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Remark on homotopy types of twisted complex projective spaces

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1 Introduction.

The main purpose of this note is to announce the recent results given in the preprints ([5], [11]) and to consider the remaining several related unsolved problems. Let $m \geq 0$ and $n \geq 2$ be integers and let $M$ be a simply-connected $2n$ dimensional finite Poincaré complex. Then it is called an $m$-twisted $\mathbb{C}\mathrm{P}^n$ if there is an isomorphism $H_*(M, \mathbb{Z}) \cong H_*(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ and $x_1 \cdot x_1 = mx_2$, where $x_k \in H^{2k}(M, \mathbb{Z}) \cong \mathbb{Z}$ $(k = 1, 2)$ denotes the generator. If $M$ is an $m$-twisted $\mathbb{C}\mathrm{P}^n$, it has the homotopy type of the form

$$M \simeq S^2 \cup_{mm} e^4 \cup e^6 \cup \ldots \cup e^{2n-2} \cup e^{2n}.$$ (1)

We denote by $\mathcal{M}^n_m$ the set consisting of all homotopy equivalence classes of $m$-twisted $\mathbb{C}\mathrm{P}^n$'s. If $n = 2$, it is easy to see that $\mathcal{M}^2_1 = \{[\mathbb{C}\mathrm{P}^2]\}$ and $\mathcal{M}^2_m = \emptyset$ if $m \neq 1$. If $n = 3$, it is known in [9] that $\text{card}(\mathcal{M}^3_m) = 2 + (-1)^m$, where $\text{card}(V)$ denotes the number of a finite set $V$. For example, if $m = 0$ or 1, then $\mathcal{M}^3_1 = \{[\mathbb{C}\mathrm{P}^3]\}$ and $\mathcal{M}^3_0 = \{[M_0], [M_1], [M_2]\}$, where $i_k : S^k \to S^2 \vee S^4$ denotes the inclusion $(k = 2, 4)$ and we take $M_0 = S^2 \times S^4 = S^2 \vee S^4 \cup_{[i_2,i_4]} e^6$, $M_1 = S^2 \vee S^4 \cup_{i_2\circ\eta_i + [i_2,i_4]} e^6$ and $m_2 = S^2 \vee S^4 \cup_{i_2\circ\eta_i + [i_2,i_4]} e^6$.

In general, we can show that $\mathcal{M}^n_m \neq \emptyset$ for any $m \geq 0$ when $n \geq 5$ is an odd integer, which is shown by using a technique of the theory of transformation groups (cf. [1]). So it seems interesting to study the set $\mathcal{M}^n_m$ when $n \geq 4$ is an even integer. More precisely, we consider the following:
Problem. Let $n \geq 4$ and $m \geq 0$ be integers.

(a) Then is the set $\mathcal{M}_m^n$ an emptyset or not? Moreover, if $\mathcal{M}_m^n \neq \emptyset$, can we determine the number $\text{card}(\mathcal{M}_m^n)$ and representatives of $\mathcal{M}_m^n$?

(b) Let $M$ be an $m$-twisted $\mathbb{C}P^n$. Then does it have the homotopy type of closed smooth manifolds of dimension $2n$?

The precise statement of this paper is as follows.

**Theorem 1.1.** Let $m \geq 0$ be an integer and let $(a, b)$ denote the greatest common divisor of positive integers $a, b$.

(i) If $m \equiv 1 \pmod{2}$, $\text{card}(\mathcal{M}_m^4) = (m, 3)$.

(ii) If $m \equiv 0 \pmod{2}$ and it is not divisible by 8, $\mathcal{M}_m^4 = \emptyset$.

(iii) If $m \equiv 0 \pmod{8}$ and $m \neq 0$, $\mathcal{M}_m^4 \neq \emptyset$ and its number is estimated as $3 \leq \text{card}(\mathcal{M}_m^4) \leq 2^5 \cdot 3 \cdot m(m, 3)$.

(iv) In particular, if $m = 0$, $\mathcal{M}_0^4 \neq \emptyset$ and its number is estimated as $3 \leq \text{card}(\mathcal{M}_m^4) \leq 2^7 \cdot 3^2$.

**Theorem 1.2.** Let $m \geq 0$, $n \geq 2$ be integers and let $M$ be an $m$-twisted $\mathbb{C}P^n$. Then it has the homotopy type of topological manifolds of dimension $2n$. In particular, if $n = 4$, then it also has the homotopy type of PL-manifolds of dimension 8.

## 2 Homotopy groups

In this section we shall give the rough idea of the proof of Theorem 1.1.

For each integer $m \geq 0$, we denote by $L_m$ the CW complex defined by $L_m = S^2 \cup_{m\eta_2} e^4$. Then we recall the following:

**Lemma 2.1.** Let $m \geq 0$ be an integer.

\[
\pi_5(L_m) = \begin{cases} 
Z \cdot b_m & \text{if } m \equiv 1 \pmod{2}, \\
Z \cdot b_m \oplus Z/4 \cdot \gamma_m & \text{if } m \equiv 2 \pmod{4}, \\
Z \cdot b_m \oplus Z/2 \cdot \gamma_m \oplus Z/2 \cdot i_*(\eta_2^3) & \text{if } m \equiv 0 \pmod{4},
\end{cases}
\]

where we take $b_m = [i, i_4]$ and $\gamma_m = i_4 \circ \eta_4$ if $m = 0$, and $2\gamma_m = i_*(\eta_2^3)$ if $m \equiv 2 \pmod{4}$. 
(ii) Let $M$ be an $m$-twisted $\mathbb{C}P^4$ and $M^{(6)}$ denote its 6-skeleton. Then there is a homotopy equivalence

$$M^{(6)} \simeq \begin{cases} X_m & \text{if } m \equiv 1 \pmod{2} \\ V_m & \text{if } m \equiv 0 \pmod{2}, \ V \in \{X, Y\} \end{cases}$$

where we take $X_m = L_m \cup_{mb_m} e^6$ and $Y_m = L_m \cup_{mb_m+i_{*}(\eta_2)} e^6$.

**Proof.** This can be proved using standard computations of homotopy groups and the method given in [9].

**Lemma 2.2.** Let $j_{1*} : \pi_7(X_m) \to \pi_7(X_m, L_m)$ denote the induced homomorphism.

(i) If $m \equiv 1 \pmod{2}$, there exists some element $\varphi_m \in \pi_7(X_m)$ such that, $j_{1*}(\varphi_m) = [\beta_m, i]_r + \epsilon_m \cdot \beta_m \circ \eta_5$, and there is an isomorphism

$$\pi_7(X_m) = \mathbb{Z}/(m, 3) \cdot j_*(f_m \circ \omega_m) \oplus \mathbb{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \oplus \mathbb{Z} \cdot \varphi_m.$$  

(ii) If $m \equiv 0 \pmod{8}$ and $m \neq 0$, there exists some element $\varphi_m \in \pi_7(X_m)$ such that, $j_{1*}(\varphi_m) = [eta_m, i]_r$, and there is an isomorphism

$$\pi_7(X_m) = \mathbb{Z} \cdot \varphi_m \oplus \mathbb{Z}/4 \cdot j_*(f_m \circ \nu') \oplus \mathbb{Z}/2 \cdot j_*(f_m \circ \sigma \circ \eta_6) \oplus \mathbb{Z}/2 \cdot (j \circ i)_*(\eta_2 \circ \omega \circ \eta_6) \oplus \mathbb{Z}/(m, 3) \cdot j_*(f_m \circ \omega_m) \oplus \mathbb{Z}/2 \cdot j_*(b_m \circ \eta_6^2) \oplus \mathbb{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \oplus \mathbb{Z}/2 \cdot \eta_5.$$

(iii) If $m = 0$, then $X_0 = S^2 \vee S^4 \vee S^6$ and there is an isomorphism

$$\pi_7(X_0) = \mathbb{Z} \cdot j_4 \circ \nu_4 \oplus \mathbb{Z} \cdot [j_2, j_6] \oplus \mathbb{Z}/2 \cdot j_2 \circ \eta_2 \circ \omega \circ \eta_6 \oplus \mathbb{Z}/2 \cdot j_6 \circ \eta_6 \oplus \mathbb{Z}/12 \cdot j_4 \circ \mathrm{E}_6 \oplus \mathbb{Z}/2 \cdot [j_2, j_4 \circ \eta_4^2] \oplus \mathbb{Z}/2 \cdot \eta_6 \circ \eta_6 \oplus \mathbb{Z}/2 \cdot [j_2 \circ \eta_2, j_4 \circ \eta_4].$$

where $j_k : S^k \to S^2 \vee S^4 \vee S^6$ $(k = 2, 4, 6)$ denote the corresponding inclusions.

**Proof.** The proof is given using standard computations of homotopy groups.

Similarly we obtain:
Lemma 2.3. Let $m \geq 0$ be an integer with $m \equiv 0 \pmod{8}$, and let $j_2 : \pi_7(Y_m) \to \pi_7(Y_m, L_m)$ be the induced homomorphism. Then there exists some element $\varphi'_m \in \pi_7(Y_m)$ such that, $j_2^*(\varphi'_m) = [\beta'_m, i)_r$, and there is an isomorphism

$$
\pi_7(Y_m) = \mathbb{Z} \cdot \varphi'_m \oplus \mathbb{Z}/4 \cdot j'_*(f_m \circ \nu') \oplus \mathbb{Z}/2 \cdot j'_*(f_m \circ \sigma \circ \eta_6) \\
\oplus \mathbb{Z}/2 \cdot j'_*(i_*(\eta_2 \circ \omega \circ \eta_6)) \oplus \mathbb{Z}/(m, 3) \cdot j'_*(f_m \circ \omega_m) \\
\oplus \mathbb{Z}/2 \cdot j'_*(b_m \circ \eta_6^2) \oplus \mathbb{Z}/m \cdot j'_*([b_m, i_*(\eta_2)]) \quad \text{if } m \neq 0,
$$

$$
\pi_7(Y_0) = \mathbb{Z} \cdot j'_*(i_4 \circ \nu_4) \oplus \mathbb{Z} \cdot \varphi'_0 \oplus \mathbb{Z}/2 \cdot j'_*([i_4 \circ \eta_4^2]) \oplus \mathbb{Z}/12 \cdot j'_*(i_4 \circ \omega \circ \eta_6) \\
\oplus \mathbb{Z}/2 \cdot j'_*([i_*(\eta_2), i_4 \circ \eta_4]) \oplus \mathbb{Z}/2 \cdot j'_*([i_*(\eta_2^2), i_4]) \\
\oplus \mathbb{Z}/2 \cdot j'_*(\eta_2 \circ \omega \circ \eta_6) \quad \text{if } m = 0.
$$

Sketch proof of Theorem 1.1. If we use some lemmas given in [10] concerning the relation between cup-products and relative Whitehead products, we can show the desired assertions. \[\square\]

3 Surgery obstructions

First, we shall give rough idea of the proof of Theorem 1.2.

Sketch proof of Theorem 1.2. Since $M$ is a finite Poincaré complex, it follows from Theorem of Spivak that there is a spherical fiber space over $M$ with fiber $S^N$ ($N$: suuficiently large). Then by using the result of Stasheff, it is classified by the map $f_M : M \to BSG$. Let us consider whether it lifts to $BSTop$ or not. Its obstructions lie in $H^k(M, \pi_{k-1}(SG/STop))$ for all $k \geq 1$. However, since $\pi_j(SG/STop) = 0$ and $H^j(M) = 0$ if $j \equiv 1 \pmod{2}$, all obstructions vanish. Hence, the map $f_m$ lifts to $BSTop$. If we recall Theorem of the type of Browder-Novikov [6], we can show that $M$ has the homotopy type of topological manifolds of dimension $2n$.

Because $\pi_{2k-1}(G/O) = 0$ for $1 \leq k \leq 4$, if $n = 4$ the map $f_m$ lifts to $BSO$ and it follows from the Browder-Novokov type Theorem ([4], Corollary 2.17) that $M$ has the homotopy type of PL-manifolds of dimension 8. \[\square\]
References


