n = 4$. Then, under Assumption 1 and 2, we have the following results:

1. *The complete network L^N is always pairwise stable.*

2. *If Condition 5 ($5c(d_C) > 4c(d_B) + c(d_D)$) is satisfied, then network transition process always reaches to L^N.*

3. *If Condition 1 holds ($c(d_B) + c(d_D) > 2c(d_C)$), then there is a cycle of network transition movement which consists of L^6, L^7, L^8, and L^9.*

4. *If both Condition 1 and the reverse inequality of Condition 5 hold ($c(d_B) + c(d_D) > 2c(d_C)$ and $4c(d_B) + c(d_D) > 5c(d_C)$), then there is a cycle of network transition movement which consists of L^7, L^8, and L^9.*

Figure 5 summarizes the results of Proposition 2 with emphasizing on the multi-valued nature of transition dynamics.

[Insert Figure 5 around here.]

Next, let us examine the pairwise stable networks for unconnected networks (L^0 to L^4) as well as connected networks. Now, suppose that the current network is L^4. In this case, if $c(d_A) - c(d_B) > 4(c(d_C) - c(d_D))$, then $\pi_C(L^4) > \pi_D(L^7)$, and C^4 player has no incentive to form a link with A^4 player. Furthermore, if $c(d_C) - c(d_D) > 4(c(d_A) - c(d_B))$, then $\pi_B(L^4) > \pi_A(L^7)$, and A^4 player has no incentive to form a link with C^4 player. As a result, either $c(d_A) - c(d_B) > 4(c(d_C) - c(d_D))$ or $c(d_C) - c(d_D) > 4(c(d_A) - c(d_B))$ makes L^4 be the pairwise stable network. Similar analysis for the possible transitions from L^4 leads us to the following result:

Proposition 3

Suppose that $n = 4$. Then, under Assumption 1 and 2, if either $c(d_A) - c(d_B) > 4(c(d_C) - c(d_D))$ or $c(d_C) - c(d_D) > 4(c(d_A) - c(d_B))$ is satisfied, L^4 is the pairwise stable network.

7 Conclusion

We have analyzed a model of Cournot oligopoly with link formation and examined the nature of equilibrium networks. It is established that for the case of four players, depending on the parameters of the model, there can arise two pairwise stable networks; one is the complete network L^N and the other is the triangular links with an isolated player called L^4.

Our network formation game defines a (multi-valued) dynamics on the set of graphs. Even for the case of three players and $n = 4$ players, link formation processes can be fairly complicated. Assuming simple linear demand and cost functions, we have shown that for the case four players, a cycle of order 3 or 4 may arise, both of which involves a wine glass form network. This network can be transferred into many other networks and hence plays the central role for the emergence of a cycle.

Although our analysis was confined to the cases of three and four players, our results may be extended to the general n player environment. We may also consider a variety of link formation games. An example is a co-author game studied by Jackson and Wolinsky (1996) and Kawamata and Tamada (2004). The game's outcome depends on the number of players directly connected to each other. We could show that different networks may form depending on the value function of the game, which describes productive externalities between players. Further generalization of the game may allow us to assess how various form of network structure arise as a result of local incentives and externalities. It is hoped that the paper gives a clue to analyze how various forms of network emerge and how the network changes to another. These analyses await future papers.

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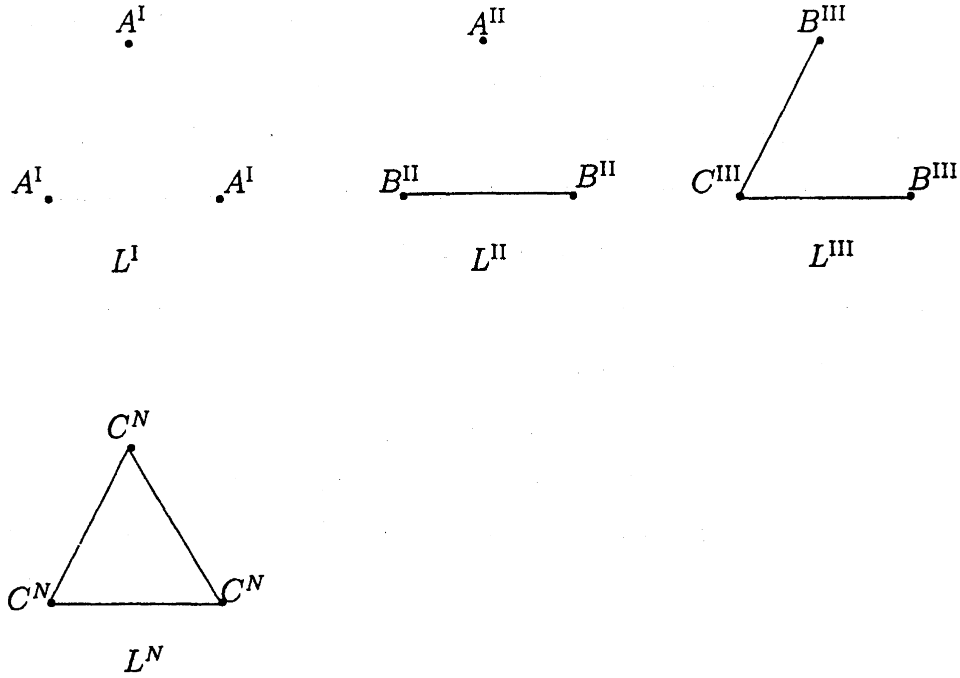
A^{I}
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C^{N}
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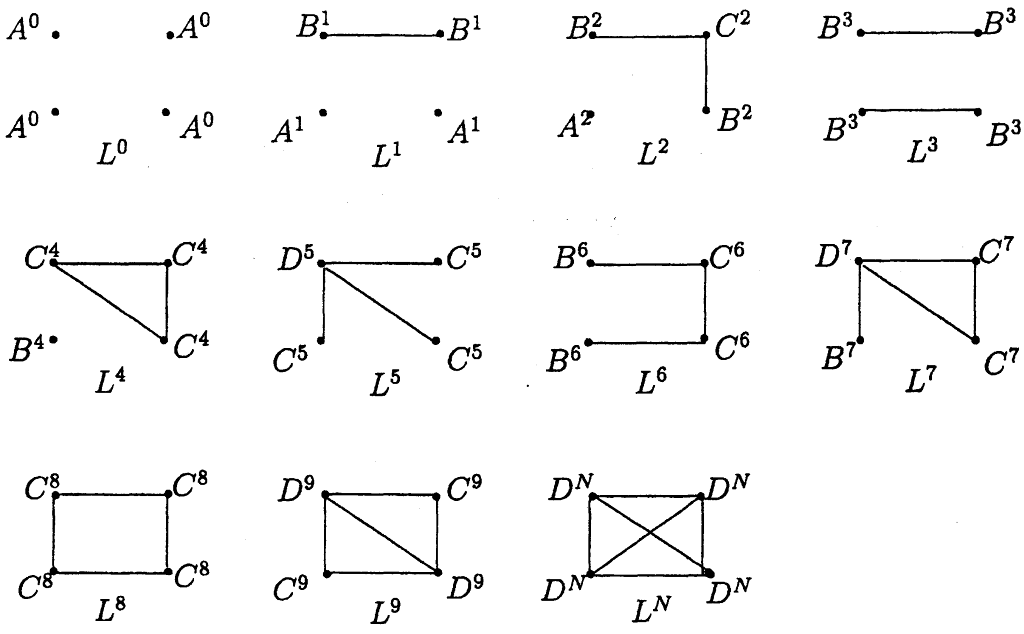
Figure 1



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L^2
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L^3
C^4 C^4 B^4 C^4
L^4
D^5 C^5 C^5 C^5
L^5
B^6 C^6 B^6 C^6
L^6
D^7 C^7 B^7 C^7
L^7
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D^9 C^9 C^9 D^9
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L^N

Figure 2

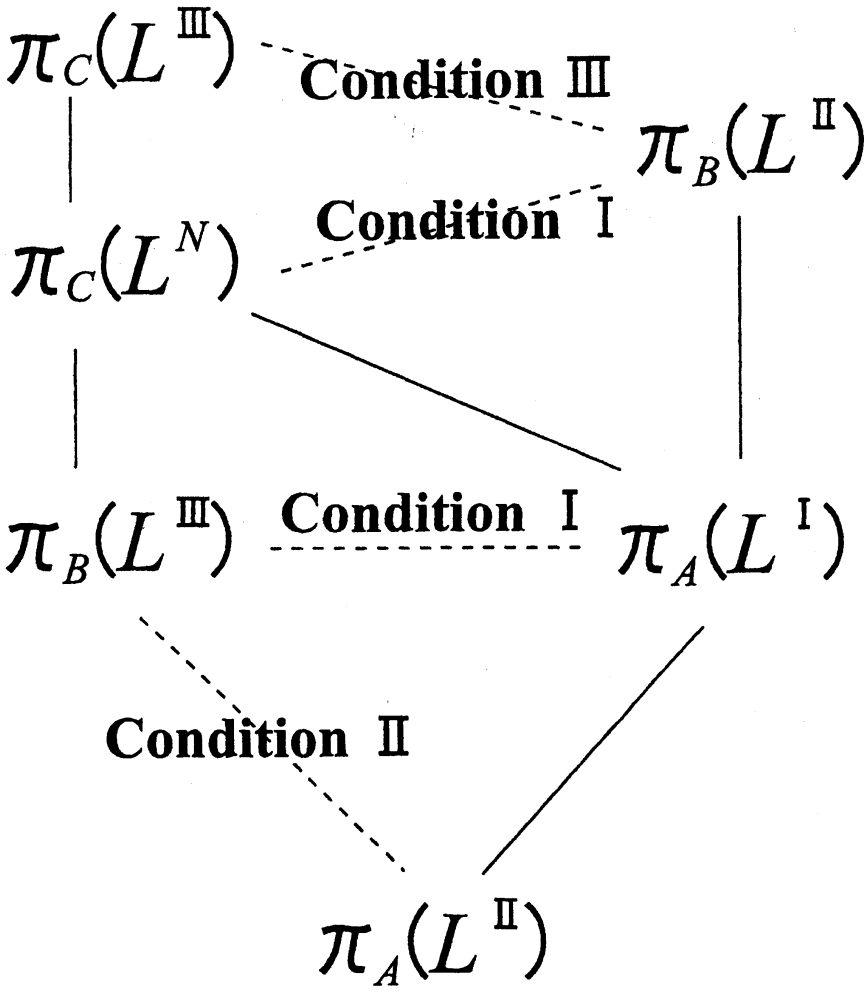


Figure 3

Comparison of equilibrium profits when $n = 3$.

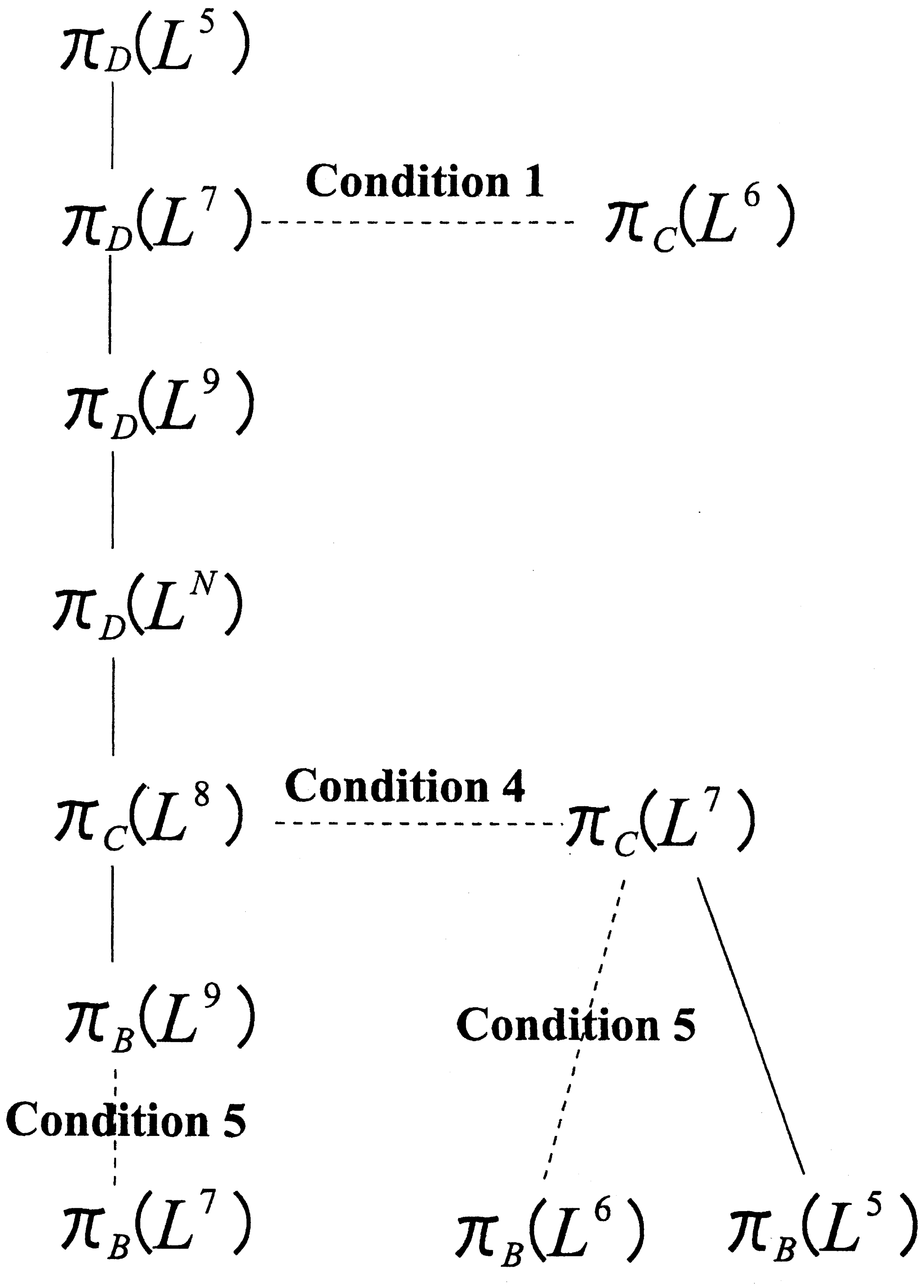
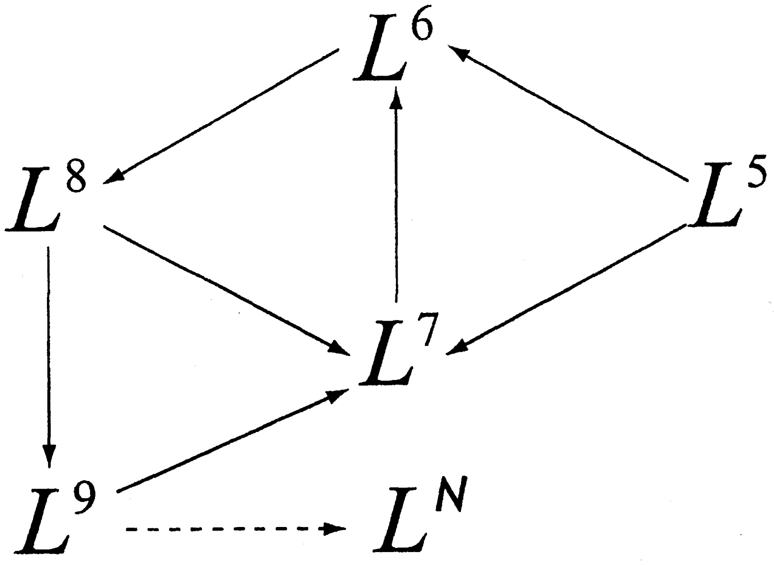


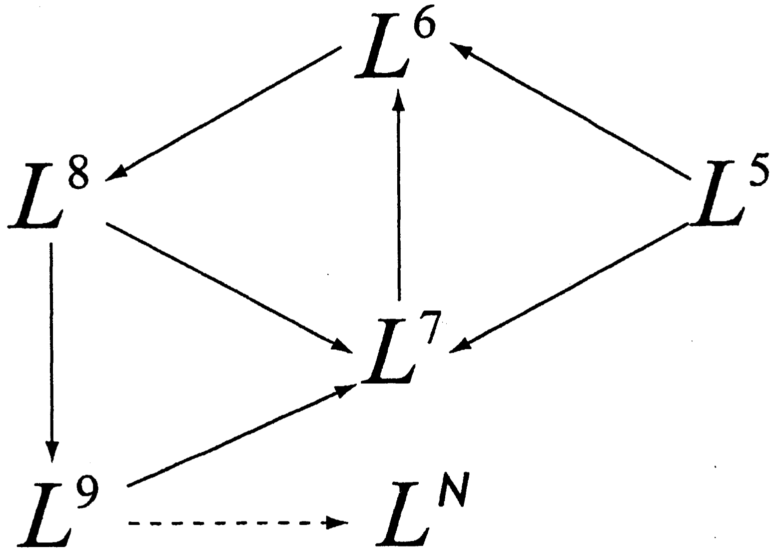
Figure 4

Comparison of equilibrium profits when $n = 4$.

Case (a) $(C(d_B) + C(d_D) > 2C(d_C)$



Case (b) $C(d_B) + C(d_D) > 2C(d_C)$, $4C(d_B) + C(d_D) > 5C(d_C)$



Case (c) $5C(d_C) > 4C(d_B) + C(d_D)$

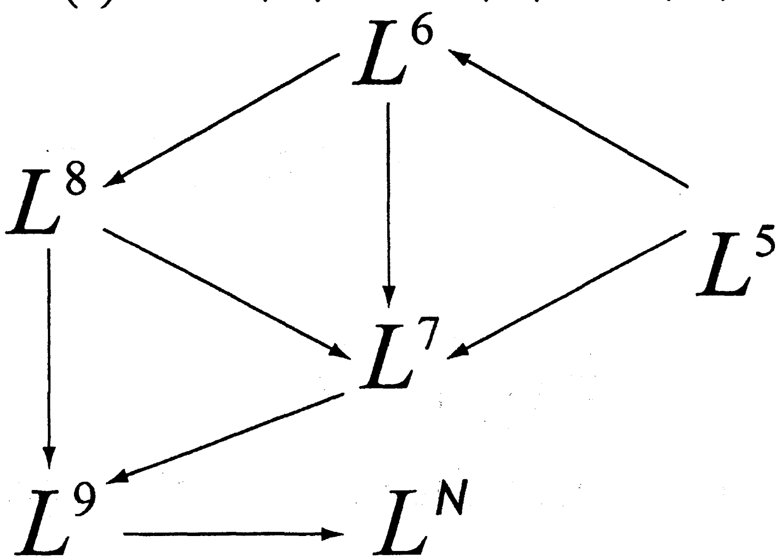


Figure 5