Completely Regular Codes in Johnson graph

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1 Introduction

In this article, we study completely regular codes in some distance regular graphs. Completely regular codes were first studied by Biggs [2] and P. Delsarte [4], but there are not yet many articles on this subject. Recently, research of completely regular codes has been developing with research of Terwilliger algebra.

We consider completely regular codes in a distance regular graph. Martin [7] conjectured that for completely regular codes in a distance regular graph, \( \gamma_i \leq \gamma_{i+1}, \beta_i \geq \beta_{i+1} \) hold. On the other hand, Koolen [6] showed \( \gamma_i < \gamma_{i+1}, \beta_i > \beta_{i+1} \) in \( H(D, 2) \). Furthermore, I conjectured \( \gamma_i < \gamma_{i+1}, \beta_i > \beta_{i+1} \) in \( J(n, d) \). In order to study this conjecture, in this paper we first classify the completely regular codes in Johnson graph \( J(4, 2), J(5, 2), J(6, 2), J(6, 3) \).

2 Distance regular graphs

A graph is a pair \( \Gamma = (V, E) \) consisting of a set \( V \), referred to as the vertex set of \( \Gamma \), and a set \( E \) of 2-subsets of \( V \), referred to as the edge set of \( \Gamma \). That is, our graphs are undirected, without loops or multiple edges. We write \( \gamma \in \Gamma \) if \( \gamma \) is a vertex of \( \Gamma \) and \( \gamma \sim \delta \) if \( \{\gamma, \delta\} \) is an edge of \( \Gamma \). The distance \( d(\gamma, \delta) \) of two vertices \( \gamma, \delta \in \Gamma \) is the length of a shortest path between \( \gamma \) and \( \delta \). \( d = \max\{d(\gamma, \delta) \mid \gamma, \delta \in \Gamma \} \) is called the diameter of \( \Gamma \). Given \( \delta \in \Gamma \), we write \( \Gamma_i(\delta) \) for the set of vertices \( \gamma \) with \( d(\gamma, \delta) = i \). In particular, \( \Gamma(\delta) = \Gamma_i(\delta) \) denotes the set of neighbours of \( \delta \). The valency \( k(\gamma) \) of a vertex \( \gamma \) is the cardinality of \( \Gamma(\gamma) \). A graph is called regular if each vertex has the same valency \( k \).
Definition 2.1 A connected graph $\Gamma = (V, E)$ is said to be a distance regular graph (DRG) if the numbers

\[ c_i = |\Gamma_{i-1}(\gamma) \cap \Gamma(\delta)|, \]
\[ a_i = |\Gamma_i(\gamma) \cap \Gamma(\delta)|, \text{ and} \]
\[ b_i = |\Gamma_{i+1}(\gamma) \cap \Gamma(\delta)| \]

are independent of the choices of $\gamma, \delta \in \Gamma$ with $d(\gamma, \delta) = i$.

The numbers $c_i, a_i$ and $b_i$ are said to be the intersection numbers of $\Gamma$.

\[ i(\Gamma) = \begin{pmatrix} * & c_1 & \cdots & c_{d-1} & c_d \\
 a_0 & a_1 & \cdots & a_{d-1} & a_d \\
b_0 & b_1 & \cdots & b_{d-1} & * \end{pmatrix} \]

is said to be an intersection array of $\Gamma$. Clearly,

\[ b_0 = k, b_d = c_0 = 0, c_1 = 1. \]

By counting edges $\{\delta, \epsilon\}$ with $d(\gamma, \delta) = i, d(\gamma, \epsilon) = i + 1$, we see that $\Gamma_i(\gamma)$ contains $k_i$ points, satisfying

\[ k_0 = 1, k_1 = k, k_{i+1} = k_i b_i / c_{i+1} \quad (i = 0, \ldots, d - 1); \]

therefore, the total number of vertices is

\[ \nu = 1 + k_1 + \cdots + k_d. \]

By counting edges $\{\delta, \epsilon\}$ with $d(\gamma, \delta) = 1$, we see that

\[ k = a_i + b_i + c_i. \]

Examples. (i) The polygons; they have intersection array
where $c_d = 2$ for the $2d$-gon and $c_d = 1$ for the $(2d + 1)$-gon.

(ii) The five Platonic solids; they have intersection array

\[
\{\begin{array}{l}
* 1 4 \\
0 2 0 \\
4 1 *
\end{array}\} \text{ (octahedron)},
\{\begin{array}{l}
* 1 2 3 \\
0 0 0 0 \\
3 2 1 *
\end{array}\} \text{ (cube)},
\{\begin{array}{l}
* 1 2 5 \\
0 2 2 0 \\
5 2 1 *
\end{array}\} \text{ (icosahedron)},
\{\begin{array}{l}
* 1 1 2 3 \\
0 0 1 1 0 0 \\
3 2 1 1 1 *
\end{array}\} \text{ (dodecahedron)}.
\]

3 Completely regular codes

A code $C$ is a set of a non-empty subset of $\Gamma$. $C \ni x$ is said to be a codeword. The number

\[
\delta(C) := \min\{d(x, y) \mid x, y \in C, x \neq y\}
\]

is called the minimum distance of $C$. The distance of $x \in \Gamma$ to $C$ is defined as

\[
d(x, C) := \min\{d(x, y) \mid y \in C\},
\]

and the number

\[
t = t_C = \max\{d(x, C) \mid x \in \Gamma\}
\]

is called the covering radius of $C$. The subconstituents of $\Gamma$ with respected to $C$ are the sets

\[
C_l := \{x \in \Gamma \mid d(x, C) = l\};
\]

in particular,
$C_0 = C, \quad C_i \neq \phi \Leftrightarrow l \leq t$

**Definition 3.1** A code $C$ is said to be a completely regular code (CRC) if the numbers

\[
\gamma_i = |C_{i-1} \cap \Gamma(x)|,
\alpha_i = |C_i \cap \Gamma(x)|, \text{ and }
\beta_i = |C_{i+1} \cap \Gamma(x)|
\]

are independent of the choices of $x \in C_i$.

The numbers $\gamma_i, \alpha_i, \beta_i$ are said to be an **intersection numbers** of $C$.

\[
\iota(C) = \begin{bmatrix}
* & \gamma_1 & \cdots & \gamma_{t-1} & \gamma_t \\
\alpha_0 & \alpha_1 & \cdots & \alpha_{t-1} & \alpha_t \\
\beta_0 & \beta_1 & \cdots & \beta_{t-1} & *
\end{bmatrix}
\]

is said to be an **intersection array** of $C$. Clearly,

$\beta_t = \gamma_0 = 0$.

By counting edges $\{\delta, \epsilon\}$ with $d(\delta, C) = i, d(\epsilon, C) = i + 1$, we see that $C_i$ contains $\kappa_i$ points, satisfying

$$
\kappa_{i+1} = \kappa_i \beta_i / \gamma_{i+1} \quad (i = 0, \ldots, t - 1);
$$

therefore, the total number of vertices is

$$
\nu = \kappa_0 + \kappa_1 + \cdots + \kappa_t.
$$

By counting edges $\{\delta, \epsilon\}$ with $d(\delta, C) = i$, we see that

$$
k = \alpha_i + \beta_i + \gamma_i.
$$
Theorem 3.1 (Neumaier[8]) For a completely regular code $C$ in a connected regular graph with intersection array

$$\iota(C) = \begin{pmatrix} * & \gamma_1 & \cdots & \gamma_{t-1} & \gamma_t \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{t-1} & \alpha_t \\ \beta_0 & \beta_1 & \cdots & \beta_{t-1} & * \end{pmatrix},$$

the subset $C_t$ is a completely regular code with intersection array

$$\iota(C_t) = \begin{pmatrix} * & \beta_{t-1} & \cdots & \beta_1 & \beta_0 \\ \alpha_t & \alpha_{t-1} & \cdots & \alpha_1 & \alpha_0 \\ \gamma_t & \gamma_{t-1} & \cdots & \gamma_1 & * \end{pmatrix}.$$

Theorem 3.2 (Suzuki[11,Proposition 2.1]) In distance-regular graph, a code \{x\} is a completely regular code.

Proof. By definition of distance-regular graph, it is clear. \hfill \Box

We say $C_t$ is a reversal of $C$. So, we consider the case where $2 \leq |C| \leq \frac{\nu}{2}$.

We say that a completely regular code $C$ with covering radius $t$ in a distance regular graph $\Gamma$ is trivial if $|C| \leq 1$, $|C_t| \leq 1$ or all the vertices of $\Gamma$ are in $C$. 
Theorem 3.3 (Koolen[6,Theorem15]) Each completely regular codes \( C \) in \( \Gamma \) has \( \gamma_i \leq \gamma_{i+1} \) and \( \beta_i \geq \beta_{i+1} \), where \( \Gamma \) is a member of one of the following families.

(i) The Hamming graphs, \( H(n,d) \),
(ii) the Johnson graphs, \( J(n,d) \),
(iii) the Grassmann graphs, \( G_q(n,d) \),
(iv) the symplectic dual polar graphs on \([C_d(q)]\),
(v) the orthogonal dual polar graphs on \([B_d(q)]\),
(vi) the orthogonal dual polar graphs on \([D_d(q)]\),
(vii) the orthogonal dual polar graphs on \([P D_{d+1}(q)]\),
(viii) the unitary dual polar graphs on \([P A_{2d}(r)]\),
(ix) the unitary dual polar graphs on \([P A_{2d-1}(r)]\),
(x) the bilinear forms graphs, \( H_q(n,d) \),
(xi) the alternating forms graphs,
(xii) the Hermitean forms graphs,
(xiii) the symmetric bilinear forms graphs,
(xiv) the quadratic forms graphs,
(xv) the folded Johnson graphs, \( \overline{J}(2m,m) \),
(xvi) the folded cubes,
(xvii) the halved cubes,
(xviii) the Doob graphs, the direct products of Shrikhande graphs and 4-cliques,
(xix) the half dual polar graphs, \( D_{m,m}(q) \),
(xx) the Ustimenko graphs, which are the distance 1-or-2 graphs of dual polar graphs on \([C_d(q)]\), and
(xxi) the Hemmeter graphs, the extended bipartite doubles of the dual polar graphs on \([C_d(q)]\).

4 Completely regular codes in Johnson graph \( J(n,d) \)

Let \( X \) be a finite set of cardinality \( n \). The Johnson graph of the \( d \)-sets in \( X \) has vertex set \( \binom{X}{d} \), the collection of \( d \)-sets of \( X \). Two vertices \( \gamma, \delta \) are adjacent whenever \( \gamma \cap \delta \) has cardinality \( d - 1 \). The Johnson graph \( J(n,d) \) has an intersection array given by
\[ b_i = (n - i)(n - d - i), \quad c_i = i^2 \quad \text{for} \quad 0 \leq i \leq d. \]

4.1 Completely regular codes in \( J(n, 2) \)

- A non-trivial CRC in Johnson graph \( J(4, 2) \):

\[
\begin{array}{c}
(1,2) \quad (1,3) \quad (1,4) \\
(2,3) \quad (2,4) \\
(3,4) \\
\end{array}
\]

\[
\begin{array}{c}
\{* \ 1 \ 4\} \\
\{0 \ 2 \ 0\} \\
\{4 \ 1 \ *\} \\
\end{array}
\]

has one of the following intersection arrays.

(i) \( |C| = 2 \)

\[
\begin{array}{c}
(1,2) \quad (1,3) \quad (1,4) \\
(3,4) \\
(2,3) \quad (2,4) \\
\end{array}
\]

\[
\begin{array}{c}
\{* \ 2\} \\
\{0 \ 2\} \\
\{4 \ *\} \\
\end{array}
\]
(ii) $|C| = 3$

- A non-trivial CRC in Johnson graph $J(5, 2)$:

    has, up to reversal, one of the following intersection arrays.
(i) $|C| = 4$

![Diagram for $|C| = 4$]

\[
\{ \begin{array}{l}
2 \end{array} \} \quad \{ 3, 4 \}
\]

(ii) $|C| = 5$

![Diagram for $|C| = 5$]

\[
\{ \begin{array}{l}
4 \end{array} \} \quad \{ 2, 2 \}
\]
• A non-trivial CRC in Johnson graph $J(6,2)$ has up to reversal, one of the following intersection arrays.

(i) $|C| = 3$
\[
\begin{array}{c}
0 & 6 \\
8 & * \\
\end{array}
\]
\[
\begin{array}{c}
* & 2 \\
2 & 4 \\
6 & 2 \\
\end{array}
\]

(ii) $|C| = 5$
\[
\begin{array}{c}
4 & 6 \\
4 & * \\
\end{array}
\]

(iii) $|C| = 6$
\[
\begin{array}{c}
* & 4 \\
2 & 4 \\
6 & * \\
\end{array}
\]

\textbf{Lemma 4.1} In Johnson graph $J(n,2)$, if $n$ is odd, then there is no non-trivial completely regular codes with $|C| = 2$.

\textbf{Proof.} For $n = 3$, there is no non-trivial completely regular codes. So, we may consider the case $n \geq 5$. If $C = \{(i,j), (i,k)\}$, we can take $(i,l), (j,m)$ in $C_1$, where $(i,l)$ is adjacent $(i,j)$ and $(i,k), (j,m)$ is adjacent $(i,j)$, contradiction. So, $C$ isn’t a completely regular code. If $C = \{(i,l), (j,m)\}$, we can take $(i,j), (i,n)$ in $C_1$, where $(i,j)$ is adjacent $(i,l)$ and $(j,m), (i,n)$ is adjacent $(i,l)$, contradiction. So, $C$ isn’t a completely regular code. Therefore, there is no non-trivial completely regular code $C$ with $|C| = 2$. \qed

\textbf{Lemma 4.2} In a Johnson graph $J(n,2)$, let $C = \{(1,2), (2,3), \ldots, (n-1,n), (n,1)\}$. Then $C$ is a completely regular code in $J(n,2)$.

\textbf{Proof.} For $n = 2$ or 3, $C$ is a trivial completely regular code. For $n = 4$, let $(i,j)$ be in $J(n,2) \setminus C$. Since $(i-1,i)$ or $(i,i+1)$ is in $C$, $d((i,j), C) = 1$. Therefore $(i,j)$ in $C_1$. Then, the intersection array is
\[
\begin{array}{c}
* & 4 \\
2 & 2(n-4) \\
2(n-3) & * \\
\end{array}
\]
so $C$ is a non-trivial completely regular code. \qed
4.2 Completely regular codes in $J(n, 3)$

- A non-trivial CRC in Johnson graph $J(6, 3)$ has one of the following intersection arrays.

(i) $|C| = 2$
\[
\begin{array}{c}
0 & 8 \\
9 & *
\end{array}
\]

(ii) $|C| = 4$
\[
\begin{array}{c}
* 2 \\
1 & 7 \\
8 & *
\end{array}
\]
$\begin{array}{c}
* 2 6 \\
3 & 5 & 3 \\
6 & 2 & *
\end{array}$

(iii) $|C| = 6$
\[
\begin{array}{c}
* 3 \\
2 & 6 \\
7 & *
\end{array}
\]

(iv) $|C| = 8$
\[
\begin{array}{c}
* 4 \\
3 & 5 \\
6 & *
\end{array}
\]

(v) $|C| = 10$
\[
\begin{array}{c}
* 3 \\
6 & 6 \\
3 & *
\end{array}
\]
$\begin{array}{c}
* 5 \\
4 & 4 \\
5 & *
\end{array}$
$\begin{array}{c}
* 6 \\
3 & 3 \\
6 & *
\end{array}$

References


