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The commutator subgroup of the general Λ-quadratic group $\mathbf{GQ}(A, \Lambda)$

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1. INTRODUCTION

Let A be a ring with the unity, $- : A \longrightarrow A$ an involution, $\lambda \in$ center(A) a symmetry, and Λ a form parameter on A in the sense of [1,Section 1]. We refer to the tuple $(A, (-, \lambda), \Lambda)$ as a *form ring*. The *general Λ-quadratic group* $\mathrm{GQ}_{2n}(A, \Lambda)$ is defined to be the matrix group corresponding to the automorphism group on the Λ-hyperbolic module $\Lambda - \boldsymbol{H}(A^n)$. The *elementary Λ-quadratic group* $\mathrm{EQ}_{2n}(A, \Lambda)$ is the subgroup of $\mathrm{GQ}_{2n}(A, \Lambda)$ generated by all elementary Λ-quadratic $2n \times 2n$-matrices. If $\boldsymbol{q}$ is an involution invariant ideal of A, then the *relative congruence subgroup* $\mathrm{GQ}_{2n}(A, \Lambda, \boldsymbol{q})$ is defined to be

$$ker[\mathrm{GQ}_{2n}(A, \Lambda) \longrightarrow \mathrm{GQ}_{2n}(A/\boldsymbol{q}, \Lambda/\boldsymbol{q})],$$

where

$$\Lambda/\boldsymbol{q} = image[\Lambda \longrightarrow A/\boldsymbol{q}],$$

and the *relative elementary subgroup* $\mathrm{EQ}_{2n}(A, \Lambda, \boldsymbol{q})$ is defined to be the normal subgroup of $\mathrm{EQ}_{2n}(A, \Lambda)$ generated by all elementary Λ-quadratic matrices belonging to $\mathrm{GQ}_{2n}(A, \Lambda, \boldsymbol{q})$. The groups $\mathrm{GQ}(A, \Lambda, \boldsymbol{q})$ and $\mathrm{EQ}(A, \Lambda, \boldsymbol{q})$ are defined to be the inductive limits of $\mathrm{GQ}_{2n}(A, \Lambda, \boldsymbol{q})$ and $\mathrm{EQ}_{2n}(A, \Lambda, \boldsymbol{q})$, respectively, as $n \to \infty$. It is evident that $\mathrm{EQ}_{2n}(A, \Lambda, \boldsymbol{q})$ is canonically embedden in $\mathrm{GQ}_{2n}(A, \Lambda, \boldsymbol{q})$ as a subgroup. The next result is given as [1, Corollary 3.9].

Theorem 1.1. *The commutator subgroup* $[\mathrm{GQ}_{2n}(A, \Lambda, \boldsymbol{q}), \mathrm{GQ}_{2n}(A, \Lambda)]$ *is equal to* $\mathrm{EQ}_{2n}(A, \Lambda, \boldsymbol{q})$.

The proof of the theorem in [1] uses the lemma:

Lemma 1.2. *Let* $G_{2n} = \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \boldsymbol{q})$ *and* $E_{2n} = \mathrm{EQ}_{2n}((A,\Lambda) \ltimes \boldsymbol{q})$. *Then*

$$[G_{2n}, G_{2n}] \cong [\mathrm{GQ}_{2n}(A,\Lambda), \mathrm{GQ}_{2n}(A,\Lambda)] \ltimes [\mathrm{GQ}_{2n}(A,\Lambda), \mathrm{GQ}_{2n}(A,\Lambda,\boldsymbol{q})]$$

and

$$E_{2n} \cong \mathrm{EQ}_{2n}(A,\Lambda) \ltimes \mathrm{EQ}_{2n}(A,\Lambda,\boldsymbol{q}).$$

In [1], isomorphisms above are obtained by implicitly identifying $\mathrm{GQ}_{2n}(A,\Lambda,\boldsymbol{q})$ with

$$\mathrm{GQ}_{2n}(A,\Lambda,\boldsymbol{q})' = ker[\mathrm{GQ}_{2n}((A,\Lambda) \ltimes \boldsymbol{q}) \longrightarrow \mathrm{GQ}_{2n}(A,\Lambda)]$$

and $\mathrm{EQ}_{2n}(A,\Lambda,\boldsymbol{q})$ with

$$\mathrm{EQ}_{2n}(A,\Lambda,\boldsymbol{q})' = ker[\mathrm{EQ}_{2n}((A,\Lambda) \ltimes \boldsymbol{q}) \longrightarrow \mathrm{EQ}_{2n}(A,\Lambda)],$$

respectively.

The purpose of this paper is to prove the lemma in a precise formulation (Lemma 1.3 below) without employing the groups $\mathrm{GQ}_{2n}(A,\Lambda,\boldsymbol{q})'$ and $\mathrm{EQ}_{2n}(A,\Lambda,\boldsymbol{q})'$ so that we can clarify the proof of Theorem 1.1.

Lemma 1.3. *Let* $G_{2n} = \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \boldsymbol{q})$ *and* $E_{2n} = \mathrm{EQ}_{2n}((A,\Lambda) \ltimes \boldsymbol{q})$. *Then there is a canonical map*

$$\psi : G_{2n} \longrightarrow \mathrm{GQ}_{2n}(A,\Lambda) \ltimes \mathrm{GQ}_{2n}(A,\Lambda,\boldsymbol{q})$$

such that the restrictions

$$\psi_{G_{2n}} : [G_{2n}, G_{2n}] \longrightarrow [\mathrm{GQ}_{2n}(A,\Lambda), \mathrm{GQ}_{2n}(A,\Lambda)] \ltimes [\mathrm{GQ}_{2n}(A,\Lambda), \mathrm{GQ}_{2n}(A,\Lambda,\boldsymbol{q})]$$

and

$$\psi_{E_{2n}} : E_{2n} \longrightarrow \mathrm{EQ}_{2n}(A,\Lambda) \ltimes \mathrm{EQ}_{2n}(A,\Lambda,\boldsymbol{q})$$

are well-defined and isomorphisms.

2. SMASH PRODUCTS OF GROUPS AND OF RINGS, Λ-QUADRATIC ELEMENTARY MATRICES

In this section we define the *smash product* of groups, one of rings and elementary matrices. They will be used in the proof Lemma 1.3.

Definition 2.1. Let Γ be a group and H a subgroup of Γ. If G is a subgroup of $N_\Gamma(H)$, we define the *smash product* $G \ltimes H$ by

$$G \ltimes H = \{(\sigma, \rho) \mid \sigma \in G, \rho \in H\}$$

with multiplication

$$(\sigma', \rho') \cdot (\sigma, \rho) = (\sigma'\sigma, (\sigma^{-1}\rho'\sigma)\rho). \tag{2.1}$$

Let $(A, (-, \lambda), \Lambda)$ be a form ring and $\boldsymbol{q}$ an involution invariant ideal of A. A *form ideal* of *level* $\boldsymbol{q}$ of (A, Λ) is a pair $(\boldsymbol{q}, \Lambda_{\boldsymbol{q}})$ where $\Lambda_{\boldsymbol{q}}$ is an additive subgroup of A such that

(1) $\{q - \lambda\bar{q} \mid q \in \boldsymbol{q}\} + \{\sum_i q_i\Lambda\bar{q}_i \mid q_i \in \boldsymbol{q}\} \subset \Lambda_{\boldsymbol{q}} \subset \boldsymbol{q} \cap \Lambda$ *and*

(2) $a\Lambda_{\boldsymbol{q}}\bar{a} \subset \Lambda_{\boldsymbol{q}}$ $(a \in A)$.

Definition 2.2. (a) Let A be a ring. If $\boldsymbol{q}$ is a both sides ideal of A, we define the *smash product ring* $A \ltimes \boldsymbol{q}$ by

$$A \ltimes \boldsymbol{q} = \{(a, q) \mid a \in A, q \in \boldsymbol{q}\}$$

with addition : $(a, q) + (a', q') = (a + a', q + q')$ and multiplication : $(a, q)(a', q') = (aa', qa' + aq' + qq')$.

(b) If $(\boldsymbol{q}, \Lambda_{\boldsymbol{q}})$ is a form ideal of (A, Λ), we define the *smash product form ring*

$$(A, \Lambda) \ltimes (\boldsymbol{q}, \Lambda_{\boldsymbol{q}}) = (A \ltimes \boldsymbol{q}, \Lambda \ltimes \Lambda_{\boldsymbol{q}})$$

where the involution on $A \ltimes \boldsymbol{q}$ is defined by $(a, q) \longmapsto (\bar{a}, \bar{q})$, and $\Lambda \ltimes \Lambda_{\boldsymbol{q}} = \{(a, q) | a \in \Lambda, q \in \Lambda_{\boldsymbol{q}}\}$. If $\Lambda_{\boldsymbol{q}} = \boldsymbol{q} \cap \Lambda$, then we shall write $(A, \Lambda) \ltimes \boldsymbol{q}$ instead of $(A, \Lambda) \ltimes (\boldsymbol{q}, \boldsymbol{q} \cap \Lambda)$.

We have the ring homomorphism

$$f : A \ltimes \boldsymbol{q} \longrightarrow A; (a, q) \longmapsto a,$$

its splitting

$$i : A \longrightarrow (A \ltimes \boldsymbol{q}); a \longmapsto (a, 0),$$

the form ring homomorphism

$$g : (A, \Lambda) \ltimes (\boldsymbol{q}, \Lambda_{\boldsymbol{q}}) \longrightarrow (A, \Lambda)$$

induced by f, and its splitting

$$j : (A, \Lambda) \longrightarrow (A, \Lambda) \ltimes (\boldsymbol{q}, \Lambda_{\boldsymbol{q}})$$

induced by i.

Let $\mathrm{M}_{n,n}(A)$ denote the set of all $n \times n$-matrices with entries in A, and $\mathrm{M}_{n,n}(\boldsymbol{q})$ the set of all $n \times n$-matrices with entries in $\boldsymbol{q}$. If $P = (p_{ij}) \in \mathrm{M}_{n,n}(A)$ and $Q = (q_{ij}) \in \mathrm{M}_{n,n}(\boldsymbol{q})$, then we have the $n \times n$-matrix (r_{ij}) with entries $r_{ij} := (p_{ij}, q_{ij}) \in A \ltimes \boldsymbol{q}$. The correspondence $\mathrm{M}_{n,n}(A) \times \mathrm{M}_{n,n}(\boldsymbol{q}) \longrightarrow \mathrm{M}_{n,n}(A \ltimes \boldsymbol{q})$; $((p_{ij}), (q_{ij})) \longmapsto (r_{ij})$, is clearly a bijection. Thus we abuse the notation (P, Q) for the assigned matrix (r_{ij}) in $\mathrm{M}_{n,n}(A \ltimes \boldsymbol{q})$. By definition, the formula of multiplication

$$(P, Q)(P', Q') = (PP', PQ' + QP' + QQ') \tag{2.2}$$

holds for (P, Q) and $(P', Q') \in \mathrm{M}_{n,n}(A \ltimes \boldsymbol{q})$.

Definition 2.3. A matrix having one form among the following $2n \times 2n$-matrices is called a Λ-*quadratic elementary matrix*.

$$\mathbf{H}(\varepsilon_{i,j}(a))\ (i \neq j, a \in A) : \quad \begin{cases} (k,k)\text{-entry} = 1 \quad (k = 1, \ldots, 2n), \\ (i,j)\text{-entry} = a, \\ (n+j, n+i)\text{-entry} = -\bar{a}, \\ \textit{all other entries} = 0. \end{cases}$$

$$\varepsilon_{n+i,j}(a)\ (i \neq j, a \in A) : \quad \begin{cases} (k,k)\text{-entry} = 1 \quad (k = 1, \ldots, 2n), \\ (i, n+j)\text{-entry} = a, \\ (j, n+i)\text{-entry} = -\bar{\lambda}\bar{a}, \\ \textit{all other entries} = 0. \end{cases}$$

$$\varepsilon_{i,n+j}(a)\ (i \neq j, a \in A): \begin{cases} (k,k)\text{-entry} = 1 \quad (k = 1, \dots, 2n), \\ (n+i, j)\text{-entry} = a, \\ (n+j, i)\text{-entry} = -\lambda \bar{a}, \\ \textit{all other entries} = 0. \end{cases}$$

$$\varepsilon_{n+i,i}(a)\ (a \in \overline{\Lambda}): \begin{cases} (k,k)\text{-entry} = 1 \quad (k = 1, \dots, 2n), \\ (i, n+i)\text{-entry} = a, \\ \textit{all other entries} = 0. \end{cases}$$

$$\varepsilon_{i,n+i}(a)\ (a \in \Lambda): \begin{cases} (k,k)\text{-entry} = 1 \quad (k = 1, \dots, 2n), \\ (n+i, i)\text{-entry} = a, \\ \textit{all other entries} = 0. \end{cases}$$

Lemma 2.4 (A. Bak [1, Lemma 3.1]). *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_{2n}(A)$ *with* $\alpha, \beta, \gamma, \delta \in \mathrm{M}_{n,n}(A)$. *Then*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GQ}_{2n}(A, \Lambda) \Longleftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\delta} & \lambda\bar{\beta} \\ \bar{\lambda}\bar{\gamma} & \bar{\alpha} \end{pmatrix}.$$

3. PROOF OF LEMMA 1.3

Throughout this section, let $P \in \mathrm{M}_{2n,2n}(A)$ and $Q \in \mathrm{M}_{2n,2n}(\boldsymbol{q})$ and therefore $(P, Q) \in \mathrm{M}_{2n,2n}(A \ltimes \boldsymbol{q})$. Let I_{2n} (or I if the context is clear) denote the identity matrix in $\mathrm{M}_{2n,2n}(A)$ and O_{2n} (or O if the context is clear) the null matrix in $\mathrm{M}_{2n,2n}(A)$.

Lemma 3.1. *The following (1) and (2) hold.*

(1) $P \in \mathrm{GL}_{2n}(A)$ *if and only if* $(P, O_{2n}) \in \mathrm{GL}_{2n}(A \ltimes \boldsymbol{q})$.

(2) *If* $P \in \mathrm{GL}_{2n}(A)$ *and* $(P, Q) \in \mathrm{GL}_{2n}(A \ltimes \boldsymbol{q})$ *then* $(I_{2n}, P^{-1}Q) \in \mathrm{GL}_{2n}(A \ltimes \boldsymbol{q})$.

Proof. Claim (1) is obvious. Suppose P and (P, Q) are as in (2). Then, since $(P, O)^{-1} = (P^{-1}, O)$,

$$(P, O)^{-1}(P, Q) = (P^{-1}, O)(P, Q) = (I, P^{-1}Q). \tag{3.1}$$

By (P^{-1}, O) and $(P, Q) \in \mathrm{GL}_{2n}(A \ltimes \boldsymbol{q})$, $(I, P^{-1}Q) \in \mathrm{GL}_{2n}(A \ltimes \boldsymbol{q})$. □

Lemma 3.2. *The following (1) and (2) hold.*

(1) $P \in \mathrm{GQ}_{2n}(A, \Lambda)$ *if and only if* $(P, O_{2n}) \in \mathrm{GQ}_{2n}((A, \Lambda) \ltimes \boldsymbol{q})$.

(2) *If* $P \in \mathrm{GQ}_{2n}(A,\Lambda)$ *and* $(P,Q) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$ *then* $(I_{2n}, P^{-1}Q) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$.

Proof. We check $P \in \mathrm{GQ}_{2n}(A,\Lambda) \Rightarrow (P,O) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$. If $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha,\beta,\gamma,\delta \in \mathrm{M}_{n,n}(A)$, then $P^{-1} = \begin{pmatrix} \bar{\delta} & \lambda\bar{\beta} \\ \bar{\lambda}\bar{\gamma} & \bar{\alpha} \end{pmatrix}$ by Lemma 2.4. The equality

$$\begin{pmatrix} (\alpha,O) & (\beta,O) \\ (\gamma,O) & (\delta,O) \end{pmatrix}\begin{pmatrix} \overline{(\delta,O)} & \lambda\overline{(\beta,O)} \\ \bar{\lambda}\overline{(\gamma,O)} & \overline{(\alpha,O)} \end{pmatrix} = \begin{pmatrix} (I,O) & (O,O) \\ (O,O) & (I,O) \end{pmatrix}$$

clearly holds. Thus $(P,O) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$. The implication "$(P,O) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q}) \Rightarrow P \in \mathrm{GQ}_{2n}(A,\Lambda)$" is similarly checked. Suppose P and (P,Q) are as in (2). Then $(I, P^{-1}Q) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$ follows from (3.1) and $(P^{-1},O),\ (P,Q) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$. □

Lemma 3.3. *If* $(I_{2n}, Q) \in \mathrm{GL}_{2n}(A \ltimes \mathfrak{q})$ *then* $I_{2n} + Q \in \mathrm{GL}_{2n}(A)$.

Proof. For the inverse matrix (I,Q') of (I,Q),

$$(I,Q)(I,Q') = (I, Q + Q' + QQ') = (I,O).$$

Thus,

$$Q + Q' + QQ' = O.$$

This implies

$$(I+Q)(I+Q') = I$$

and hence

$$I + Q \in \mathrm{GL}_{2n}(A).$$

□

Lemma 3.4. *If* $(I_{2n}, Q) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$ *then* $I_{2n} + Q \in \mathrm{GQ}_{2n}(A,\Lambda)$.

Proof. Suppose $(I,Q) \in \mathrm{GQ}_{2n}((A,\Lambda) \ltimes \mathfrak{q})$. Writing $Q = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$, with $x,y,u,v \in \mathrm{M}_{n,n}(\mathfrak{q})$, we have the equality

$$\begin{pmatrix} (I,x) & (O,y) \\ (O,u) & (I,v) \end{pmatrix}\begin{pmatrix} \overline{(I,v)} & \lambda\overline{(O,y)} \\ \bar{\lambda}\overline{(O,u)} & \overline{(I,x)} \end{pmatrix} = \begin{pmatrix} (I,O) & (O,O) \\ (O,O) & (I,O) \end{pmatrix}.$$

This provides the equality

$$\begin{aligned}(3.2)\qquad &\begin{pmatrix}(I,\bar{v}+x+x\bar{v}+\bar{\lambda}y\bar{u}) & (O,\lambda\bar{y}+\lambda x\bar{y}+y+y\bar{x})\\ (O,u+u\bar{v}+\bar{\lambda}\bar{u}+\bar{\lambda}v\bar{u}) & (I,\lambda u\bar{y}+\bar{x}+v+v\bar{x})\end{pmatrix}\\ &=\begin{pmatrix}(I,O) & (O,O)\\ (O,O) & (I,O)\end{pmatrix}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\begin{pmatrix}I+x & y\\ u & I+v\end{pmatrix}\begin{pmatrix}\overline{I+v} & \lambda\bar{y}\\ \bar{\lambda}\bar{u} & \overline{I+x}\end{pmatrix}\\ &=\begin{pmatrix}I+\bar{v}+x+x\bar{v}+\bar{\lambda}y\bar{u} & \lambda\bar{y}+\lambda x\bar{y}+y+y\bar{x}\\ u+u\bar{v}+\bar{\lambda}\bar{u}+\bar{\lambda}v\bar{u} & I+\lambda u\bar{y}+\bar{x}+v+v\bar{x}\end{pmatrix}\\ &=\begin{pmatrix}I & O\\ O & I\end{pmatrix}\ \text{by (3.2).}\end{aligned}$$

By Lemma 2.4, $I+Q=\begin{pmatrix}I+x & y\\ u & I+v\end{pmatrix}$ lies in $\mathrm{GQ}_{2n}(A,\Lambda)$. □

Lemma 3.5. *If* $(P,Q)\in \mathrm{GQ}_{2n}((A,\Lambda)\ltimes \mathfrak{q})$, *then* $(P, I_{2n}+P^{-1}Q)\in \mathrm{GQ}_{2n}(A,\Lambda)\ltimes \mathrm{GQ}_{2n}(A,\Lambda,\mathfrak{q})$.

Proof. If $(P,Q)\in \mathrm{GQ}_{2n}((A,\Lambda)\ltimes\mathfrak{q})$, then $P\in \mathrm{GQ}_{2n}(A,\Lambda)$ clearly. By Lemma 3.2 (2), we obtain $(I,P^{-1}Q)\in \mathrm{GQ}_{2n}((A,\Lambda)\ltimes\mathfrak{q})$. Then by Lemma 3.4, $I+P^{-1}Q\in \mathrm{GQ}_{2n}(A,\Lambda,\mathfrak{q})$. □

We define the map

$$\psi:\mathrm{GQ}_{2n}((A,\Lambda)\ltimes\mathfrak{q})\longrightarrow \mathrm{GQ}_{2n}(A,\Lambda)\ltimes \mathrm{GQ}_{2n}(A,\Lambda,\mathfrak{q});$$

$$(P,Q)\longmapsto (P,I+P^{-1}Q).$$

The well-definedness follows from Lemma 3.5.

Lemma 3.6. *The map* ψ *is a homomorphism.*

Proof. If (P,Q) and (P',Q') belong to $\mathrm{GQ}_{2n}((A,\Lambda)\ltimes\mathfrak{q})$, then by (2.2),

$$\psi((P,Q)(P',Q'))=(PP',I+P'^{-1}Q+P'^{-1}P^{-1}QP'+P'^{-1}P^{-1}QQ').$$

On the other hand,

$$\begin{aligned}\psi(P,Q)\psi(P',Q') &= (P, I+P^{-1}Q)\cdot(P', I+P'^{-1}Q')\\ &= (PP', P'^{-1}(I+P^{-1}Q)P'(I+P'^{-1}Q')) \text{ by } (2.1)\\ &= (PP', I+P'^{-1}Q+P'^{-1}P^{-1}QP'+P'^{-1}P^{-1}QQ').\end{aligned}$$

Thus $\psi((P,Q)(P',Q')) = \psi(P,Q)\psi(P',Q')$. □

Lemma 3.7. *If $A \in \mathrm{GQ}_{2n}(A,\Lambda)$ and $B \in \mathrm{GQ}_{2n}(A,\Lambda,q)$, then $(A, AB-A) \in \mathrm{GQ}_{2n}((A,\Lambda)\ltimes q)$.*

Proof. If $B = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ with $\alpha,\beta,\gamma,\delta \in \mathrm{M}_{n,n}(q)$, then the equality

$$\begin{pmatrix}(I,\alpha-I) & (O,\beta)\\ (O,\gamma) & (I,\delta-I)\end{pmatrix}\begin{pmatrix}\overline{(I,\delta-I)} & \lambda\overline{(O,\beta)}\\ \bar\lambda\overline{(O,\gamma)} & \overline{(I,\alpha-I)}\end{pmatrix}$$
$$= \begin{pmatrix}(I,\alpha\bar\delta+\bar\lambda\beta\bar\gamma-I) & (O,\lambda\alpha\bar\beta+\beta\bar\alpha)\\ (O,\gamma\bar\delta+\bar\lambda\delta\bar\gamma) & (I,\lambda\gamma\bar\beta+\delta\bar\alpha-I)\end{pmatrix} = \begin{pmatrix}(I,O) & (O,O)\\ (O,O) & (I,O)\end{pmatrix}.$$

holds. By Lemma 2.4, $(I, B-I) \in \mathrm{GQ}_{2n}((A,\Lambda)\ltimes q)$.

Next, if $A = \begin{pmatrix}x & y\\ u & v\end{pmatrix}$ with $x,y,u,v \in \mathrm{M}_{n,n}(A)$, then (A,O) clearly belong to $\mathrm{GQ}_{2n}((A,\Lambda)\ltimes q)$. Thus by (2.2),

$$(A, AB-A) = (A,O)(I,B-I) \in \mathrm{GQ}_{2n}((A,\Lambda)\ltimes q).$$

□

We define the map

$$\phi : \mathrm{GQ}_{2n}(A,\Lambda)\ltimes \mathrm{GQ}_{2n}(A,\Lambda,q) \longrightarrow \mathrm{GQ}_{2n}((A,\Lambda)\ltimes q);$$

$$(A,B) \longmapsto (A, AB-A).$$

The well-definedness of the map follows from Lemma 3.7.

Lemma 3.8. *The map ϕ is a homomorphism.*

Proof. If (A,B) and (A',B') belong to $\mathrm{GQ}_{2n}(A,\Lambda)\ltimes\mathrm{GQ}_{2n}(A,\Lambda,q)$, then by (2.1),

$$\phi((A,B)\cdot(A',B')) = \phi(AA', A'^{-1}BA'B') = (AA', ABA'B'-AA').$$

On the other hand,

$$\begin{aligned}
&\phi(A,B)\phi(A',B')\\
&=(A,AB-A)(A',A'B'-A')\\
&=(AA',A(A'B'-A')+(AB-A)A'+(AB-A)(A'B'-A'))\\
&=(AA',AA'B'-AA'+ABA'-AA'+ABA'B'-ABA'-AA'B'+AA')\\
&=(AA',ABA'B'-AA').
\end{aligned}$$

Thus $\phi((A,B)(A',B'))=\phi(A,B)\phi(A',B')$. □

Lemma 3.9. *The compositions $\psi\circ\phi$ and $\phi\circ\psi$ of the maps ψ and ϕ are the identity maps.*

Proof. By deinition, $(\psi\circ\phi)(A,B)=\psi(A,AB-A)=(A,B)$ and $(\phi\circ\psi)(A,Q)=\phi(A,I-A^{-1}Q)=(A,Q)$. □

We have shown

$$\mathrm{GQ}_{2n}((A,\Lambda)\ltimes \boldsymbol{q})\cong \mathrm{GQ}_{2n}(A,\Lambda)\ltimes \mathrm{GQ}_{2n}(A,\Lambda,\boldsymbol{q}).$$

We define the map

$$\psi_E:\mathrm{EQ}_{2n}((A,\Lambda)\ltimes \boldsymbol{q})\longrightarrow \mathrm{EQ}_{2n}(A,\Lambda)\ltimes \mathrm{EQ}_{2n}(A,\Lambda,\boldsymbol{q})$$

to be the restriction of ψ, and

$$\phi_E:\mathrm{EQ}_{2n}(A,\Lambda)\ltimes \mathrm{EQ}_{2n}(A,\Lambda,\boldsymbol{q})\longrightarrow \mathrm{EQ}_{2n}((A,\Lambda)\ltimes \boldsymbol{q})$$

to be restriction of ϕ.

The well-defindness of ψ_E is checked: for example in the case of

$$H(\varepsilon_{ij}(x))=\left(\begin{array}{ccc|ccc}
1 & & x_{ij} & & & \\
 & \ddots & & & & \\
 & & 1 & & & \\
\hline
 & & & 1 & & \\
 & & & & \ddots & \\
 & & & -\bar{x}_{ji} & & 1
\end{array}\right)\in \mathrm{EQ}_{2n}((A,\Lambda)\ltimes \boldsymbol{q})$$

with $(x_{ij}=(a_{ij},q_{ij}))$,

$$\psi_E(H(\varepsilon_{ij}(x)))$$
$$= \left(\left(\begin{array}{ccc|ccc} 1 & & a_{ij} & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & -\bar{a}_{ji} & & 1 \end{array} \right), \left(\begin{array}{ccc|ccc} 1 & & q_{ij} & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & -\bar{q}_{ji} & & 1 \end{array} \right) \right)$$
$$\in \mathrm{EQ}_{2n}(A, \Lambda) \ltimes \mathrm{EQ}_{2n}(A, \Lambda, \boldsymbol{q}).$$

The well-defindness of ϕ_E is checked as follows. For example in the case of

$$(H(\varepsilon_{ij}(a)), \varepsilon_{i,n+j}(q)) \in \mathrm{EQ}_{2n}(A, \Lambda) \ltimes \mathrm{EQ}_{2n}(A, \Lambda, \boldsymbol{q}),$$

by (2.2), the equality

$$\phi_E(H(\varepsilon_{ij}(a)) \,,\, \varepsilon_{i,n+j}(q))$$
$$= \left(\begin{pmatrix} I_n + (a_{ij}) & 0 \\ 0 & I_n + (-\bar{a}_{ji}) \end{pmatrix}, \begin{pmatrix} 0 & (q_{ij}) + (\lambda \bar{q}_{ji}) + (-\lambda a \bar{q}_{ii}) \\ 0 & 0 \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} I_n + (a_{ij}) & 0 \\ 0 & I_n + (-\bar{a}_{ji}) \end{pmatrix}, O \right) \left(I, \begin{pmatrix} 0 & (q_{ij}) + (-\lambda \bar{q}_{ji}) \\ 0 & 0 \end{pmatrix} \right)$$

holds. Since these two matrices belong to $\mathrm{EQ}_{2n}((A, \Lambda) \ltimes \boldsymbol{q})$,

$$\phi_E(H(\varepsilon_{ij}(a)), \varepsilon_{i,n+j}(q)) \in \mathrm{EQ}_{2n}((A, \Lambda) \ltimes \boldsymbol{q}).$$

By Lemma 3.9, the compositions $\psi_E \circ \phi_E$ and $\phi_E \circ \psi_E$ of the maps ψ_E and ϕ_E are clearly the identity maps. Thus, we have shown

$$\mathrm{EQ}_{2n}((A, \Lambda) \ltimes \boldsymbol{q}) \cong \mathrm{EQ}_{2n}(A, \Lambda) \ltimes \mathrm{EQ}_{2n}(A, \Lambda, \boldsymbol{q}).$$

References

[1] A. Bak, K-Theory of Forms, Princeton University Press, Princeton, 1981.