<table>
<thead>
<tr>
<th>Title</th>
<th>Remark on homotopy types of twisted complex projective spaces (Transformation Group Theory and Surgery)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yamaguchi, Kohhei</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1393: 140-144</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25898">http://hdl.handle.net/2433/25898</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Remark on homotopy types of twisted complex projective spaces

電気通信大学 山口耕平 (Kohhei Yamaguchi)
University of Electro-Communications

1 Introduction.

The main purpose of this note is to announce the recent results given in the preprints ([5], [11]) and is to consider the remaining several related unsolved problems. Let $m \geq 0$ and $n \geq 2$ be integers and let $M$ be a simply-connected $2n$ dimensional finite Poincaré complex. Then it is called an $m$-twisted $\mathbb{C}\mathbb{P}^{n}$ if there is an isomorphism $H_{*}(M, \mathbb{Z}) \cong H_{*}(\mathbb{C}\mathbb{P}^{n}, \mathbb{Z})$ and $x_{1} \cdot x_{1} = mx_{2}$, where $x_{k} \in H^{2k}(M, \mathbb{Z}) \cong \mathbb{Z}(k = 1, 2)$ denotes the generator. If $M$ is an $m$-twisted $\mathbb{C}\mathbb{P}^{n}$, it has the homotopy type of the form

\[ M \simeq S^{2} \cup_{m_{0}} e^{4} \cup e^{6} \cup \cdots \cup e^{2n-2} \cup e^{2n}. \] (1)

We denote by $\mathcal{M}_{m}^{n}$ the set consisting of all homotopy equivalence classes of $m$-twisted $\mathbb{C}\mathbb{P}^{n}$'s. If $n = 2$, it is easy to see that $\mathcal{M}_{1}^{2} = \{[\mathbb{C}\mathbb{P}^{2}]\}$ and $\mathcal{M}_{m}^{2} = \emptyset$ if $m \neq 1$. If $n = 3$, it is known in [9] that $\text{card}(\mathcal{M}_{m}^{3}) = 2 + (-1)^{m}$, where $\text{card}(V)$ denotes the number of a finite set $V$. For example, if $m = 0$ or 1, then $\mathcal{M}_{1}^{3} = \{[\mathbb{C}\mathbb{P}^{3}]\}$ and $\mathcal{M}_{0}^{3} = \{[M_{0}], [M_{1}], [M_{2}]\}$, where $i_{k} : S^{k} \rightarrow S^{2} \vee S^{4}$ denotes the inclusion ($k = 2, 4$) and we take $M_{0} = S^{2} \times S^{4} = S^{2} \vee S^{4} \cup_{[i_{2}, i_{4}]} e^{6}$, $M_{1} = S^{2} \vee S^{4} \cup_{i_{4} \oplus [i_{2}, i_{4}]} e^{6}$ and $M_{2} = S^{2} \vee S^{4} \cup_{i_{2} \oplus i_{4} + [i_{2}, i_{4}]} e^{6}$. 

In general, we can show that $\mathcal{M}_{m}^{n} \neq \emptyset$ for any $m \geq 0$ when $n \geq 5$ is an odd integer, which is shown by using a technique of the theory of transformation groups (cf. [1]). So it seems interesting to study the set $\mathcal{M}_{m}^{n}$ when $n \geq 4$ is an even integer. More precisely, we consider the following:
Problem. Let $n \geq 4$ and $m \geq 0$ be integers.

(a) Then is the set $\mathcal{M}_m^n$ an emptyset or not? Moreover, if $\mathcal{M}_m^n \neq \emptyset$, can we determine the number $\text{card}(\mathcal{M}_m^n)$ and representatives of $\mathcal{M}_m^n$?

(b) Let $M$ be an $m$-twisted $\mathbb{C}P^n$. Then does it have the homotopy type of closed smooth manifolds of dimension $2n$?

The precise statement of this paper is as follows.

**Theorem 1.1.** Let $m \geq 0$ be an integer and let $(a, b)$ denote the greatest common divisor of positive integers $a$, $b$.

(i) If $m \equiv 1 \pmod{2}$, $\text{card}(\mathcal{M}_m) = (m, 3)$.

(ii) If $m \equiv 0 \pmod{2}$ and it is not divisible by $8$, $\mathcal{M}_m = \emptyset$.

(iii) If $m \equiv 0 \pmod{8}$ and $m \neq 0$, $\mathcal{M}_m^4 \neq \emptyset$ and its number is estimated as $3 \leq \text{card}(\mathcal{M}_m^4) \leq 2^5 \cdot 3 \cdot m(m, 3)$.

(iv) In particular, if $m = 0$, $\mathcal{M}_0^4 \neq \emptyset$ and its number is estimated as $3 \leq \text{card}(\mathcal{M}_0^4) \leq 2^7 \cdot 3^2$.

**Theorem 1.2.** Let $m \geq 0$, $n \geq 2$ be integers and let $M$ be an $m$-twisted $\mathbb{C}P^n$. Then it has the homotopy type of topological manifolds of dimension $2n$. In particular, if $n = 4$, then it also has the homotopy type of PL-manifolds of dimension $8$.

### 2 Homotopy groups

In this section we shall give the rough idea of the proof of Theorem 1.1.

For each integer $m \geq 0$, we denote by $L_m$ the CW complex defined by $L_m = S^2 \cup_{m \eta_2} e^4$. Then we recall the following:

**Lemma 2.1.** Let $m \geq 0$ be an integer.

$$
\pi_5(L_m) = \begin{cases} 
Z \cdot b_m & \text{if } m \equiv 1 \pmod{2}, \\
Z \cdot b_m \oplus Z/4 \cdot \gamma_m & \text{if } m \equiv 2 \pmod{4}, \\
Z \cdot b_m \oplus Z/2 \cdot \gamma_m \oplus Z/2 \cdot i_*(\eta_2^3) & \text{if } m \equiv 0 \pmod{4},
\end{cases}
$$

where we take $b_m = [i, i^4]$ and $\gamma_m = i_4 \circ \eta_4$ if $m = 0$, and $2\gamma_m = i_*(\eta_2^3)$ if $m \equiv 2 \pmod{4}$. 

(ii) Let $M$ be an $m$-twisted $\mathbb{C}P^4$ and $M^{(6)}$ denote its 6-skeleton. Then there is a homotopy equivalence

$$M^{(6)} \simeq \begin{cases} X_m & \text{if } m \equiv 1 \, (\mod 2) \\ V_m & \text{if } m \equiv 0 \, (\mod 2), \ V \in \{X, Y\} \end{cases}$$

where we take $X_m = L_m \cup mb_m \cdot e^6$ and $Y_m = L_m \cup mb_m + i_*(\eta_2) \cdot e^6$.

**Proof.** This can be proved using standard computation of homotopy groups and the method given in [9].

**Lemma 2.2.** Let $j_* : \pi_7(X_m) \to \pi_7(X_m, L_m)$ denote the induced homomorphism. 

(i) If $m \equiv 1 \, (\mod 2)$, there exists some element $\varphi_m \in \pi_7(X_m)$ such that, $j_*(\varphi_m) = [\beta_m, i]_r + \epsilon_m \cdot \beta_m \circ \eta_5$, and there is an isomorphism

$$\pi_7(X_m) = \mathbb{Z}/(m, 3) \cdot j_*(f_m \circ \omega_m) \oplus \mathbb{Z}/m \cdot j_*(b_m \circ \eta_5) \oplus \mathbb{Z} \cdot \varphi_m.$$

(ii) If $m \equiv 0 \, (\mod 8)$ and $m \neq 0$, there exists some element $\varphi_m \in \pi_7(X_m)$ such that, $j_*(\varphi_m) = [\beta_m, i]_r$, and there is an isomorphism

$$\pi_7(X_m) = \mathbb{Z} \cdot \varphi_m \oplus \mathbb{Z}/4 \cdot j_*(f_m \circ \nu') \oplus \mathbb{Z}/2 \cdot j_*(f_m \circ \sigma \circ \eta_6) \oplus \mathbb{Z}/2 \cdot (j \circ i)_*(\eta_2 \circ \omega \circ \eta_6) \oplus \mathbb{Z}/(m, 3) \cdot j_*(f_m \circ \omega_m) \oplus \mathbb{Z}/2 \cdot j_*([b_m, i_*(\eta_2)]) \oplus \mathbb{Z}/2 \cdot \tilde{\eta}_5.$$

(iii) If $m = 0$, then $X_0 = S^2 \vee S^4 \vee S^6$ and there is an isomorphism

$$\pi_7(X_0) = \mathbb{Z} \cdot j_4 \circ \nu_4 \oplus \mathbb{Z} \cdot [j_2, j_6] \oplus \mathbb{Z}/2 \cdot j_2 \circ \eta_2 \circ \omega \circ \eta_6 \oplus \mathbb{Z}/2 \cdot j_6 \circ \eta_6 \oplus \mathbb{Z}/12 \cdot j_4 \circ E \omega \oplus \mathbb{Z}/2 \cdot [j_2, j_4 \circ \eta_4] \oplus \mathbb{Z}/2 \cdot [j_2 \circ \eta_2, j_4 \circ \eta_4] \oplus \mathbb{Z}/2 \cdot [j_2 \circ \eta_2^2, j_4],$$

where $j_k : S^k \to S^2 \vee S^4 \vee S^6$ $(k = 2, 4, 6)$ denote the corresponding inclusions.

**Proof.** The proof is given using standard computations of homotopy groups.

Similarly we obtain:
Lemma 2.3. Let $m \geq 0$ be an integer with $m \equiv 0 \pmod{8}$, and let $j_2 : \pi_7(Y_m) \to \pi_7(Y_m, L_m)$ be the induced homomorphism. Then there exists some element $\varphi'_m \in \pi_7(Y_m)$ such that, $j_2 \ast (\varphi'_m) = [\beta'_m, i]_r$, and there is an isomorphism

\[
\pi_7(Y_m) = \mathbb{Z} \cdot \varphi'_m \oplus \mathbb{Z}/4 \cdot j'_* (f_m \circ \nu) \oplus \mathbb{Z}/2 \cdot j'_* (f_m \circ \sigma \circ \eta_6) \\
\oplus \mathbb{Z}/2 \cdot j'_* (i_*(\eta_2 \circ \omega \circ \eta_6)) \oplus \mathbb{Z}/(m, 3) \cdot j'_* (f_m \circ \omega_m) \\
\oplus \mathbb{Z}/2 \cdot j'_* (b_m \circ \eta_6^2) \oplus \mathbb{Z}/m \cdot j'_* ([b_m, i_*(\eta_7)]) \quad \text{if } m \neq 0,
\]

\[
\pi_7(Y_0) = \mathbb{Z} \cdot j'_* (i_4 \circ \nu_4) \oplus \mathbb{Z} \cdot \varphi'_0 \oplus \mathbb{Z}/2 \cdot j'_* ([i, i_4 \circ \eta_4^2]) \oplus \mathbb{Z}/12 \cdot j'_* (i_4 \circ E \omega) \\
\oplus \mathbb{Z}/2 \cdot j'_* ([i_*(\eta_2), i_4 \circ \eta_4]) \oplus \mathbb{Z}/2 \cdot j'_* ([i_*(\eta_2^2), i_4]) \\
\oplus \mathbb{Z}/2 \cdot j'_* (\eta_2 \circ \omega \circ \eta_6) \quad \text{if } m = 0.
\]

Sketch proof of Theorem 1.1. If we use some lemmas given in [10] concerning the relation between cup-products and relative Whitehead products, we can show the desired assertions. \(\square\)

3 Surgery obstructions

First, we shall give rough idea of the proof of Theorem 1.2.

Sketch proof of Theorem 1.2. Since $M$ is a finite Poincaré complex, it follows from Theorem of Spivak that there is a spherical fiber space over $M$ with fiber $S^N$ ($N$: sufficiently large). Then by using the result of Stasheff, it is classified by the map $f_M : M \to BSG$. Let us consider whether it lifts to $BSTop$ or not. Its obstructions lie in $H^k(M, \pi_{k-1}(SG/STop))$ for all $k \geq 1$. However, since $\pi_j(SG/STop) = 0$ and $H^j(M) = 0$ if $j \equiv 1 \pmod{2}$, all obstructions vanish. Hence, the map $f_m$ lifts to $BSTop$. If we recall Theorem of the type of Browder-Novikov [6], we can show that $M$ has the homotopy type of topological manifolds of dimension $2n$.

Because $\pi_{2k-1}(G/O) = 0$ for $1 \leq k \leq 4$, if $n = 4$ the map $f_m$ lifts to $BSO$ and it follows from the Browder-Novokov type Theorem ([4], Corollary 2.17) that $M$ has the homotopy type of PL-manifolds of dimension 8. \(\square\)
References


