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<td>Author(s)</td>
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Kyoto University
An exact sequence of Grothendieck-Witt rings
(Grothendieck-Witt 環の完全系列)

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1. INTRODUCTION

Throughout this paper let $G$ denote a finite group, $\Theta$ a finite $G$-set, and $R$ a commutative ring with multiplicative unit.

C. B. Thomas [13] defined the Hermitian representation ring $G_1(R, G)$ and showed that the Wall group $L_n(\mathbb{Z}[G], w)$ is a module over $G_1(\mathbb{Z}, G)$, providing the orientation homomorphism $w$ is trivial. A. Dress defined the Grothendieck-Witt rings $GW_0(R, G)$ and $GW(G, R)$ in [7, p. 742] and [8, p. 294], respectively (cf. [12, p. 2356]) as quotient rings of $G_1(R, G)$. By [8, Theorem 5], we can see that the canonical epimorphism $GW(G, \mathbb{Z}) \rightarrow GW_0(\mathbb{Z}, G)$ is actually an isomorphism. For the induction theory of equivariant surgery obstruction groups, the authors have defined in [2, Section 2] the (generalized) Grothendieck-Witt ring $GW_0(R, G, \Theta)$. Details of the induction theory of equivariant surgery obstruction groups are described in [12] and [10]. Applications to equivariant surgery are given in [11, Section 6] and [4]. Let $\mathfrak{S}_2$ denote the group of order 2 with generator $\tau$. Give the cartesian product $\Theta \times \Theta$ the diagonal $G$-action and the $\mathfrak{S}_2$-action:


Keywords and phrases. Grothendieck-Witt ring, Hermitian module, Quillen submodule, exact sequence.
$(\tau, (x, y)) \mapsto (y, x)$ for $x, y \in \Theta$. Let $\text{Map}_{G \times \mathfrak{S}_2}(\Theta \times \Theta, R)$ denote the ring of all $G \times \mathfrak{S}_2$-maps from $\Theta \times \Theta$ to $R$, where $R$ has the trivial $G \times \mathfrak{S}_2$-action. The goal of this article is to prove the following theorem.

**Theorem 1.** Let $R$ be a principal ideal domain. Then the sequence of canonical homomorphisms

$$0 \longrightarrow \text{GW}_0(R, G) \longrightarrow \text{GW}_0(R, G, \Theta) \longrightarrow \text{Map}_{G \times \mathfrak{S}_2}(\Theta \times \Theta, R) \longrightarrow 0$$

is split exact.

We remark that $\text{GW}_0(R, G)$ and $\text{GW}_0(R, G, \Theta)$ are rings with multiplicative unit and the canonical homomorphism $\text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)$ preserves multiplication, but not the multiplicative unit. The $R$-rank of $\text{Map}_{G \times \mathfrak{S}_2}(\Theta \times \Theta, R)$ was computed by Mitsuaki Kubo in his Master thesis for $G = A_5$ and by XianMeng Ju [9] for $G = \text{SL}(2, 5)$.

The definitions of the Grothendieck-Witt rings above are recalled in Section 2 for the reader's convenience. Theorem 1 is proved in Section 3.

**Acknowledgements**

The first author gratefully acknowledges the support of INTAS 00-0566. The second author would like to acknowledge the support of the Grant-in-Aid for Scientific Research (Kakenhi) No. 15540076.

**2. Definition of the Grothendieck-Witt rings**

In this section we recall the definitions of the Grothendieck-Witt rings used in the current paper and the canonical homomorphisms

$$\text{GW}_0(R, G) \longrightarrow \text{GW}_0(R, G, \Theta) \longrightarrow \text{Map}_{G \times \mathfrak{S}_2}(\Theta \times \Theta, R).$$

The reader can refer to Section 4 of [12] for details.

A *Hermitian $R[G]$-module* is a pair $(M, B)$ consisting of a finitely generated $R$-projective $R[G]$-module $M$ and a symmetric $G$-invariant $R$-bilinear map $B : M \times M \rightarrow R$. The map $B$ is called nonsingular if the associated map $M \rightarrow \text{Hom}_R(M, R); x \mapsto B(x, -)$, is a
bijection. A \(\Theta\)-positioned Hermitian \(R[G]\)-module is a triple \((M, B, \alpha)\) consisting of a Hermitian module \((M, B)\) and a \(G\)-map \(\alpha : \Theta \to M\). If \(B\) is nonsingular then \((M, B)\) and \((M, B, \alpha)\) are also called nonsingular. The \(G\)-map \(\alpha\) is called trivial (resp. totally isotropic) if \(\alpha(t) = 0\) for all \(t \in \Theta\) (resp. \(B(\alpha(t), \alpha(t')) = 0\) for all \(t, t' \in \Theta\)). Let \(\mathcal{H}(R, G, \Theta)\) denote the category of all nonsingular \(\Theta\)-positioned Hermitian \(R[G]\)-modules \((M, B, \alpha)\), where the morphisms \((M, B, \alpha) \to (M', B', \alpha')\) are isomorphisms \(f : M \to M'\) such that \(B'(f(x), f(y)) = B(x, y)\) for all \(x, y \in M\) and the diagram

\[
\begin{array}{ccc}
\Theta & \xrightarrow{\alpha} & M \\
\downarrow \alpha' & & \downarrow f \\
M' & & \\
\end{array}
\]

commutes. Let \(\mathcal{H}(R, G, \Theta)_{\text{triv}}\) (resp. \(\mathcal{H}(R, G, \Theta)^{t\text{-iso}}\)) denote the full subcategory of \(\mathcal{H}(R, G, \Theta)\) consisting of all \((M, B, \alpha) \in \mathcal{H}(R, G, \Theta)\) such that \(\alpha\) is trivial (resp. totally isotropic).

The orthogonal sum

\[
(M, B, \alpha) \oplus (M', B', \alpha'), \quad (= (M'', B'', \alpha'') \text{ say})
\]

of \((M, B, \alpha), (M', B', \alpha') \in \mathcal{H}(R, G, \Theta)\) is defined by \(M'' = M \oplus M', B''((x, x'), (y, y')) = B(x, y) + B'(x', y')\) for \(x, y \in M\) and \(x', y' \in M'\), and \(\alpha''(t) = (\alpha(t), \alpha'(t))\) for \(t \in \Theta\). The tensor product

\[
(M, B, \alpha) \otimes (M', B', \alpha'), \quad (= (M'', B'', \alpha'') \text{ say})
\]

is defined by \(M'' = M \otimes M', B''(x \otimes x', y \otimes y') = B(x, y)B'(x', y')\) for \(x, y \in M\) and \(x', y' \in M'\), and \(\alpha''(t) = \alpha(t) \otimes \alpha'(t)\) for \(t \in \Theta\). \(\mathcal{H}(R, G, \Theta)_{\text{triv}}\) and \(\mathcal{H}(R, G, \Theta)^{t\text{-iso}}\) are closed under orthogonal sum as well as tensor product. Let \(\text{KH}_0(R, G, \Theta)\) (resp. \(\text{KH}_0(R, G, \Theta)_{\text{triv}}, \text{KH}_0(R, G, \Theta)^{t\text{-iso}}\)) denote the Grothendieck group of the category \(\mathcal{H}(R, G, \Theta)\) (resp. \(\mathcal{H}(R, G, \Theta)_{\text{triv}}, \mathcal{H}(R, G, \Theta)^{t\text{-iso}}\)) with respect to orthogonal sum.

Let \((M, B, \alpha) \in \mathcal{H}(R, G, \Theta)\). An \(R[G]\)-submodule \(U\) of \(M\) is called a Quillen submodule of \((M, B, \alpha)\) if \(U\) is an \(R\)-direct summand of \(M\) such that \(B(U, U) = 0\) and \(\alpha(\Theta) \subseteq U\). In this case, \(((M, B, \alpha), U)\) is called a Quillen pair. For any \((M, B, \alpha) \in \mathcal{H}(R, G, \Theta)\),

\[
\Delta M = \{(x, x) \in M \oplus M \mid x \in M\}
\]
is a Quillen submodule of 

$$(M, B, \alpha) \oplus (M, -B, \alpha).$$

If $((M, B, \alpha), U)$ is a Quillen pair, we obtain $(U^\perp/U, B^\perp, \alpha_0) \in \mathcal{H}(R, G, \Theta)$ where

$$U^\perp = \{ y \in M \mid B(x, y) = 0 \ \forall x \in U \}$$

$$B^\perp(x + U, y + U) = B(x, y) \text{ for } x, y \in U^\perp$$

$$\alpha_0(t) = 0 + U \in U^\perp/U \text{ for } t \in \Theta.$$

Define the Grothendieck-Witt group (which will be also referred to as the Grothendieck-Witt ring)

$$\text{GW}_0(R, G, \Theta) \text{ (resp. } \text{GW}_0(R, G, \Theta)^{\text{triv}}, \text{GW}_0(R, G, \Theta)^{t-\text{iso}})$$

by

$$\text{GW}_0(R, G, \Theta) = \text{KH}_0(R, G, \Theta)/((M, B, \alpha) - (U^\perp/U, B^\perp, \alpha_0))$$

(resp. $\text{GW}_0(R, G, \Theta)^{\text{triv}} = \text{KH}_0(R, G, \Theta)^{\text{triv}}/((M, B, \alpha) - (U^\perp/U, B^\perp, \alpha_0))$)

$$\text{GW}_0(R, G, \Theta)^{t-\text{iso}} = \text{KH}_0(R, G, \Theta)^{t-\text{iso}}/((M, B, \alpha) - (U^\perp/U, B^\perp, \alpha_0)))$$

where $((M, B, \alpha), U)$ runs over all Quillen pairs in $\mathcal{H}(R, G, \Theta)$ (resp. $\mathcal{H}(R, G, \Theta)^{\text{triv}}, \mathcal{H}(R, G, \Theta)^{t-\text{iso}}$). Note that

$$[M, -B, \alpha] = -[M, B, \alpha]$$

in $\text{GW}_0(R, G, \Theta)$. $\text{GW}_0(R, G, \Theta)$, $\text{GW}_0(R, G, \Theta)^{\text{triv}}$ and $\text{GW}_0(R, G, \Theta)^{t-\text{iso}}$ are commutative rings and the first two have multiplicative units. The Grothendieck-Witt ring $\text{GW}_0(R, G)$ of A. Dress is obtained as $\text{GW}_0(R, G, \emptyset)$. By definition, there are canonical homomorphisms

$$\text{GW}_0(R, G) \longrightarrow \text{GW}_0(R, G, \Theta)^{\text{triv}} \longrightarrow \text{GW}_0(R, G, \Theta)^{t-\text{iso}} \longrightarrow \text{GW}_0(R, G, \Theta)$$

and the first homomorphism is an isomorphism. In addition, we have a canonical retraction

$$\text{GW}_0(R, G, \Theta) \rightarrow \text{GW}_0(R, G); \ [M, B, \alpha] \mapsto [M, B].$$

We define the homomorphism

$$\kappa: \text{GW}_0(R, G, \Theta) \rightarrow \text{Map}_{G \times \mathcal{S}_2}(\Theta \times \Theta, R)$$
by

$$
\kappa([M, B, \alpha])(t, t') = B(\alpha(t), \alpha(t')) \text{ for } t, t' \in \Theta.
$$

3. Proof of Theorem 1

We have already proved the exactness of the sequence

$$
0 \longrightarrow GW_0(R, G) \longrightarrow GW_0(R, G, \Theta) \longrightarrow \text{Map}_{G \times S_2}(\Theta \times \Theta, R)
$$

in Proposition 2.1 of [2]. Thus, in order to prove Theorem 1, it suffices to show the homomorphism $\kappa$ splits.

Let $f : \Theta \times \Theta \rightarrow R$ be a $G \times S_2$-map. We assign a $\Theta$-positioned Hermitian $R[G]$-module $(M, B, \alpha)$ to $f$ as follows. Let $\Theta'$ be a copy of the $G$-set $\Theta$. For each element $x \in \Theta$, let $x'$ stand for the copy in $\Theta'$ of $x$. Let $M$ be the free $R$-module with basis $\Theta \Pi \Theta'$, namely $M = R[\Theta] \oplus R[\Theta']$. Let $B : M \times M \rightarrow R$ be the $R$-bilinear map satisfying $B(x, y) = f(x, y), B(x, y') = \delta_{x,y}, B(x', y) = \delta_{x,y}$ and $B(x', y') = 0$ for all $x, y \in \Theta$, where

$$
\delta_{x,y} = \begin{cases} 
1 & \text{if } x = y' \\
0 & \text{if } x \neq y.
\end{cases}
$$

Since $f$ is $G$-equivariant and symmetric, $B$ is $G$-invariant and symmetric. Clearly, $B$ is nonsingular. Define $\alpha : \Theta \rightarrow M$ by $\alpha(x) = (x, 0) \in R[\Theta] \oplus R[\Theta']$ for $x \in \Theta$. Obviously, $\alpha$ is a $G$-map. The assignment $f \mapsto [M, B, \alpha]$ defines a homomorphism

$$
\sigma : \text{Map}_{G \times S_2}(\Theta \times \Theta, R) \rightarrow GW_0(R, G, \Theta).
$$

Since

$$
\kappa([M, B, \alpha])(x, y) = B(\alpha(x), \alpha(y)) = B((x, 0), (y, 0)) = f(x, y),
$$

the homomorphism $\sigma$ is a splitting of $\kappa$.

References


