DEFINABLE $C^r G$ TRIVIALITY OF $G$ INVARIANT PROPER
DEFINABLE $C^r$ MAPS

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ABSTRACT. Let $G$ be a compact definable $C^r$ group and $1 \leq r < \infty$. We prove that
every $G$ invariant proper definable $C^r$ onto submersion from an affine definable $C^r G$
manifold to $\mathbb{R}$ is definably $C^r G$ trivial.

1. INTRODUCTION

M. Coste and M. Shiota [1] proved that a proper Nash onto submersion from an affine
Nash manifold to $\mathbb{R}$ is Nash trivial. This Nash category is a special case of the definable $C^r$
category and it coincides with the definable $C^\infty$ category based on $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ [16].
General reference on o-minimal structures are [2], [5], see also [15]. Further properties
and constructions of them are studied in [3], [4], [6], [12] and there are uncountably many
o-minimal expansions of $\mathcal{R}$ [13]. Equivariant definable category is studied in [7], [9], [10],

Let $G$ be a definable $C^r$ group, $X$ a definable $C^r G$ manifold and $1 \leq r < \infty$. Suppose
that $f$ is a $G$ invariant definable $C^r$ function from $X$ to $\mathbb{R}$. We say that $f$ is definably $C^r G$
trivial if there exist a definable $C^r G$ manifold $F$ and a definable $C^r G$ map $h : X \to F$
such that $H = (f, h) : X \to \mathbb{R} \times F$ is a definable $C^r G$ diffeomorphism. If $f$ is definably
$C^r G$ trivial, then for any $y \in \mathbb{R}$, $f^{-1}(y)$ is definably $C^r G$ diffeomorphic to $F$ and there
exists a definable $C^r G$ diffeomorphism $\phi : X \to \mathbb{R} \times f^{-1}(y)$ such that $f = p \circ \phi$, where
$p : \mathbb{R} \times f^{-1}(y) \to \mathbb{R}$ denotes the projection.

A map $\psi : M \to N$ between topological spaces is proper if for any compact set $C \subset N$,$\psi^{-1}(C)$ is compact.

We consider an equivariant definable $C^r$ version of [1] and an equivariant version of [1].

Theorem 1.1. Let $G$ be a compact definable $C^r$ group and $X$ an affine definable $C^r G$
manifold and $1 \leq r < \infty$. Then every $G$ invariant proper definable $C^r$ onto submersion
$f : X \to \mathbb{R}$ is definably $C^r G$ trivial.

Let $X = \{y = 0\} \cup \{xy = 1\} \subset \mathbb{R}^2, Y = \{y = 0\} \subset \mathbb{R}^2$ and $f : X \to Y, f(x, y) = x$.
Then $f$ is a polynomial onto submersion and it is not definably trivial. Thus proper
condition is necessary.

The projection onto $S^n$ of the tangent bundle of the standard $n$-dimensional sphere $S^n$
with the standard $O(n + 1)$ action for $n \geq 8$ is not piecewise definably $C^r G$ trivial. Thus
$G$ invariant condition is necessary.

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trivial, Nash $G$ trivial.
Corollary 1.2. Let $G$ be a finite group and $X$ an affine Nash $G$ manifold. Then every $G$ invariant proper Nash onto submersion from $X$ to $\mathbb{R}$ is Nash $G$ trivial.

2. PROOF OF RESULTS

The following is a result on piecewise definable $C^rG$ triviality of $G$ invariant submersive definable $C^r$ maps [9].

Theorem 2.1 (1.1 [9]). (Piecewise definable $C^rG$ triviality). Let $X$ be an affine definable $C^rG$ manifold, $Y$ a definable $C^r$ manifold and $1 \leq r < \infty$. Suppose that $f : X \to Y$ is a $G$ invariant submersive definable $C^r$ map. Then there exist a finite decomposition \( \{ T_i \}_{i=1}^k \) of $Y$ into definable $C^r$ submanifolds and definable $C^rG$ diffeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \leq i \leq k)$, where $p_i$ denotes the projection $T_i \times f^{-1}(y_i) \to T_i$ and $y_i \in T_i$.

The following is existence of a definable $C^rG$ tubular neighborhood of a definable $C^rG$ submanifold of a representation of $G$ when $1 \leq r < \infty$.

Proposition 2.2 ([8]). If $1 \leq r < \infty$, then every definable $C^rG$ submanifold $X$ of a representation $\Omega$ of $G$ has a definable $C^rG$ tubular neighborhood $(U, \theta)$ of $X$ in $\Omega$, namely $U$ is a $G$ invariant definable open neighborhood of $X$ in $\Omega$ and $\theta : U \to X$ is a definable $C^rG$ map with $\theta|X = id_X$.

Note that if $r = \infty$ or $\omega$, then Proposition 2.2 is already known in [11].

Proof of Theorem 1.1. Applying Theorem 2.1, we have a partition $-\infty = a_0 < a_1 < a_2 < \cdots < a_j < a_{j+1} = \infty$ of $\mathbb{R}$ and definable $C^rG$ diffeomorphisms $\phi_i : f^{-1}((a_i, a_{i+1})) \to (a_i, a_{i+1}) \times f^{-1}(y_i)$ with $f|f^{-1}((a_i, a_{i+1})) = p_i \circ \phi_i$, $(0 \leq i \leq j)$, where $p_i$ denotes the projection $(a_i, a_{i+1}) \times f^{-1}(y_i) \to (a_i, a_{i+1})$ and $y_i \in (a_i, a_{i+1})$.

Now we prove that for each $a_i$ with $1 \leq i \leq j$, there exist an open interval $I_i$ containing $a_i$ and a definable $C^rG$ map $\pi_i : f^{-1}(I_i) \to f^{-1}(a_i)$ such that $F_i = (f, \pi_i) : f^{-1}(I_i) \to I_i \times f^{-1}(a_i)$ is a definable $C^rG$ diffeomorphism. By Proposition 2.2, we have a definable $C^rG$ tubular neighborhood $(U_i, \pi_i)$ of $f^{-1}(a_i)$ in $X$. Since $f$ is proper, there exists an open interval $I_i$ containing $a_i$ such that $f^{-1}(I_i) \subset U_i$. Note that if $f$ is not proper, then such an open interval does not always exist. Hence shrinking $I_i$, if necessary, $F_i = (f, \pi_i) : f^{-1}(I_i) \to I_i \times f^{-1}(a_i)$ is the required definable $C^rG$ diffeomorphism.

By the above argument, we have a finite family of $\{ J_i \}_{i=1}^l$ of open intervals and definable $C^rG$ diffeomorphisms $h_i : f^{-1}(J_i) \to J_i \times f^{-1}(y_i)$, $(1 \leq i \leq l)$, such that $y_i \in J_i$, $\bigcup_{i=1}^l J_i = \mathbb{R}$ and the composition of $h_i$ with the projection $J_i \times f^{-1}(y_i)$ onto $J_i$ is $f|f^{-1}(J_i)$.

Now we glue these trivializations to get a global one. We can suppose that $i \geq 2$, $U_{i-1} \cap J_i = (a, b)$ and $k_{i-1} : f^{-1}(U_{i-1}) \to U_{i-1} \times f^{-1}(y_i)$ is a definable $C^rG$ diffeomorphism with $f|f^{-1}(U_{i-1}) = proj_{i-1} \circ k_{i-1}$, where $U_{i-1} = \bigcup_{s=1}^{i-1} J_s$ and $proj_{i-1}$ denotes the projection $U_{i-1} \times f^{-1}(y_i) \to U_{i-1}$. Take $z \in (a, b) = U_{i-1} \cap J_i$. Then since $f^{-1}(y_i) \cong f^{-1}(z) \cong f^{-1}(y_1)$, $f^{-1}(y_1)$ is definably $C^rG$ diffeomorphic to $f^{-1}(y_i)$. Hence we may assume that $h_i$ is a definable $C^rG$ diffeomorphism from $f^{-1}(J_i)$ to $J_i \times f^{-1}(y_i)$. Then we have a definable $C^rG$ diffeomorphism $k_{i-1} \circ h_{i-1} : (a, b) \times f^{-1}(y_1) \to (a, b) \times f^{-1}(y_i), (t, x) \mapsto (t, q(t, x))$. 

TOMOHIRO KAWAKAMI

103
Take a $C^r$ Nash function $u : \mathbb{R} \to \mathbb{R}$ such that $u = \frac{a+b}{2}$ on $(-\infty, \frac{3}{4}a + \frac{1}{4}b]$ and $u = id$ on $[\frac{1}{4}a + \frac{3}{4}b, \infty)$. Let

$$H : (a, b) \times f^{-1}(y_1) \to f^{-1}((a, b)), H(t, x) = k_{i-1}^{-1}(t, q(u(t), x)).$$

Then $H$ is a definable $C^rG$ diffeomorphism such that $H = h_i^{-1}$ if $\frac{3}{4}a + \frac{1}{4}b \leq t \leq b$ and $H = k_{i-1}^{-1} \circ (id \times \psi)$ if $a \leq t \leq \frac{3}{4}a + \frac{1}{4}b$, where $\psi : f^{-1}(y_1) \to f^{-1}(y_1), \psi(x) = q(\frac{a+b}{2}, x)$. Thus we can define

$$k_i : f^{-1}(U_i) \to U_i \times f^{-1}(y_1),$$

$$k_i(x) = \begin{cases} (id \times \psi)^{-1} \circ k_{i-1}(x), & f(x) \leq \frac{3}{4}a + \frac{1}{4}b \\ H^{-1}(x), & \frac{3}{4}a + \frac{1}{4}b \leq f(x) \leq b \\ h_i(x), & f(x) > b \end{cases}.$$

Then $k_i$ is a definable $C^rG$ diffeomorphism. Therefore $k_i$ is the required definable $C^rG$ diffeomorphism. 

By [14] and 4.3 [9], we have the following proposition.

**Proposition 2.3.** Let $G$ be a finite group, $f$ a $C^r$ Nash $G$ map between affine Nash $G$ manifolds and $1 \leq r < \infty$. Then $f$ is approximated by a Nash $G$ map.

**Proof of Corollary 1.2.** By Theorem 1.1, we have a $C^r$ Nash $G$ diffeomorphism $F = (f, \phi) : X \to \mathbb{R} \times f^{-1}(y)$ such that $f = p \circ F$, where $p : \mathbb{R} \times f^{-1}(y) \to \mathbb{R}$ denotes the projection. By Proposition 2.3, we have a Nash $G$ map $\psi : X \to f^{-1}(y)$ as an approximation of $\phi$. If this approximation is sufficiently close, then $H = (f, \psi) : X \to \mathbb{R} \times f^{-1}(y)$ is the required Nash $G$ diffeomorphism. 

**References**


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