数理解析研究所講究録 1393 巻 2004 年 115-121



# $(\mathbb{Z}_2)^k$-ACTIONS AND A LINEAR INDEPENDENCE CONDITION

ZHI LÜ

## 1. BASIC BACKGROUND

Throughout the following, $G=(\mathbb{Z}_2)^k$ (i.e., 2-torus of rank $k\geq 0$ where $k=0$ means that $G$ is trivial), and all manifolds are smooth, closed and unoriented. Here *unoriented* means that no requirements of orientability or orientedness are imposed.

Now let us review the development history of $G$-manifolds from the viewpoint of bordism theory.

(I) The case in which $G$ is trivial.

In the 1950s, Thom invented the unoriented bordism theory, so that all closed manifolds are classified in terms of bordism and are understood very well. Simply speaking, the unoriented bordism classes of all closed manifolds form a polynomial algebra $\mathfrak{N}_*=\sum_{n\geq 0}\mathfrak{N}_n$ over $\mathbb{Z}_2$ with generators $x_n, n\neq 2^s-1$. In even dimensions the $x_n$ can be chosen to be the real projective spaces $\mathbb{R}P^n$, and in odd dimensions the $x_n$ can explicitly be chosen to be Dold manifolds. Note that Stong [S4] constructed Stong manifolds so $x_n$ can be chosen to a Stong manifold whenever $n$ is even or odd.

(II) The case in which $G$ is non-trivial.

In the 1960s, Conner and Floyd applied the bordism theory to $G$-manifolds and established the equivariant bordism theory, so that $G$-manifolds are understood very well in many respects in terms of bordism (see, [C4], [CF]). For example, they showed that

(1) The equivariant bordism class of any involution (i.e., $\mathbb{Z}_2$-action) on a closed manifold is determined by that of normal bundle to its fixed point set.

(2) Any closed $G$-manifold $M$ with $M^G$ empty bounds.

(3) There cannot be $G$-actions fixing only an isolated point.

Along this line (or from the viewpoint of bordism), the further development with respect to $G$-manifolds is stated as follows.

Let $(\Phi, M^n)$ be a $G$-action on a closed manifold, and $F$ its fixed point set. Then $F$ is the disjoint union of submanifolds of $M^n$. By $\dim F$ we denotes the dimension of the component of $F$ of largest dimension.

(A) The case where $G=\mathbb{Z}_2$ (i.e., involution).

(i) In 1967, Boardman [B] proved $\frac{5}{2}$-theorem that if $\dim M>\frac{5}{2}\dim F$, then $M^n$ bounds.

(ii) In 1973, Stong [S4] introduced the group $J_n^r$, which consists of all those closed manifolds in $\mathfrak{N}_n$ admitting an involution whose fixed point set has constant-codimension $r$. Many authors studied such $J_n^r$. For example, Capobianco [C1][C2] determined the group structure of $J_n^r$ when $r=3,4$; Iwata [I] determined the group structure of $J_n^r$ when $r=5$; Wada [W1] determined the

RIMS-Workshop, 27 May 2004

group structure of $J_n^r$ when $r = 6$; Wu [W3] determined the group structure of $J_n^r$ when $r \leq 35$; finally, Yue [Y4] completely determined the group structure of $J_n^r$ for all $r$.

In 1989, Wu [W2] introduced the group $J_n^{r_1,\dots,r_l}$ in the general sense. which consists of all those closed manifolds in $\mathfrak{N}_n$ admitting an involution whose fixed point sets have codimensions $r_1, \dots, r_l$. Up to now, the group structure of $J_n^{r_1,\dots,r_l}$ is yet not determined completely. Some works with respect to this problem can be found in [LW1],[LW2],[L1], [L2], and [LL1].

(iii) In 1978, Kosniowski and Stong [KS1] gave a formula of calculating Stitefel-Whitney numbers of $M^n$ in terms of fixed data $\nu \to F = \bigsqcup_r \nu^r \to F^{n-r}$, which is stated as follows.

**Kosniowski-Stong formula I:** *If $f(x_1, \cdots, x_n)$ is any symmetric polynomial over $\mathbb{Z}_2$ in $n$ variables of degree at most $n$, then*

$$f(x_1, \cdots, x_n)[M^n] = \sum_r \frac{f(1+y_1, \cdots, 1+y_r, z_1, \cdots, z_{n-r})}{\prod_{i=1}^r (1+y_i)}[F^{n-r}]$$

*where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x), \sigma_i(y)$, and $\sigma_i(z)$ by the Stiefel-Whitney classes $W_i(M^n)$, $W_i(\nu^r)$, and $W_i(F^{n-r})$ respectively, and taking the value of the resulting cohomology class on the fundamental homology class of $M^n$ or $F^{n-r}$.*

Using this formula, some classical results can be reproved. For example, Smith theorem (i.e., $\chi(M^n) \equiv \chi(F) \mod 2$), Boardman $\frac{5}{2}$-theorem (note that actually, a stronger result can be obtained, i.e., if $\dim M > \frac{5}{2}\dim F$, then $M^n$ with $\mathbb{Z}_2$-action bounds equivariantly), and some results by Conner and Floyd [CF], and by tom Dieck [D1]. In addition, some new results can be obtained. For example, Kosniowski and Stong showed that if $(\Phi, M)$ is an involution fixing a constant-dimensional fixed point set such that $\dim M > 2\dim F$, then $(\Phi, M)$ bounds equivariantly. Also, Lü [L3] showed that if $(\Phi, M)$ is an involution with $\dim M > 2\dim F$ (note that here $F$ is not restricted to be constant dimensional), then $(\Phi, M)$ bounds equivariantly, and especially, $2\dim F$ is the best possible upper bound of $\dim M$ if $(\Phi, M)$ is nonbounding.

(B) The case where $G = (\mathbb{Z}_2)^k$ with $k \geq 1$.

(i) In 1970, Stong [S3] generalized the above result (1) of Conner-Floyd into the general case, i.e., the equivariant cobordism class of any $(\mathbb{Z}_2)^k$-action on a closed manifold is determined by that of normal bundle to its fixed point set. Further, he showed that any closed $G$-manifold $M$ with $M^G$ empty doesn't only bounds, but also bounds equivariantly.

(ii) In 1971, tom Dieck [D2] studied characteristic numbers of $G$-manifolds, and also obtained some integrality theorems. Then applying obtained results to $G$-manifolds fixing only isolated points, he gave a necessary and sufficient condition that some isolated points with given $G$-representations are the fixed data of a $G$-manifold.

(iii) In 1979, Kosniowski and Stong [KS2] obtained the formula in the general case of $(\mathbb{Z}_2)^k$-actions, stated as follows.

**Kosniowski-Stong formula II:** *If $f(\alpha; x; x^i)$ is of degree less than or equal to $n$, then*

$$f(\alpha; x; x^i)[M] = \sum_F \frac{f(\alpha; z, \alpha + y; \alpha + v)}{\prod(\alpha + y)}[F]$$

*in the quotient filed of $H^*(BG; \mathbb{Z}_2)$ .*

Note: The above formulae I, II were given by Kawakubo [K] independently.

Using this formula, Kosniowski and Stong showed that if $(\Phi, M^n)$ is a $G$-action with the fixed point set $F = \bigsqcup_r F^{n-r}$ satisfying the condition that each part $F^{n-r}$ is connected and if $\dim M > (2^{k+1} - 1)\dim F$, then $(\Phi, M^n)$ bounds equivariantly. Note that this inequality is not the best possible. And Kosniowski and Stong also posed the following question:

*Suppose that* $(\Phi, M)$ *is a* $G$*-action fixing a connected submanifold* $F$*. Then (a) if* $\dim M > 2^k \dim F$, $(\Phi, M)$ *bounds equivariantly; (b) if* $\dim M = 2^k \dim F$, $(\Phi, M)$ *is equivariantly cobordant to the action* $(twist, F \times \cdots \times F)$ *on the product of* $2^k$ *copies of* $F$ *which interchanges factors.*

This question was solved by Pergher [P3] in 2002.

(iv) In 1992, Pergher [P1] introduced the group $J^r_{n,k}$, which consists of those closed manifolds in $\mathfrak{N}_n$ admitting a $G$-action with fixed point set of codimension $r$. This is directly the generalization of $J^r_n$ for $k = 1$. When $k > 1$, generally it is difficult to determine the structure of $J^r_{n,k}$. Some works in this respect can be found in [P7], [S1], [S2], [WWD], [WWM].

(v) In 2001, Lü [L4] introduced a linear independence condition for the fixed point set. With the help of the condition, argument can be carried out without the connectedness restriction of fixed point set. For example, with the help of the condition, one can analyze the following two kinds of $G$-actions: (1) $G$-actions with trivial normal bundle of fixed point set; (2) $G$-actions with $w(F) = 1$. Note that in this talk, I will mainly introduce the linear independence condition, and a result for $G$-actions with $w(F) = 1$.

**Fundamental Problem:** *To classify all G-manifolds in terms of bordism.*

Unfortunately, the fundamental problem is far from solved even if $G$ is equal to the simplest $\mathbb{Z}_2$ group. Also, many works are restricted to be $G = \mathbb{Z}_2$.

With respect to the fundamental problem, the following two ways are often used mainly.

One is that given a known closed manifold $M$, to classify all possible $G$-actions on $M$. The other one is that given a known closed manifold $F$, to classify all possible $G$-actions fixing $F$. Some works in this respect can be found in [C3], [HT1], [HT2], [LL2], [L5], [L6], [P2], [P4], [P5], [P6], [P8], [PS], [R], [S5], [Y1], [Y2], [Y3].

*Kosniowski-Stong Conjecture:* Any involution with $w(F) = 1$ is cobordant to a polynomial formed by involutions $(T_s, \mathbb{R}P^{2^s})$ defined by

$$T_s : [x_0, x_1, ..., x_{2^s}] \longmapsto [-x_0, x_1, ..., x_{2^s}].$$

## 2. A LINEAR INDEPENDENCE CONDITION

Suppose that $(\phi, M^n)$ is a $G$-action on a closed manifold $M^n$ and let $F = \sqcup_d F^{n-d}$ be its fixed point set.

Let $\mathrm{Hom}(G, \mathbb{Z}_2)$ be the set of homomorphisms $\rho : G \to \mathbb{Z}_2 = \{+1, -1\}$, which consists of $2^k$ distinct homomorphisms. One agrees to let $\rho_0$ denote the trivial element in $\mathrm{Hom}(G, \mathbb{Z}_2)$, i.e., $\rho_0(g) = 1$ for all $g \in G$. Every irreducible real representation of $G$ is one-dimensional and has the form $\lambda_\rho : G \times \mathbb{R} \longrightarrow \mathbb{R}$ with $\lambda_\rho(g, r) = \rho(g) \cdot r$ for some $\rho$. $\lambda_{\rho_0}$ is the trivial representation corresponding to $\rho_0$.

Let $EG \longrightarrow BG$ be the universal principal $G$-bundle, where $BG = EG/G = (\mathbb{R}P^\infty)^k$ is the classifying space of $G$. It is well-known that

$$H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, ..., a_k]$$

with the $a_i$ one-dimensional generators. In particular, all nonzero elements of $H^1(BG; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k$ consist of $2^k - 1$ polynomials of degree one in $\mathbb{Z}_2[a_1, ..., a_k]$, i.e.,

$$a_1, ..., a_k,$$

$$a_1 + a_2, ..., a_1 + a_k, a_2 + a_3, ..., a_2 + a_k, ...,$$

$$\cdots$$

$$a_1 + \cdots + a_{k-1}, ..., a_2 + a_3 + \cdots + a_k,$$

$$a_1 + a_2 + \cdots + a_k.$$

These polynomials of degree one correspond to all nontrivial elements of $\mathrm{Hom}(G, \mathbb{Z}_2)$ (note that actually $H^1(BG; \mathbb{Z}_2) \cong \mathrm{Hom}(G, \mathbb{Z}_2)$), and so for a convenience, they are denoted by $\alpha_\rho$ for $\rho \in \mathrm{Hom}(G, \mathbb{Z}_2)$ with $\rho \neq \rho_0$. Also, one agrees to let $\alpha_{\rho_0} = 0$, the zero element of $H^1(BG; \mathbb{Z}_2)$.

For each part $F^{n-d}$ of $F$, write $F^{n-d} = \bigsqcup_{i=1}^{\ell_d} F_i^{n-d}$. Then the restriction to each connected component $F_i^{n-d}$ of $F^{n-d}$ of the tangent bundle of $M^n$ decomposes into subbundles under the action of $G$

$$\tau_M|_{F_i^{n-d}} \cong \tau_{F_i^{n-d}} \oplus \bigoplus_{\rho \neq \rho_0} \nu_{\rho,d}$$

where $\nu_{\rho,d}$ is the subbundle on which $G$ acts via $\lambda_\rho$, and the subbundle on which $G$ acts trivially is identified with the tangent bundle of $F_i^{n-d}$. Let $q^i_{\rho,d} = \dim \nu_{\rho,d}$, so that $d = \sum_{\rho \neq \rho_0} q^i_{\rho,d}$ and one obtains the sequence $\{q_{\rho,d} | \rho \neq \rho_0\}$ (called the *normal dimensional sequence*). The collection

$$C_{F^{n-d}} = \{\{q^i_{\rho,d} | \rho \neq \rho_0\} | i = 1, \cdots, \ell_d\}$$

of such sequences occuring in $F^{n-d}$ will be called the *normal dimensional sequence set* of $F^{n-d}$.

Generally, all sequences of $C_{F^{n-d}}$ may not be distinct if $F^{n-d}$ is disconnected. However, $(\Phi, M^n)$ is equivariantly cobordant to a $G$-action such that all elements of the normal dimensional sequence set of the $(n-d)$-dimensional part $F'^{n-d}$ of its fixed point set are distinct. In fact, one may form a connected sum for those connected components in $F^{n-d}$ with the same normal dimensional sequence when $n - d > 0$, and one may cancel pairs of components with the same normal dimensional sequence when $n - d = 0$. This doesn't change the $(\mathbb{Z}_2)^k$-action $(\Phi, M^n)$ up to equivariant cobordism. Thus, without loss of generality one may assume that the part of $F^{n-d}$ with the same normal dimensional sequence is connected, so all sequences of $C_{F^{n-d}}$ are distinct.

DEFINITION. We say that the $(n-d)$-dimensional part $F^{n-d}$ of $F$ possesses the *linear independence property* if its normal dimensional sequence set

$$C_{F^{n-d}} = \{\{q^i_{\rho,d} | \rho \neq \rho_0\} | i = 1, \cdots, \ell_d\}$$

has the following property:

$$\frac{1}{\prod_{\rho \neq \rho_0} \alpha_\rho^{q^1_{\rho,d}}}, \cdots, \frac{1}{\prod_{\rho \neq \rho_0} \alpha_\rho^{q^{\ell_d}_{\rho,d}}}$$

are linearly independent in the quotient field of $\mathbb{Z}_2[a_1, ..., a_k]$.

**Theorem 2.1.** *Suppose that $(\Phi, M^n)$ is a smooth $(\mathbb{Z}_2)^k$-action on a closed smooth n-dimensional manifold such that each part $F^p$ of the fixed point set $F$ possesses the linear independence property, and $w(F) = 1$. If $\dim M^n > 2^k \dim F$, then $(\Phi, M^n)$ bounds equivariantly.*

Note. (1) When $k = 1$, as shown in [L3], $2 \dim F$ is the best possible upper bound of $\dim M$ if the involution $(\Phi, M)$ with $w(F) = 1$ doesn't bound. For the general case, Example 1 will show that $2^k \dim F$ is still the best possible upper bound of $\dim M$ if $(\Phi, M)$ doesn't bound.

(2) Example 2 will show that the condition which each part $F^p$ of the fixed point set $F$ possesses the linear independence property is necessary.

**Example 1.** Let us begin with the involution $(T, \mathbb{R}P^2)$ given by $[x_0; x_1; x_2] \longmapsto [-x_0; x_1; x_2]$, which fixes the disjoint union of a point and a real projective 1-space $\mathbb{R}P^1$. Then the product

$$(\underbrace{T \times \cdots \times T}_{\ell}, \underbrace{\mathbb{R}P^2 \times \cdots \times \mathbb{R}P^2}_{\ell})$$

of $\ell$ copies of $(T, \mathbb{R}P^2)$ forms a new involution, and its fixed point set is $\bigsqcup_{i=0}^{\ell} \binom{\ell}{i} \mathbb{R}P^1 \underbrace{\times \cdots \times}_{i} \mathbb{R}P^1$, where $\mathbb{R}P^1 \underbrace{\times \cdots \times}_{i} \mathbb{R}P^1$ means a point if $i = 0$. This new involution is cobordant to an involution $(\Phi_1, M_1^{2\ell})$ having fixed set $F = F^\ell \bigsqcup F^{\ell-1} \bigsqcup \cdots \bigsqcup F^0$ with $\dim M^{2\ell} = 2 \dim F$ and $w(F) = 1$, where

$$F^p = \begin{cases} \mathbb{R}P^1 \underbrace{\times \cdots \times}_{p} \mathbb{R}P^1 & \text{if } \binom{\ell}{p} \not\equiv 0 \mod 2 \\ \text{empty} & \text{if } \binom{\ell}{p} \equiv 0 \mod 2. \end{cases}$$

Consider $M_1^{2\ell} \times M_1^{2\ell}$ with two involutions $t_1$ = twist and $t_2 = \Phi_1 \times \Phi_1$. The fixed point set of this $(\mathbb{Z}_2)^2$-action $(\Phi_2, M_2^{4\ell})$ is the fixed point set of $\Phi_1$ in the diagonal copy of $M_1^{2\ell}$ which is $F = F^\ell \bigsqcup F^{\ell-1} \bigsqcup \cdots \bigsqcup F^0$, which has $w(F) = 1$ and $\dim M_2^{4\ell} = 2^2 \dim F$. Squaring this example gives examples for all $(\mathbb{Z}_2)^k$-actions. Actually, if $(\Phi_{k-1}, M_{k-1}^{2^{k-1}\ell})$ is a $(\mathbb{Z}_2)^{k-1}$-action fixing $F = F^\ell \bigsqcup F^{\ell-1} \bigsqcup \cdots \bigsqcup F^0$, then the twist and the diagonal $(\mathbb{Z}_2)^{k-1}$-action induced by $\Phi_{k-1}$ on $M_{k-1}^{2^{k-1}\ell} \times M_{k-1}^{2^{k-1}\ell}$ produce a $(\mathbb{Z}_2)^k$-action $(\Phi_k, M_k^{2^k\ell})$ whose fixed set is still $F = F^\ell \bigsqcup F^{\ell-1} \bigsqcup \cdots \bigsqcup F^0$, and $\dim M_k^{2^k\ell} = 2^k \dim F$. Also, the linear independence for the fixed point set is obvious since $F^p$ is connected for each $p$. However, $(\Phi_k, M_k^{2^k\ell})$ is nonbounding for every value of $\dim F = \ell$ and every $k$.

**Example 2.** Consider the standard $(\mathbb{Z}_2)^2$-action $(\Phi_0, \mathbb{R}P^2)$ given by

$$[x_0, x_1, x_2] \longmapsto [x_0, g_1 x_1, g_2 x_2],$$

which fixes three isolated points, where $(g_1, g_2) \in (\mathbb{Z}_2)^2$. Then the diagonal action on the product of $2\ell'$ copies of $(\Phi_0, \mathbb{R}P^2)$ is also a $(\mathbb{Z}_2)^2$-action denoted by $(\Phi, M^{4\ell'})$, and the fixed point set of this action is formed by $3^{2\ell'}$ isolated points. Furthermore, by using the construction as in Example 1 to $(\Phi, M^{2\ell'})$, one may obtain a $(\mathbb{Z}_2)^k$-action $(\Psi, M^{2^k\ell'})$, which fixes $3^{2\ell'}$ isolated points. Now, the diagonal action on the product of $(\Psi, M^{2^k\ell'})$ and $(\Phi_k, M_k^{2^k\ell})$ in Example 1 produces a $(\mathbb{Z}_2)^k$-action $(\Phi', M^{2^k(\ell+\ell')})$ fixing the disjoint union $F'$ of $3^{2\ell'}$ copies of $F = F^\ell \bigsqcup F^{\ell-1} \bigsqcup \cdots \bigsqcup F^0$. However, $(\Phi', M^{2^k(\ell+\ell')})$ never bounds although $\dim M^{2^k(\ell+\ell')} = 2^k(\ell + \ell') > 2^k \dim F'$ for $\ell' > 0$ and $w(F') = 1$. This is because each $p$-dimensional part of $F'$ doesn't satisfy the linear independence property.

## References


[B] J.M. Boardman, *On manifolds with involution*, Bull. Amer. Math. Soc. **73** (1967), 136–138.

[C1] F.L. Capobianco, *Manifolds with involution whose fixed set has codimension four*, Proc. Amer. Math. Soc. **61** (1976), no. 1, 157–162.

[C2] F.L. Capobianco, *Cobordism classes represented by fiberings with fiber $RP(2k+1)$*, Michigan Math. J. **24** (1977), no. 2, 185–192.

[C3] F.L. Capobianco, *Stationary points of $(\mathbb{Z}_2)^k$-actions*, Proc. Amer. Math. Soc. **61** (1976), 377-380.

[C4] P.E. Conner, *Differentiable Periodic Maps*, 2nd Edition, **738**, Springer-Verlag, 1979.

[CF] P.E. Conner and E.E. Floyd, *Differentiable periodic maps*, Ergebnisse Math. Grenzgebiete, N. F., Bd. **33**, Springer-Verlag, Berlin, 1964.

[D1] T. tom Dieck, *Fixpunkte vertauschbarer Involutionen*, Arch. Math. **17** (1966), 395-399.

[D2] T. tom Dieck, *Characteristic numbers of G-manifolds. I*, Invent. Math. **13** (1971), 213-224.

[HT1] D. Hou and B. Torrence, *Involutions fixing the disjoint union of odd-dimensional projective spaces*, Canad. Math. Bull. **37** (1994), no. 1, 66–74.

[HT2] D. Hou and B. Torrence, *Involutions fixing the disjoint union of copies of even projective space*, Acta Math. Sinica (N.S.) **12** (1996), no. 2, 162–166.

[I] K. Iwata, *Involutions with fixed point set of constant codimension*, Proc. Amer. Math. Soc. **83** (1981), no. 4, 829–832.

[K] K. Kawakubo, *Global and local equivariant characteristic numbers of G-manifolds*, J. Math. Soc. Japan **32** (1980), 301-323.

[KS1] C. Kosniowski and R.E. Stong, *Involutions and characteristic numbers*, Topology **17** (1978), 309-330.

[KS2] C. Kosniowski and R.E. Stong, *$(\mathbb{Z}_2)^k$-actions and characteristic numbers*, Indiana Univ. Math. J. **28** (1979), 723-743.

[LL1] Z. Lü and X.B. Liu, *Manifolds with involution whose fixed point set has variable codimension*, Topology Appl. **70** (1996), no. 1, 57–65.

[LL2] Z. Lü and X.B. Liu, *Involutions fixing the disjoint union of 3-real projective space with Dold manifold*, Kodai Math. J. **23** (2000), no. 2, 187–213.

[L1] Z. Lü, *Involutions with fixed point sets of variable codimension* (Chinese), Chinese Ann. Math. Ser. A **16** (1995), no. 1, 16–21.

[L2] Z. Lü, *Smooth involutions for which the highest-dimensional component of the fixed point set has dimension $n-5$* (Chinese), Acta Math. Sinica **38** (1995), no. 5, 583–588.

[L3] Z. Lü, *Involutions with $W(F)=1$*, Proc. Amer. Math. Soc. **128** (2000), 307-313.

[L4] Z. Lü, *$(\mathbb{Z}_2)^k$-actions with disconnected fixed point set*, Topology Appl. **114** (2001), 227-233.

[L5] Z. Lü, *Involutions fixing $\mathbb{RP}^{\mathrm{odd}} \sqcup P(h,i)$. I*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4539–4570.

[L6] Z. Lü, *Involutions fixing $\mathbb{RP}^{\mathrm{odd}} \sqcup P(h,i)$. II*, Trans. Amer. Math. Soc. **356** (2004), no. 4, 1291-1314.

[L7] Z. Lü, *$(\mathbb{Z}_2)^k$-actions with $w(F)=1$*, preprint.

[LW1] Z.Z. Liu and Z.D. Wu, *Determination of the groups $J^{6,k_1,\cdots,k_l}$* (Chinese), Acta Math. Sinica **37** (1994), no. 3, 296–300.

[LW2] Z.Z. Liu and Z.D. Wu, *Manifolds with an involution whose set of fixed points has nonconstant codimension* (Chinese), Acta Math. Sinica **37** (1994), no. 4, 444–448.

[P1] P.L.Q. Pergher, *$(\mathbb{Z}_2)^k$-actions with fixed point set of constant codimension*, Topology Appl. **46** (1992), 55-64.

[P2] P.L.Q. Pergher, *$(\mathbb{Z}_2)^k$-actions whose fixed data has a section*, Trans. Amer. Math. Soc. **353** (2001), 175-189.

[P3] P.L.Q. Pergher, *On $(\mathbb{Z}_2)^k$-actions*, Topology Appl. **117** (2002), 105-112.

[P4] P.L.Q. Pergher, *$(\mathbb{Z}_2)^k$-actions fixing a product of spheres and a point*, Canad. Math. Bull. **38** (1995), no. 3, 366–372.

[P5] P.L.Q. Pergher, *The union of a connected manifold and a point as fixed set of commuting involutions*, Topology Appl. **69** (1996), no. 1, 71–81.

[P6] P.L.Q. Pergher, *Involutions fixing an arbitrary product of spheres and a point*, Manuscripta Math. **89** (1996), no. 4, 471–474.

[P7] P.L.Q. Pergher, *Bordism of two commuting involutions*, Proc. Amer. Math. Soc. **126** (1998), no. 7, 2141–2149.

[P8] P.L.Q. Pergher, *$Z_2^k$-actions fixing $\{\text{point}\} \cup V^n$*, Fund. Math. **172** (2002), no. 1, 83–97.

[PS] P.L.Q. Pergher and R.E. Stong, *Involutions fixing (point) $\cup F^n$*, Transform. Groups **6** (2001), no. 1, 79–86.

[R] D.C. Royster, *Involutions fixing the disjoint union of two projective spaces*, Indiana Univ. Math. J. **29** (1980), 267–276.

[S1] R.J. Shaker, *Constant codimension fixed sets of commuting involutions*, Proc. Amer. Math. Soc. **121** (1994), no. 1, 275–281.

[S2] R.J. Shaker, *Dold manifolds with $(Z_2)^k$-action*, Proc. Amer. Math. Soc. **123** (1995), no. 3, 955–958.

[S3] R.E. Stong, *Equivariant bordism and $(\mathbb{Z}_2)^k$-actions*, Duke Math. J. **37** (1970), 779-785.

[S4] R.E. Stong, *On fibering of cobordism classes*, Trans. Amer. Math. Soc. **178** (1973), 431–447.

[S5] R.E. Stong, *Involutions fixing projective spaces*, Michigan Math. J. **13** (1966), 445–447.

[W1] T. Wada, *Manifolds with involution whose fixed point set has codimension six*, Bull. Yamagata Univ. Natur. Sci. **10** (1981) 193–205.

[W2] Z.D. Wu, *Manifolds with involution*, Acta Math. Sinica (N.S.) **5** (1989), no. 4, 302–306.

[W3] Z.D. Wu, *Manifolds with involutions whose fixed point set has constant codimension*, Sci. China Ser. A **33** (1990), no. 7, 784–791.

[WWD] Y.Y. Wang, Z.D. Wu and Y.H. Ding, *Commuting involutions with fixed point set of constant codimension*, Acta Math. Sin. (Engl. Ser.) **15** (1999), no. 2, 181–186.

[WWM] Y.Y. Wang, Z.D. Wu and K. Ma, *$(Z_2)^k$-actions with fixed point set of constant codimension $2^k+1$*, Proc. Amer. Math. Soc. **128** (2000), no. 5, 1515–1521.

[Y1] H.J. Yang, *Involution number sequence and dimensions of components in fixed point set of involution*, Sci. China Ser. A **34** (1991), no. 11, 1302–1306.

[Y2] H.J. Yang, *Smooth involutions on $RP(2k+1)$*, (Chinese) Acta Math. Sinica **35** (1992), no. 1, 39–44.

[Y3] H.J. Yang, *On the fixed point sets of smooth involutions on the products of spheres*, Proc. Amer. Math. Soc. **124** (1996), no. 6, 1941–1947.

[Y4] Q.Q. Yue, *Manifolds with constant codimensional involution*, Sci. China Ser. A **35** (1992), no. 7, 837–841.

INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI, 200433, PEOPLE'S REPUBLIC OF CHINA.

*E-mail address*: zlu@@fudan.edu.cn

*Recent address*: DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN.