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DEFINABLE $C^G$ TRIVIALITY OF $G$ INVARIANT PROPER DEFINABLE $C^r$ MAPS

TOMOHIRO KAWAKAMI

ABSTRACT. Let $G$ be a compact definable $C^r$ group and $1 \leq r < \infty$. We prove that every $G$ invariant proper definable $C^r$ onto submersion from an affine definable $C^G$ manifold to $\mathbb{R}$ is definably $C^G$ trivial.

1. INTRODUCTION

M. Coste and M. Shiota [1] proved that a proper Nash onto submersion from an affine Nash manifold to $\mathbb{R}$ is Nash trivial. This Nash category is a special case of the definable $C^r$ category and it coincides with the definable $C^\infty$ category based on $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ [16]. General reference on o-minimal structures are [2], [5], see also [15]. Further properties and constructions of them are studied in [3], [4], [6], [12] and there are uncountably many o-minimal expansions of $\mathcal{R}$ [13]. Equivariant definable category is studied in [7], [9], [10], [11].

Let $G$ be a definable $C^r$ group, $X$ a definable $C^G$ manifold and $1 \leq r < \infty$. Suppose that $f$ is a $G$ invariant definable $C^r$ function from $X$ to $\mathbb{R}$. We say that $f$ is definably $C^G$ trivial if there exist a definable $C^G$ manifold $F$ and a definable $C^r$ map $h : X \rightarrow F$ such that $H = (f, h) : X \rightarrow \mathbb{R} \times F$ is a definable $C^G$ diffeomorphism. If $f$ is definably $C^G$ trivial, then for any $y \in \mathbb{R}$, $f^{-1}(y)$ is definably $C^G$ diffeomorphic to $F$ and there exists a definable $C^G$ diffeomorphism $\phi : X \rightarrow \mathbb{R} \times f^{-1}(y)$ such that $f = p \circ \phi$, where $p : \mathbb{R} \times f^{-1}(y) \rightarrow \mathbb{R}$ denotes the projection.

A map $\psi : M \rightarrow N$ between topological spaces is proper if for any compact set $C \subset N$, $\psi^{-1}(C)$ is compact.

We consider an equivariant definable $C^r$ version of [1] and an equivariant version of [1].

Theorem 1.1. Let $G$ be a compact definable $C^r$ group and $X$ an affine definable $C^G$ manifold and $1 \leq r < \infty$. Then every $G$ invariant proper definable $C^r$ onto submersion $f : X \rightarrow \mathbb{R}$ is definably $C^G$ trivial.

Let $X = \{y = 0\} \cup \{xy = 1\} \subset \mathbb{R}^2$, $Y = \{y = 0\} \subset \mathbb{R}^2$ and $f : X \rightarrow Y$, $f(x, y) = x$. Then $f$ is a polynomial onto submersion and it is not definably trivial. Thus proper condition is necessary.

The projection onto $S^n$ of the tangent bundle of the standard $n$-dimensional sphere $S^n$ with the standard $O(n + 1)$ action for $n \geq 8$ is not piecewise definably $C^G$ trivial. Thus $G$ invariant condition is necessary.

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Corollary 1.2. Let \( G \) be a finite group and \( X \) an affine Nash \( G \) manifold. Then every \( G \) invariant proper Nash onto submersion from \( X \) to \( \mathbb{R} \) is Nash trivial.

2. Proof of results

The following is a result on piecewise definable \( C^r \) triviality of \( G \) invariant submersive definable \( C^r \) maps [9].

Theorem 2.1 (1.1 [9]). (Piecewise definable \( C^r \) triviality). Let \( X \) be an affine definable \( C^r \) manifold, \( Y \) a definable \( C^r \) manifold and \( 1 \leq r < \infty \). Suppose that \( f : X \rightarrow Y \) is a \( G \) invariant submersive definable \( C^r \) map. Then there exist a finite decomposition \( \{ T_i \}_{i=1}^k \) of \( Y \) into definable \( C^r \) submanifolds and definable \( C^r \) diffeomorphisms \( \phi_i : f^{-1}(T_i) \rightarrow T_i \times f^{-1}(y_i) \) such that \( f|f^{-1}(T_i) = p_i \circ \phi_i \), \( (1 \leq i \leq k) \), where \( p_i \) denotes the projection \( T_i \times f^{-1}(y_i) \rightarrow T_i \) and \( y_i \in T_i \).

The following is existence of a definable \( C^r \) tubular neighborhood of a definable \( C^r \) submanifold of a representation of \( G \) when \( 1 \leq r < \infty \).

Proposition 2.2 ([8]). If \( 1 \leq r < \infty \), then every definable \( C^r \) submanifold \( X \) of a representation \( \Omega \) of \( G \) has a definable \( C^r \) tubular neighborhood \( (U, \theta) \) of \( X \) in \( \Omega \), namely \( U \) is a \( G \) invariant definable open neighborhood of \( X \) in \( \Omega \) and \( \theta : U \rightarrow X \) is a definable \( C^r \) map with \( \theta|X = id_X \).

Note that if \( r = \infty \) or \( \omega \), then Proposition 2.2 is already known in [11].

Proof of Theorem 1.1. Applying Theorem 2.1, we have a partition \( -\infty = a_0 < a_1 < a_2 < \cdots < a_j < a_{j+1} = \infty \) of \( \mathbb{R} \) and definable \( C^r \) diffeomorphisms \( \phi_i : f^{-1}((a_i, a_{i+1})) \rightarrow (a_i, a_{i+1}) \times f^{-1}(y_i) \) with \( f|f^{-1}((a_i, a_{i+1})) = p_i \circ \phi_i \), \( (0 \leq i \leq j) \), where \( p_i \) denotes the projection \( (a_i, a_{i+1}) \times f^{-1}(y_i) \rightarrow (a_i, a_{i+1}) \) and \( y_i \in (a_i, a_{i+1}) \).

Now we prove that for each \( a_i \) with \( 1 \leq i \leq j \), there exist an open interval \( I_i \) containing \( a_i \) and a definable \( C^r \) map \( \pi_i : f^{-1}(I_i) \rightarrow f^{-1}(a_i) \) such that \( F_i = (f, \pi_i) : f^{-1}(I_i) \rightarrow I_i \times f^{-1}(a_i) \) is a definable \( C^r \) diffeomorphism. By Proposition 2.2, we have a definable \( C^r \) tubular neighborhood \( (U_i, \pi_i) \) of \( f^{-1}(a_i) \) in \( X \). Since \( f \) is proper, there exists an open interval \( I_i \) containing \( a_i \) such that \( f^{-1}(I_i) \subset U_i \). Note that if \( f \) is not proper, then such an open interval does not always exist. Hence shrinking \( I_i \), if necessary, \( F_i = (f, \pi_i) : f^{-1}(I_i) \rightarrow I_i \times f^{-1}(a_i) \) is the required definable \( C^r \) diffeomorphism.

By the above argument, we have a finite family of \( \{ J_i \}_{i=1}^l \) of open intervals and definable \( C^r \) diffeomorphisms \( h_i : f^{-1}(J_i) \rightarrow J_i \times f^{-1}(y_i), \ (1 \leq i \leq l) \), such that \( y_i \in J_i, U_{i-1} \cap J_i = (a, b) \) and \( k_{i-1} : f^{-1}(U_{i-1}) \rightarrow U_{i-1} \times f^{-1}(y_i) \) is a definable \( C^r \) diffeomorphism with \( f|f^{-1}(U_{i-1}) = \text{proj}_{i-1} \circ k_{i-1} \), where \( U_{i-1} = \bigcup_{s=1}^{i-1} J_s \) and \( \text{proj}_{i-1} \) denotes the projection \( U_{i-1} \times f^{-1}(y_i) \rightarrow U_{i-1} \). Take \( z \in (a, b) = U_{i-1} \cap J_i \). Then since \( f^{-1}(y_i) \approx f^{-1}(z) \approx f^{-1}(y_1) \), \( f^{-1}(y_i) \) is definably \( C^r \) diffeomorphic to \( f^{-1}(y_1) \). Hence we may assume that \( \hat{h}_i \) is a definable \( C^r \) diffeomorphism from \( f^{-1}(J_i) \) to \( J_i \times f^{-1}(y_1) \). Then we have a definable \( C^r \) diffeomorphism

\[ k_{i-1} \circ h_i^{-1} : (a, b) \times f^{-1}(y_1) \rightarrow (a, b) \times f^{-1}(y_1), (t, x) \mapsto (t, q(t, x)). \]
Take a $C^r$ Nash function $u : \mathbb{R} \to \mathbb{R}$ such that $u = \frac{a+b}{2}$ on $(-\infty, \frac{3}{4}a + \frac{1}{4}b]$ and $u = id$ on $[\frac{1}{4}a + \frac{3}{4}b, \infty)$. Let

$$H : (a, b) \times f^{-1}(y_1) \to f^{-1}((a, b)), \quad H(t, x) = k_{i-1}^{-1}(t, q(u(t), x)).$$

Then $H$ is a definable $C^r G$ diffeomorphism such that $H = h_i^{-1}$ if $\frac{3}{4}a + \frac{3}{4}b \leq t \leq b$ and $H = k_{i-1}^{-1} \circ (id \times \psi)$ if $a \leq t \leq \frac{3}{4}a + \frac{1}{4}b$, where $\psi : f^{-1}(y_1) \to f^{-1}(y_1), \psi(x) = q(\frac{a+b}{2}, x)$. Thus we can define

$$k_i : f^{-1}(U_{i}) \to U_{i} \times f^{-1}(y_1),$$

$$k_i(x) = \begin{cases} (id \times \psi)^{-1} \circ k_{i-1}(x), & f(x) \leq \frac{3}{4}a + \frac{1}{4}b \\ H^{-1}(x), & \frac{3}{4}a + \frac{1}{4}b \leq f(x) \leq b \\ k_{i}(x), & f(x) > b \end{cases}.$$

Then $k_i$ is a definable $C^r G$ diffeomorphism. Therefore $k_i$ is the required definable $C^r G$ diffeomorphism. □

By [14] and 4.3 [9], we have the following proposition.

**Proposition 2.3.** Let $G$ be a finite group, $f$ a $C^r$ Nash $G$ map between affine Nash $G$ manifolds and $1 \leq r < \infty$. Then $f$ is approximated by a Nash $G$ map.

**Proof of Corollary 1.2.** By Theorem 1.1, we have a $C^r$ Nash $G$ diffeomorphism $F = (f, \phi) : X \to \mathbb{R} \times f^{-1}(y)$ such that $f = p \circ F$, where $p : \mathbb{R} \times f^{-1}(y) \to \mathbb{R}$ denotes the projection. By Proposition 2.3, we have a Nash $G$ map $\psi : X \to f^{-1}(y)$ as an approximation of $\phi$. If this approximation is sufficiently close, then $H = (f, \psi) : X \to \mathbb{R} \times f^{-1}(y)$ is the required Nash $G$ diffeomorphism. □

**References**


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