A universal bound for a covering in regular posets
and its application to pool testing

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Let $V(n)$ be the set of all $2^n$ subsets of the set $N_n = \{1, 2, \ldots, n\}$ and $V_i(n) = \{x \in V(n) : |x| = i\}$. For a fixed $i = 1, \ldots, n$, consider a covering operator $F : V_i(n) \rightarrow V(n)$ such that $x \subseteq F(x)$ for any $x \in V_i(n)$. Let $C = \{F(x) : x \in V_i(n)\}$. For any $1 \leq T \leq \binom{n}{i}$, consider the decreasing continuous function $g_i(T) = k + \frac{k+1}{i}(1 - \alpha)$ where $k$ and $\alpha$ are uniquely defined by the conditions $T \binom{k}{i} = \alpha \binom{n}{i}$, $k \in \{i, \ldots, n\}$, and $1 - \frac{1}{k+1} < \alpha \leq 1$.

Using averaging and linear programing it is proved that

$$\frac{1}{\binom{n}{i}} \sum_{x \in V_i(n)} |F(x)| \geq g_i(|C|) \geq \frac{n}{\sqrt{|C|}}$$

with the first inequality as an equality if and only if $C$ is a Steiner $S(i, \{k, k+1\}, n)$ design with a specified distance distribution. A generalization of this result to the case of monotone left-regular $n$-posets is given. One of motivating applications is the problem of reconstructing an unknown binary vector $x$ of length $n$ using pool testing under the condition that the vectors $x$ are

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distributed with probabilities $p^{|x|}(1 - p)^{n-|x|}$ where $x \in V(n)$ denotes the indices of the ones (active items) in $x$. The bound above implies that the expected number of items which remain unresolved after application in parallel of arbitrary $r$ pools is not less than

$$n \sum_{i=1}^{n} \binom{n}{i} p^i (1 - p)^{n-i} 2^{-r} - np.$$ 

This improves upon an information theoretic bound for the minimum average number $E(n, p)$ of tests to reconstruct an unknown $x$ using two-stage pool testing and allows determination of the asymptotic behavior of $E(n, p)$ up to a positive constant factor as $n \to \infty$ under some restrictions upon $p = p(n)$. 