2-ELEMENTS OUTSIDE OF THE DRESS SUBGROUP OF TYPE 2

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1. Introduction

Let $G$ be a finite group. We denote by $\pi(G)$ the set of prime divisors of the order of $G$. For a prime $p$, we denote by the symbol $O^p(G)$, called the Dress subgroup of $G$ of type $p$, the smallest normal subgroup of $G$ such that $\pi(G/O^p(G)) \subseteq \{p\}$. We denote by $\mathcal{P}(G)$ the set of subgroups $P$ of $G$ of prime power order, possibly 1 and by $\mathcal{L}(G)$ the set of subgroups $H$ of $G$ containing the Dress subgroup $O^p(G)$ of type $p$ for some prime $p$.

We say that a $G$-module $V$ is $\mathcal{L}(G)$-free if $\dim V^{O^p(G)} = 0$ holds for any prime $p$. Here a $G$-module means a $\mathbb{R}[G]$-module which is finite dimensional over $\mathbb{R}$. We denote by $\mathcal{D}(G)$ the set of all pairs $(P, H)$ of subgroups of $G$ such that $P < H \leq G$ and $P$ is of prime power order. A $G$-module $V$ is called a gap $G$-module if $V$ is $\mathcal{L}(G)$-free and the number

$$\dim V^P - 2 \dim V^H$$

is positive for any pair $(P, H) \in \mathcal{D}(G)$. A finite group $G$ is called a gap group if there exists a gap $G$-module and is called a nongap group otherwise.

A finite group $G$ is an Oliver group, if $G$ has no isthmus series of subgroups of the form

$$P < H < G$$

where $|\pi(P)| \leq 1$, $|\pi(G/H)| \leq 1$ and $H/P$ is cyclic. A finite group $G$ has a fixed point free smooth action on a disk if and only if $G$ is an Oliver group ([5]). Furthermore, Oliver has completely decided which a smooth compact manifold is the fixed point set of a smooth action on a disk ([6]). On the other hand, Laitinen and Morimoto ([2]) has shown that a finite group $G$ has a smooth one fixed point action of a sphere

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if and only if \( G \) is an Oliver group. We do not know which a smooth manifold of positive dimension is the fixed point set of a smooth action on a sphere. For an Oliver group \( G \) which is a gap group, one can apply equivariant surgery to convert an appropriate smooth action of \( G \) on a disk \( D \) into a smooth action of \( G \) on a sphere \( S \) with \( S^G = M = D^G \), where \( \dim M > 0 \) (cf. [3, Corollary 0.3]). Thus it is important to ask whether a given group \( G \) is a gap group.

2. Centralizers of 2-elements outside of the Dress subgroup of type 2

Let \( G \) be a finite group. An element \( x \) of \( G \) is a 2-element if the order of \( x \) is a power of 2 or equals to 1. Let \( K \) be a normal subgroup of \( G \) with \( K \geq O^2(G) \).

For an element \( x \) of \( G \), we denote by \( \psi(x) \) the set of odd primes \( q \) such that there exists a subgroup \( N \) of \( G \) satisfying \( x \in N \) and \( O^q(N) \neq N \). We define a subset \( E_2(G, K) \) of \( G \setminus K \) as the set of involutions (elements of order 2) \( x \) such that either \( |\psi(x)| > 1 \) or \( |\pi(C_G(x))| = |\pi(O^2(C_G(x)))| = 2 \) holds, and define \( E_4(G, K) \) as the subset of 2-elements \( x \) of \( G \setminus K \) of order \( \geq 4 \) with \( |\psi(x)| > 0 \). Set \( E(G, K) = E_2(G, K) \cup E_4(G, K) \) (cf. [8]). Note that \( E_2(G, K) = \emptyset \) if \( K \neq O^2(G) \). We define sets \( E_2^g(G, K) \), \( E_4^g(G, K) \) and \( E^g(G, K) \) as follows. The set \( E_2^g(G, K) \) consists of 2-elements \( x \) of \( G \setminus K \) of order \( > 2 \) such that \( C_G(x) \) is not a 2-group. The set \( E_4^g(G, K) \) consists of involutions \( x \) of \( G \setminus K \) such that \( |\pi(O^2(C_G(x)))| \geq 2 \) holds. Set \( E^g(G, K) = E_2^g(G, K) \cup E_4^g(G, K) \). Note that the sets \( E_2^g(G, K) \), \( E_4^g(G, K) \) and \( E^g(G, K) \) are subsets of \( E_2(G, K) \), \( E_4(G, K) \) and \( E(G, K) \) respectively.

We set

\[
\mathcal{D}^2(G) = \left\{ (P, H) \in \mathcal{D}(G) \mid [H : P] = [O^2(G)H : O^2(G)P] = 2 \text{ and } O^q(G)P = G \text{ for all odd primes } q \right\}.
\]

(cf. [4]) and set

\[
\mathcal{D}^2(G, K) = \left\{ (P, H) \in \mathcal{D}^2(G) \mid H \not\leq K \right\}.
\]

According to Laitinen and Morimoto [2], we denote by \( V(G) \) the \( G \)-module

\[
(\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \in \pi(G)} (\mathbb{R}[G/O^p(G)] - \mathbb{R}).
\]
If $G$ is a group of prime power order, then $V(G) = \{0\}$ holds. Laitinen and Morimoto [2, Theorems 2.3 and B] have shown that $V(G)$ is an $\mathcal{L}(G)$-free $G$-module such that
\[ \dim V(G)^P - 2 \dim V(G)^H \]
is nonnegative for any pair $(P, H) \in \mathcal{D}(G)$ and is zero only if either $(P, H) \in \mathcal{D}^2(G, \emptyset)$ or $P \in \mathcal{L}(G)$. Note that $P \notin \mathcal{L}(G)$ for $(P, H) \in \mathcal{D}(G)$ if $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint.

**Theorem 1.** Let $G$ be a finite group such that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint. Let $K$ be a subgroup of $G$ with index 2. Then the following claims are equivalent.

1. $E^g(G, K)$ is empty.
2. $E(G, K)$ is empty.
3. There exist pairs $(P_j, H_j) \in \mathcal{D}^2(G, K)$ such that
\[ \sum_j \left( \dim V^{P_j} - 2 \dim V^{H_j} \right) = 0 \]
for any $\mathcal{L}(G)$-free $G$-module $V$.

**Corollary 2.** If $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint, then either sets $E(G, O^2(G))$ and $E^g(G, O^2(G))$ are both empty or both nonempty.

### 3. Nongap groups

Let $G$ be a finite group such that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint. The group $G$ is a gap group if and only if any subgroup $K$ of $G$ with $K > O^2(G)$ is a gap group. Therefore it is easy to see the following result by Theorem 1.

**Theorem 3.** Let $G$ be a finite group and let $K$ be a gap subgroup of $G$ with index 2. Then the following claims are equivalent.

1. $E^g(G, K)$ is empty.
2. $E(G, K)$ is empty.
3. $G$ is a nongap group.

Now, assume that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Recall that if $\mathcal{P}(G) \cap \mathcal{L}(G) \neq \emptyset$, then $G$ is a nongap group.
Proposition 4. Let $G$ be a finite group such that $O^2(G) \neq G$ and $P(G) \cap L(G) = \emptyset$, and let $K$ be a subgroup of $G$ such that $[G, K] = 2$. Suppose that $E^2(G, K) = \emptyset$. Let $G_2$ be a Sylow 2-subgroup of $G$. Then it holds the followings.

1. If two elements $x$ and $y$ of $G_2$ outside of $K$ are conjugate in $G$, then they are conjugate in $G_2$.

2. \[
\sum_{(x)_{G}} \frac{2}{|C_G(x)|} = 1, \text{ where } (x)_G \text{ runs over conjugacy classes in } G \text{ represented by elements of } G_2 \text{ outside of } K.
\]

3. \[
\sum_{(C)_{G}} \frac{|C|}{|N_G(C)|} = 1, \text{ where } (C)_G \text{ runs over conjugacy classes in } G \text{ represented by cyclic groups } C \text{ of } G_2 \text{ with } CK = G.
\]

Proof. For an element $x$ of $G \setminus K$, we denote by $x_2$ the involution of the cyclic subgroup generated by $x$. As $E^2_2(G)$ is empty, $x_2$ is an element outside of $K$. Recall that if two elements $x$ and $y$ of $G \setminus K$ are conjugate in $G$, namely $x = g^{-1}yg$, for some $g \in G$, then $x_2 = g^{-1}y_2g$ and thus $g \in C_G(x_2)$. Since $E^2_2(G, K)$ is empty and

\[
\sum_{(x)_{G \setminus G \setminus K}} \frac{|G|}{|C_G(x)|} = |G| - |K| = \frac{|G|}{2},
\]

we have

\[
1 = \sum_{(x)_{G \setminus G \setminus K}} \frac{2}{|C_G(x)|} = \left( \sum_{(x)_{G \setminus G \setminus K}} \right) \frac{2}{|C_G(x)|} + \left( \sum_{(x)_{G \setminus G \setminus K}} \right) \frac{2}{|C_G(x)|} + \left( \sum_{(x)_{G \setminus G \setminus K}} \right) \frac{2}{|C_G(x)|}
\]

\[
= \sum_{(x)_{G \setminus G \setminus K}} \left( \frac{2}{|C_G(y)|} + \sum_{\substack{(x)_{G \setminus G \setminus K} \in (x)_{G \setminus G \setminus K} \setminus \{x\} \setminus \{y\}}} \frac{2}{|C_G(x)|} \right) + \sum_{(x)_{G \setminus G \setminus K}} \frac{2}{|C_G(x)|}
\]

\[
= \sum_{(x)_{G \setminus G \setminus K}} \sum_{\substack{(x)_{G \setminus G \setminus K} \in (x)_{G \setminus G \setminus K} \setminus \{x\} \setminus \{y\}}} \frac{2}{|C_G(y)(x)|} + \sum_{(x)_{G \setminus G \setminus K}} \frac{2}{|C_G(x)|}.
\]

Set $L(y) = O^2(C_G(y))(y) \cong O^2(C_G(y)) \times \langle y \rangle$. Let $B(y)$ (resp. $C(y)$) be the set of conjugacy classes in $C_G(y)$ which are represented by elements of $L(y) \setminus O^2(C_G(y))$ (resp. $O^2(C_G(y))$). Note that if two elements $x$ and $x'$ of $G$ outside of $K$ with $x_2 = x'_2$
are conjugate in $G$, then they are conjugate in $C_{G}(x_2)$. Therefore we obtain that

$$1 = \sum_{y \in G} \sum_{z \in G} \frac{2}{|C_{G}(y)(x)|} + \sum_{y \in G} \frac{2}{|C_{G}(x)|}$$

$$= \sum_{y \in G} \sum_{z \in G} \frac{2}{|C_{G}(y)(x^2)|} + \sum_{y \in G} \frac{2}{|C_{G}(x)|}$$

(5)

$$= \sum_{y \in G} \sum_{z \in G} \frac{2}{|C_{G}(y)(z)|} + \sum_{y \in G} \frac{2}{|C_{G}(x)|}.$$  

Let $A$ be the set of conjugacy classes $(x)_{G_2}$ in $G_2$ represented by elements of $G_2 \setminus (G_2 \cap K)$. As $E_4^2(G, K)$ is empty, we have $C_{G}(x)$ for $x \in G \setminus K$ with $|x| = 2^r > 2$ is a 2-group. Furthermore by using the assumption that $E_2^2(G, K)$ is empty again, the last number at (5) equals to

$$\sum_{y \in G} \frac{2|O^2(C_{G}(y))|}{|C_{G}(y)|} + \sum_{y \in G} \frac{2}{|C_{G}(x)|}$$

(6)

$$= \sum_{y \in G} \frac{2}{|C_{G}(y)|} + \sum_{y \in G} \frac{2}{|C_{G}(x)|} \leq \sum_{y \in G} \frac{2}{|C_{G}(y)|} = 1,$$

where $C_{G}(x)$ (resp. $C_{G}(y)$) is a Sylow 2-subgroup of $C_{G}(x)$ (resp. $C_{G}(y)$). Therefore any inequality or equality in (6) must be equality and thus if $x, y \in G_2$ are conjugate in $G$, then they are conjugate in $G_2$.

**Theorem 7.** Let $G$ be a nongap group satisfying that $P(G) \cap L(G) = \emptyset$ and that $[G : O^2(G)] = 2$. Let $G_2$ be a Sylow 2-subgroup of $G$. Suppose the order of $G$ is divisible by 4. Then it holds the followings.

1. If $x$ and $y$ are involutions of $G_2 \setminus K$, then $xy \in [G_2, G_2]$.
2. There exists an element $x$ of $G_2 \setminus K$ such that $|x| > 2$.
3. The group generated by all involutions of $G_2$ outside of $K$ is a proper subgroup of $G_2$.

**Theorem 8.** Let $G$ be a finite group satisfying that $P(G) \cap L(G) = \emptyset$ and that $G/[G, G]$ is not a 2-group. If $G$ is a nongap group, then $O^2(G)$ is of odd order.

**Proof.** If $G$ is perfect, then $G$ is a gap group. Suppose that $G/[G, G]$ is of even order. Let $K$ be a subgroup of $G$ such that $K > O^2(G)$, $[K : O^2(G)] = 2$ and $O^2(K/O^2(K))$ is isomorphic to $O^2(G/O^2(G))$. If $G$ is a nongap group, then $K$ is
also a nongap group. There exist no 2-elements, not involutions, of \( K \) outside of \( O^2(K) \). If there might exist such an element \( x \), then \( x \) lies in \( E(K, O^2(K)) \) which implies that \( K \) is a gap group by Theorem 1. Therefore, the group generated by all involutions of \( K_2 \) outside of \( K \) is just \( K_2 \), where \( K_2 \) is a Sylow 2-subgroup of \( K \). By Theorem 7 (3), the order of \( K \) is not divisible by 4. Since \([K : O^2(K)] = 2\), the order of \( O^2(K) = O^2(G) \) is odd. □

**Corollary 9.** Let \( G \) be a finite group satisfying that \( \mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset \) and that \( G/[G, G] \) is not a 2-group. If \( G \) is a nongap group, then \( G \) is solvable.

**Proof.** By Theorem 8, the Dress group \( O^2(G) \) of type 2 is of odd order. Recall that \( G/O^2(G) \) is a 2-group. By Burnside's theorem, \( O^2(G) \) and \( G/O^2(G) \) are both solvable. Thus \( G \) is solvable. □

Note that a finite group \( G \) such that \( \mathcal{P}(G) \cap \mathcal{L}(G) \neq \emptyset \) is solvable.

### 4. Direct product

**Lemma 10.** Let \( G \) be a finite group such that \( O^2(G) \neq G \) and \( \mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset \), and let \( K \) be a subgroup of \( G \) such that \([G, K] = 2\). If all elements of \( H \) outside of \( K \) are 2-elements, then

\[
\sum_{(C)_{G}} |N_{G}(C)/C|^{-1} |(H \setminus G)^C| = 1
\]

where \((C)_{G}\) runs over conjugacy classes in \( G \) represented by cyclic groups \( C \) of \( G \) with \( CK = G \).

We define \( E^d(G, K) \) as the set of 2-elements \( x \) of \( G \) outside of \( K \) such that \( C_G(x) \) is not a 2-group. Note that \( E^d(G, K) \) is a subset of \( E^d(G, K) \). There exist finite groups \( G \) so that \([G : O^2(G)] = 2\) and \( E^d(G, O^2(G)) \) is empty. A solvable group SmallGroup(1920, 239651) and a nonsolvable group SmallGroup(1344, 11427) both satisfy such conditions. (cf. [1])

**Proposition 11.** Let \( G \) be a finite group such that \( O^2(G) \neq G \) and \( \mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset \), and let \( K \) be a subgroup of \( G \) such that \([G, K] = 2\). Suppose that \( E^d(G, K) = \emptyset \). Let \( G_2 \) be a Sylow 2-subgroup of \( G \) and let \( C \) be a cyclic subgroup of \( G \) with \( CK = G \). Then it holds the followings.
(1) If a subgroup of $G_2$ intersects with any conjugacy class $(x)_G$ represented by elements of $G_2$ outside of $K$, then it is just $G_2$.

(2) $|(G_2 \backslash G)/N_G(C)| = 1$ holds. In particular, $(G_2 \backslash G)^C = G_2 \backslash G N_G(C)$, if $C < G_2$.

**Proof.** Let $C$ be a cyclic subgroup of $G$ with $CK = G$. By assumption, $(H \backslash G)^C$ is nonempty. By Proposition 4 (3), we obtain that

$$\sum_{(C)_G} |N_G(C)/C|^{-1} |(H \backslash G)^C| \geq \sum_{(C)_G} \frac{|C|}{|N_G(C)|} = 1,$$

where $(C)_G$ runs over conjugacy classes in $G$ represented by cyclic groups $C$ of $G_2$ with $CK = G$. Furthermore as $C$ is a 2-group, we obtain that

$$\sum_{(C)_G} |N_G(C)/C|^{-1} |(H \backslash G)^C| = \sum_{(C)_G} \frac{|C|}{|N_G(C)|} = 1$$

by Lemma 10 and thus

$$|(H \backslash G)^C| = 1.$$ 

Take an element $a \in G$ such that $aCa^{-1} \leq H$. Then we have

$$(H \backslash G)^C \supseteq H \backslash N_G(H)a.$$ 

Supposing that $H \neq G_2$, it holds $N_G(H) \neq H$, which implies $|(H \backslash G)^C| \geq 2$. □

**Theorem 12.** Let $G$ be a finite group satisfying that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint, $|O^2(G)|$ is even and $G/O^2(G)$ is cyclic. Let $K$ be a subgroup of $G$ with index 2. Then the following claims are equivalent.

1. $E^d(G, K)$ is nonempty.
2. $G \times G$ is a gap group.
3. $G^k = \underbrace{G \times \cdots \times G}_{k \text{ times}}$ is a gap group for $k \geq 2$.

Note that $G^k$ is a nongap group for any $k \geq 1$ if $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are not disjoint, since $\mathcal{P}(G^k)$ and $\mathcal{L}(G^k)$ are not disjoint. The assumption that $|O^2(G)|$ is even is need.

**Remark 13.** Let $p$, $q$ and $r$ be odd primes with $p \neq q$. Let $G = D_{2pq} \times C_r$ be the direct product group of a dihedral group $D_{2pq}$ of order $2pq$ and a cyclic group $C_r$ of order $r$. Then it holds that $E^d(G, O^2(G))$ is nonempty, $O^2(G)$ is of order odd and $G^k$ is a nongap group for any $k \geq 1$. 

Corollary 14. Let $G$ be a finite group satisfying that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, $|O^2(G)|$ is even and $[G : O^2(G)] = 2$. Let $k > 1$ be an integer. Then we have the following claims:

1. $G$ and $G^k$ are gap groups $\iff E^d(G, O^2(G)) \neq \emptyset$.
2. $G^k$ is a gap group and $G$ is a nongap group $\iff E^d(G, O^2(G)) = \emptyset$.
3. $G^k$ (and $G$) are nongap groups $\iff E^d(G, O^2(G)) = \emptyset$.

5. Wreath product

Let $K$ and $L$ be finite groups. We denote by $K \int L$ the semidirect product group $K^{[L]} \rtimes L$ such that $L$ acts on $K^{[L]}$ by permutation:

$$1 \rightarrow K^{[L]} \rightarrow K \int L \rightarrow L \rightarrow 1$$

Proposition 15. Let $G$ be a finite group satisfying that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and that $G/O^2(G)$ is cyclic. Let $K$ be a subgroup of $G$ with index 2. If $G \int C_n$ is a gap group for a 2-power integer $n$, then $E^d(G, K)$ is nonempty, where $C_n$ is a cyclic group of order $n$.

Let $G = \text{SmallGroup}(1344, 11427)$. It is a nonsolvable group satisfying that $[G : O^2(G)] = 2$ and $E^d(G, O^2(G)) = \emptyset$. By Corollary 9, $G \int C_n$ is a gap group for any integer $n > 1$, not a 2-power.

Theorem 16. Let $G$ be a finite group satisfying that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint. For any subgroup $K$, $O^2(G) \triangleleft K \leq G$, possessing a cyclic quotient $K/O^2(G)$, the set $E(K, K_0)$ is nonempty, if and only if $G$ is a gap group, where $K_0$ is a subgroup of $K$ with index 2.

Corollary 17. Let $G$ be a finite group satisfying that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint and $[G : O^2(G)] = 2$. The set $E(G, O^2(G))$ is nonempty if and only if $G$ is a gap group.

Before closing this section, we show the following theorem:
Theorem 18. Let $G$ be a finite group satisfying that $G/O^2(G)$ is cyclic, $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, $E^d(G,K) \neq \emptyset$ and that $O^2(G)$ is of even order, where $K$ is a subgroup of $G$ with index 2. For any nontrivial finite group $L$, the wreath product group $G \wr L$ is a gap group.

First we show the assertion in the case where $L = C_2$:

Lemma 19. Let $G$ and $K$ be finite groups as in Theorem 18. For a cyclic subgroup $C = C_2$ of order 2, the wreath product group $G \wr C$ is a gap group.

Proof. Let $\pi: G \rightarrow (G \wr C)/O^2(G \wr C) \cong (G/O^2(G)) \wr C$ be an epimorphism. If $\pi^{-1}(\pi(\langle x \rangle))$ is a gap group for any nontrivial 2-element $x$ of $(G/O^2(G)) \wr C$, then $G \wr C$ is a gap group. Note that $O^2(G \wr C) = O^2(G)^2 = O^2(G) \times O^2(G)$. Let $f$ be a generator of $C$. Let $h$ be a 2-element of $G$ outside of $K$ such that $C_G(h)$ is not a 2-group. Recall that $G \times G$ is a gap group by Theorem 12. It suffices to show that

$$N := \langle O^2(G)^2, (h_1, h_2)f \rangle$$

is a gap group for any elements $h_1$ and $h_2$ of $\langle h \rangle$. Note that

$$(h_1, h_2)f^2 = (h_1h_2, h_2h_1).$$

We obtain that

$$C_{G_2}((h_1, h_2)f) = \left\langle (h_1, h_2)f, (a, h_1^{-1}ah_1) \mid a \in C_{O^2(G)}(h_1h_2) \right\rangle.$$ 

As $[G : O^2(G)] = 2$, the group $C_{O^2(G)}(h)$ is not a 2-group. Thus $C_{G_2}((h_1, h_2)f)$ is not a 2-group by $C_{O^2(G)}(h_1h_2) \geq C_{O^2(G)}(h)$. Let

$$N_0 := \langle O^2(G)^2, (h_1, h_2, h_2h_1) \rangle$$

be a subgroup of $N$ with index 2. We show that $E^d(N, N_0)$ is nonempty. If $(h_1, h_2)f$ is an involution, then $(h_1, h_2)f$ lies in $E^d_2(N, N_0)$. Suppose that $(h_1, h_2)f$ is a 2-group by $C_{O^2(G)}(h_1h_2) \geq C_{O^2(G)}(h)$. Let

$$N_0 := \langle O^2(G)^2, (h_1h_2, h_2h_1) \rangle$$

be a subgroup of $N$ with index 2. We show that $E^d(N, N_0)$ is nonempty. If $(h_1, h_2)f$ is not an involution, then $(h_1, h_2)f$ lies in $E^d_2(N, N_0)$. Suppose that $(h_1, h_2)f$ is an involution.

Then it follows $h_1 = h_2$ which is an involution. In this case, $C_{G_2}((h_1, h_2)f)$ is isomorphic to $O^2(G)$ and thus $(h_1, h_2)f$ lies in $E^d_2(N, N_0)$. Therefore $E^d(N, N_0)$ is nonempty. Since $N_0$ is a subgroup of $G \times G$ with 2-power index, $N_0$ is a gap group. Then $N$ is a gap group by combining Theorems 1 and 16. □

Proof of Theorem 18. Let $\pi: G \rightarrow L$ be an epimorphism. If $\pi^{-1}(\pi(\langle x \rangle))$ is a gap group for any 2-element $x$ of $G \wr L$ outside of $O^2(G \wr L)$, then $G \wr L$ is a gap group.
group. As $G^{(U)}$ is a gap group by Theorem 12, it suffices to show that $\pi^{-1}(C)$ is a gap group for any nontrivial cyclic group $C$. Let $C = C_n$ be a cyclic subgroup of $L$ of order $n > 1$. Note that $|O^2(G \int C)|$ is even and $\mathcal{P}(G \int C) \cap \mathcal{L}(G \int C) = \emptyset$ since there is a subgroup of $G \int C$ isomorphic to $G$. Thus if $n$ is not a 2-power integer, then $G \int C$ is a gap group by Corollary 9.

Assume that $n$ is a 2-power integer, say $2^k$. We show that $G \int C$ is a gap group by induction on $k$. In the case where $n = 2$, the assertion follows from Lemma 19. Let $m = 2^{k-1} \geq 2$ and let $C_m$ be a cyclic subgroup of $C$ with index 2.

Suppose that $G \int C_m$ is a gap group for any $G$ as in Theorem 18. Note that $\rho^{-1}(C_m) = G^2 \int C_m$, where $\rho: G \int C \to C$ is an epimorphism. $\rho^{-1}(C_m)$ is isomorphic to a subgroup of the gap group $(G \int C_m)^2$ with 2-power index and thus is a gap group.

Let $h$ be a 2-element of $G$ outside of $K$ such that $C_G(h)$ is not a 2-group. Let $h_j$ be an element of $\langle h \rangle$ for each $j = 1, \ldots, n$ and let $f$ be a generator of $C$. Consider the subgroup

$$N := \langle O^2(G)^n, (h_1, \ldots, h_n)f \rangle.$$

Let $N_0$ be a subgroup of $N$ with index 2. As $N_0$ is a subgroup of $\rho^{-1}(C_m)$ with 2-power index, it is a gap group. Thus it suffices to show that $E^2(N, N_0)$ is nonempty. We show that $(h_1, \ldots, h_n)f$ lies in $E^2(N, N_0)$. We have

$$C_{O^2(G)}((h_1, \ldots, h_n)f)$$

$$= \langle (a, h_1^{-1}ah_1, (h_1h_2)^{-1}a(h_1h_2), \ldots, (h_1 \ldots h_{n-1})^{-1}a(h_1 \ldots h_{n-1}))$$

$$\mid a \in C_{O^2(G)}(h_1h_2 \ldots h_n) \rangle.$$

The group $C_{O^2(G)}(h_1h_2 \ldots h_n)$ contains the group $C_{O^2(G)}(h)$ and thus it is not a 2-group. As the element $(h_1, \ldots, h_n)f$ is not an involution, it lies in $E^2(N, N_0)$ and then $N$ is a gap group.

The group $G \int C$ is a gap group, since any subgroup $N$, $O^2(G)^n \triangleleft N \leq G \int C$, possessing a cyclic quotient $N/O^2(G)^n$ is a gap group.

\[\square\]

References


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