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Completely Regular Codes in Johnson graph

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1 Introduction

In this article, we study completely regular codes in some distance regular graphs. Completely regular codes were first studied by Biggs [2] and P. Delsarte [4], but there are not yet many articles on this subject. Recently, research of completely regular codes has been developing with research of Terwilliger algebra.

We consider completely regular codes in a distance regular graph. Martin [7] conjectured that for completely regular codes in a distance regular graph, $\gamma_i \leq \gamma_{i+1}$, $\beta_i \geq \beta_{i+1}$ hold. On the other hand, Koolen [6] showed $\gamma_i < \gamma_{i+1}$, $\beta_i > \beta_{i+1}$ in $H(D, 2)$. Furthermore, I conjectured $\gamma_i < \gamma_{i+1}$, $\beta_i > \beta_{i+1}$ in $J(n, d)$.

In order to study this conjecture, in this paper we first classify the completely regular codes in Johnson graph $J(4, 2)$, $J(5, 2)$, $J(6, 2)$, $J(6, 3)$.

2 Distance regular graphs

A graph is a pair $\Gamma = (V, E)$ consisting of a set $V$, referred to as the vertex set of $\Gamma$, and a set $E$ of 2-subsets of $V$, referred to as the edge set of $\Gamma$. That is, our graphs are undirected, without loops or multiple edges. We write $\gamma \in \Gamma$ if $\gamma$ is a vertex of $\Gamma$ and $\gamma \sim \delta$ if $\{\gamma, \delta\}$ is an edge of $\Gamma$. The distance $d(\gamma, \delta)$ of two vertices $\gamma, \delta \in \Gamma$ is the length of a shortest path between $\gamma$ and $\delta$. $d = \max\{d(\gamma, \delta) | \gamma, \delta \in \Gamma\}$ is called the diameter of $\Gamma$. Given $\delta \in \Gamma$, we write $\Gamma_i(\delta)$ for the set of vertices $\gamma$ with $d(\gamma, \delta) = i$. In particular, $\Gamma(\delta) = \Gamma_1(\delta)$ denotes the set of neighbours of $\delta$. The valency $k(\gamma)$ of a vertex $\gamma$ is the cardinality of $\Gamma(\gamma)$. A graph is called regular if each vertex has the same valency $k$. 
Definition 2.1 A connected graph $\Gamma = (V, E)$ is said to be a distance regular graph (DRG) if the numbers

\[ c_i = |\Gamma_{i-1}(\gamma) \cap \Gamma(\delta)|, \]
\[ a_i = |\Gamma_i(\gamma) \cap \Gamma(\delta)|, \text{ and} \]
\[ b_i = |\Gamma_{i+1}(\gamma) \cap \Gamma(\delta)| \]

are independent of the choices of $\gamma, \delta \in \Gamma$ with $d(\gamma, \delta) = i$.

The numbers $c_i$, $a_i$ and $b_i$ are said to be the intersection numbers of $\Gamma$.

\[ \iota(\Gamma) = \begin{bmatrix} * & c_1 & \hdots & c_{d-1} & c_d \\ a_0 & a_1 & \hdots & a_{d-1} & a_d \\ b_0 & b_1 & \hdots & b_{d-1} & * \end{bmatrix} \]

is said to be an intersection array of $\Gamma$. Clearly,

\[ b_0 = k, b_d = c_0 = 0, c_1 = 1. \]

By counting edges $\{\delta, \epsilon\}$ with $d(\gamma, \delta) = i, d(\gamma, \epsilon) = i + 1$, we see that $\Gamma_i(\gamma)$ contains $k_i$ points, satisfying

\[ k_0 = 1, k_1 = k, k_{i+1} = k_i b_i / c_{i+1}, \quad (i = 0, \ldots, d - 1); \]

therefore, the total number of vertices is

\[ \nu = 1 + k_1 + \cdots + k_d. \]

By counting edges $\{\delta, \epsilon\}$ with $d(\gamma, \delta) = 1$, we see that

\[ k = a_i + b_i + c_i. \]

Examples. (i) The polygons; they have intersection array
where $c_d = 2$ for the $2d$-gon and $c_d = 1$ for the $(2d + 1)$-gon.

(ii) The five Platonic solids; they have intersection array

\[
\begin{pmatrix}
  * & 1 & * \\
  0 & 2 & 0 \\
  3 & * & *
\end{pmatrix}
\text{(tetrahedron)},
\begin{pmatrix}
  * & 1 & 4 \\
  0 & 2 & 0 \\
  4 & 1 & *
\end{pmatrix}
\text{(octahedron)},
\begin{pmatrix}
  * & 1 & 2 & 3 \\
  0 & 0 & 0 & 0 \\
  3 & 2 & 1 & *
\end{pmatrix}
\text{(cube)},
\begin{pmatrix}
  * & 1 & 2 & 5 \\
  0 & 2 & 2 & 0 \\
  5 & 2 & 1 & *
\end{pmatrix}
\text{(icosahedron)},
\begin{pmatrix}
  * & 1 & 1 & 1 & 2 & 3 \\
  0 & 0 & 1 & 1 & 0 & 0 \\
  3 & 2 & 1 & 1 & 1 & *
\end{pmatrix}
\text{(dodecahedron)}.
\]

3 Completely regular codes

A code $C$ is a set of a non-empty subset of $\Gamma$. $C \ni x$ is said to be a codeword. The number

$$\delta(C) := \min \{d(x, y) \mid x, y \in C, x \neq y\}$$

is called the minimum distance of $C$. The distance of $x \in \Gamma$ to $C$ is defined as

$$d(x, C) := \min \{d(x, y) \mid y \in C\},$$

and the number

$$t = t_C = \max \{d(x, C) \mid x \in \Gamma\}$$

is called the covering radius of $C$. The subconstituents of $\Gamma$ with respected to $C$ are the sets

$$C_l := \{x \in \Gamma \mid d(x, C) = l\};$$

in particular,
\[ C_0 = C, \quad C_i \neq \phi \iff l \leq t \]

**Definition 3.1** A code \( C \) is said to be a completely regular code (CRC) if the numbers

\[
\begin{align*}
\gamma_i &= |C_{i-1} \cap \Gamma(x)|, \\
\alpha_i &= |C_i \cap \Gamma(x)|, \text{ and} \\
\beta_i &= |C_{i+1} \cap \Gamma(x)|
\end{align*}
\]

are independent of the choices of \( x \in C_i \).

The numbers \( \gamma_i, \alpha_i, \beta_i \) are said to be an *intersection numbers* of \( C \).

\[
\iota(C) = \left\{ \begin{array}{cccc}
* & \gamma_1 & \cdots & \gamma_{i-1} & \gamma_i \\
\alpha_0 & \alpha_1 & \cdots & \alpha_{i-1} & \alpha_i \\
\beta_0 & \beta_1 & \cdots & \beta_{i-1} & * \\
\end{array} \right\}
\]

is said to be an *intersection array* of \( C \). Clearly,

\[ \beta_i = \gamma_0 = 0. \]

By counting edges \( \{\delta, \epsilon\} \) with \( d(\delta, C) = i, \ d(\epsilon, C) = i + 1 \), we see that \( C_i \) contains \( \kappa_i \) points, satisfying

\[ \kappa_{i+1} = \kappa_i \beta_i / \gamma_{i+1} \quad (i = 0, \ldots, t - 1); \]

therefore, the total number of vertices is

\[ \nu = \kappa_0 + \kappa_1 + \cdots + \kappa_t. \]

By counting edges \( \{\delta, \epsilon\} \) with \( d(\delta, C) = i \), we see that

\[ k = \alpha_i + \beta_i + \gamma_i. \]
Theorem 3.1 (Neumaier[8]) For a completely regular code $C$ in a connected regular graph with intersection array

$$\iota(C) = \left\{ \begin{array}{lllll} * & \gamma_1 & \ldots & \gamma_{t-1} & \gamma_t \\ \alpha_0 & \alpha_1 & \ldots & \alpha_{t-1} & \alpha_t \\ \beta_0 & \beta_1 & \ldots & \beta_{t-1} & * \end{array} \right\},$$

the subset $C_t$ is a completely regular code with intersection array

$$\iota(C_t) = \left\{ \begin{array}{lllll} * & \beta_{t-1} & \ldots & \beta_1 & \beta_0 \\ \alpha_t & \alpha_{t-1} & \ldots & \alpha_1 & \alpha_0 \\ \gamma_t & \gamma_{t-1} & \ldots & \gamma_1 & * \end{array} \right\}.$$

Theorem 3.2 (Suzuki[11,Proposition 2.1]) In distance-regular graph, a code $\{x\}$ is a completely regular code.

Proof. By definition of distance-regular graph, it is clear. $\square$

We say $C_t$ is a reversal of $C$. So, we consider the case where $2 \leq |C| \leq \frac{\nu}{2}$.

We say that a completely regular code $C$ with covering radius $t$ in a distance regular graph $\Gamma$ is trivial if $|C| \leq 1$, $|C_t| \leq 1$ or all the vertices of $\Gamma$ are in $C$. 

Theorem 3.3 (Koolen[6,Theorem15]) Each completely regular codes $C$ in $\Gamma$ has $\gamma_i \leq \gamma_{i+1}$ and $\beta_i \geq \beta_{i+1}$, where $\Gamma$ is a member of one of the following families.

(i) The Hamming graphs, $H(n,d)$,
(ii) the Johnson graphs, $J(n,d)$,
(iii) the Grassmann graphs, $G_q(n,d)$,
(iv) the symplectic dual polar graphs on $[C_d(q)]$,
(v) the orthogonal dual polar graphs on $[D_d(q)]$,
(vi) the orthogonal dual polar graphs on $[\mathbb{P} D_{d+1}(q)]$,
(vii) the unitary dual polar graphs on $[\mathbb{P} A_{2d-1}(r)]$,
(viii) the bilinear forms graphs, $H_q(n,d)$,
(ix) the alternating forms graphs,
(x) the Hermitean forms graphs,
(xi) the symmetric bilinear forms graphs,
(xii) the quadratic forms graphs,
(xiii) the folded Johnson graphs, $\overline{J}(2m,m)$,
(xiv) the folded cubes,
(xv) the halved cubes,
(xvi) the Doob graphs, the direct products of Shrikhande graphs and 4-cliques,
(xvii) the half dual polar graphs, $D_{m,m}(q)$,
(xviii) the Ustimenko graphs, which are the distance 1-or-2 graphs of dual polar graphs on $[C_d(q)]$, and
(xix) the Hemmeter graphs, the extended bipartite doubles of the dual polar graphs on $[C_d(q)]$.

4 Completely regular codes in Johnson graph $J(n,d)$

Let $X$ be a finite set of cardinality $n$. The Johnson graph of the $d$-sets in $X$ has vertex set $\binom{X}{d}$, the collection of $d$-sets of $X$. Two vertices $\gamma, \delta$ are adjacent whenever $\gamma \cap \delta$ has cardinality $d - 1$. The Johnson graph $J(n,d)$ has an intersection array given by
\[ b_i = (n - i)(n - d - i), \quad c_i = i^2 \quad \text{for} \quad 0 \leq i \leq d. \]

4.1 Completely regular codes in \( J(n, 2) \)

- A non-trivial CRC in Johnson graph \( J(4, 2) \):

\[
\begin{array}{c}
(1,2) \quad (1,3) \\
(2,3) \quad (2,4) \\
(1,4) \\
(3,4)
\end{array}
\]

\[
\{ \begin{array}{c}
* \quad 1 \quad 4 \\
0 \quad 2 \quad 0 \\
4 \quad 1 \quad * \\
\end{array}
\}
\]

has one of the following intersection arrays.

(i) \( |C| = 2 \)

\[
\begin{array}{c}
(1,2) \quad (1,3) \\
(3,4) \quad (2,3) \\
(1,4) \\
(2,4)
\end{array}
\]

\[
\{ \begin{array}{c}
* \quad 2 \\
0 \quad 2 \\
4 \quad * \\
\end{array}
\}
\]
(ii) $|C| = 3$

\[
\begin{array}{c}
(1,2) \quad (1,4) \quad (2,3) \quad (2,4) \\
(1,3) \quad (2,3) \quad (3,4)
\end{array}
\]

\[
\begin{array}{c}
 \{* & 2 \\
 2 & 2 \\
 2 & *
\end{array}
\]

- A non-trivial CRC in Johnson graph $J(5,2)$:

\[
\begin{array}{c}
(1,2) \quad (1,3) \quad (1,4) \quad (1,5) \quad (2,3) \quad (2,4) \\
(2,5) \quad (3,4) \quad (3,5) \quad (4,5)
\end{array}
\]

\[
\begin{array}{c}
 \{* & 1 & 4 \\
 0 & 3 & 2 \\
 6 & 2 & *
\end{array}
\]

has, up to reversal, one of the following intersection arrays.
(i) $|C| = 4$

(ii) $|C| = 5$

---

(i) $|C| = 4$

(ii) $|C| = 5$
A non-trivial CRC in Johnson graph $J(6,2)$ has up to reversal, one of the following intersection arrays.

(i) $|C| = 3$
\[
\begin{array}{ll}
* & 2 \\
0 & 6 \\
8 & *
\end{array},
\begin{array}{ll}
* & 2 \\
2 & 4 \\
6 & 2 \\
\end{array}
\]

(ii) $|C| = 5$
\[
\begin{array}{ll}
* & 2 \\
4 & 6 \\
4 & *
\end{array}
\]

(iii) $|C| = 6$
\[
\begin{array}{ll}
* & 4 \\
2 & 4 \\
6 & *
\end{array}
\]

**Lemma 4.1** In Johnson graph $J(n,2)$, if $n$ is odd, then there is no non-trivial completely regular codes with $|C| = 2$.

**Proof.** For $n = 3$, there is no non-trivial completely regular codes. So, we may consider the case $n \geq 5$. If $C = \{(i,j),(i,k)\}$, we can take $(i,l),(j,m)$ in $C_1$, where $(i,l)$ is adjacent $(i,j)$ and $(i,k),(j,m)$ is adjacent $(i,j)$, contradiction. So, $C$ isn't a completely regular code. If $C = \{(i,l),(j,m)\}$, we can take $(i,j),(i,n)$ in $C_1$, where $(i,j)$ is adjacent $(i,l)$ and $(j,m),(i,n)$ is adjacent $(i,l)$, contradiction. So, $C$ isn't a completely regular code. Therefore, there is no non-trivial completely regular code $C$ with $|C| = 2$. \(\square\)

**Lemma 4.2** In a Johnson graph $J(n,2)$, let $C = \{(1,2),(2,3),\ldots,(n-1,n),(n,1)\}$. Then $C$ is a completely regular code in $J(n,2)$.

**Proof.** For $n = 2$ or 3, $C$ is a trivial completely regular code. For $n = 4$, let $(i,j)$ be in $J(n,2) \setminus C$. Since $(i-1,i)$ or $(i,i+1)$ is in $C$, $d((i,j),C) = 1$. Therefore $(i,j)$ in $C_1$. Then, the intersection array is
\[
\begin{array}{ll}
* & 4 \\
2 & 2(n-4) \\
2(n-3) & *
\end{array},
\]
so $C$ is a non-trivial completely regular code. \(\square\)
4.2 Completely regular codes in $J(n, 3)$

- A non-trivial CRC in Johnson graph $J(6, 3)$ has one of the following intersection arrays.

(i) $|C| = 2$
$$\begin{array}{l}
\{ * 1 \} \\
\{ 0 8 \} \\
\{ 9 * \}
\end{array}$$

(ii) $|C| = 4$
$$\begin{array}{l}
\{ * 2 \} \\
\{ 1 7 \} \\
\{ 8 * \} \\
\{ * 2 6 \} \\
\{ 3 5 3 \} \\
\{ 6 2 * \}
\end{array}$$

(iii) $|C| = 6$
$$\begin{array}{l}
\{ * 3 \} \\
\{ 2 6 \} \\
\{ 7 * \}
\end{array}$$

(iv) $|C| = 8$
$$\begin{array}{l}
\{ * 4 \} \\
\{ 3 5 \} \\
\{ 6 * \}
\end{array}$$

(v) $|C| = 10$
$$\begin{array}{l}
\{ * 3 \} \\
\{ 6 6 \} \\
\{ 3 * \} \\
\{ * 5 \} \\
\{ 4 4 \} \\
\{ 3 3 \} \\
\{ * 6 \} \\
\{ 5 * \} \\
\{ 6 * \}
\end{array}$$

References


