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</thead>
<tbody>
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Kyoto University
Some inequalities for distance-regular graphs

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1 Definitions

All graphs considered are undirected finite graphs without loops or multiple edges. Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph with usual shortest path distance $\partial_{\Gamma}$. The diameter of $\Gamma$, denoted by $d$, is the maximal distance of two vertices in $\Gamma$. Let $u \in \Gamma$. We denote by $\Gamma_{j}(u)$ the set of vertices which are at distance $j$ from $u$. For $x, y \in V\Gamma$ with $\partial_{\Gamma}(x, y) = i$, let

\[ C_{i}(x, y) = \Gamma_{i-1}(x) \cap \Gamma_{1}(y), \]
\[ A_{i}(x, y) = \Gamma_{i}(x) \cap \Gamma_{1}(y), \]
\[ B_{i}(x, y) = \Gamma_{i+1}(x) \cap \Gamma_{1}(y). \]

A connected graph $\Gamma$ is said to be distance-regular if

\[ c_{i} = |C_{i}(x, y)|, \quad a_{i} = |A_{i}(x, y)| \quad \text{and} \quad b_{i} = |B_{i}(x, y)| \]

depend only on $i = \partial_{\Gamma}(x, y)$ rather than individual vertices.
The numbers $c_i, a_i$ and $b_i$ are called the intersection numbers of $\Gamma$. In particular, $k := b_0$ is called the valency of $\Gamma$. The array

$$i(\Gamma) = \begin{bmatrix} * & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & * \end{bmatrix}$$

is called the intersection array of $\Gamma$. Define

$$r = r(\Gamma) := \max \{ i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1) \}.$$

Let $\Gamma = (V, E)$ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Define $R_i := \{(x, y) \mid \partial_r(x, y) = i\}$. Then $(V, \{R_i\}_{0 \leq i \leq d})$ is an association scheme of class $d$ and

$$p_{i,j}^t := \# \{ z \mid \partial_r(x, z) = i, \partial_r(y, z) = j \},$$

where $\partial_r(x, y) = t$.

Conversely, for an association scheme $(X, \{R_i\}_{0 \leq i \leq d})$ satisfying $P$-polynomial condition, the graph $(X, R_1)$ is a distance-regular graph. (See [1] and [3].)

## 2 The Odd graph and the doubled Odd graph

Let $m$ be a positive integer and $X$ a set of $2m + 1$ elements. For $0 \leq t \leq m$ define

$$X_t := \{ Y \subseteq X \mid \#Y = t \}.$$

The Odd graph $O_{m+1}$ is a graph with

$$V \Gamma = X_m, \quad E \Gamma = \{ (Y, Y') \mid Y \cap Y' = \emptyset \}.$$

The doubled Odd graph $2O_{m+1}$ is a graph with

$$V \Gamma = X_m \cup X_{m+1}, \quad E \Gamma = \{ (Y, Z) \mid Y \in X_m, Z \in X_{m+1}, Y \subset Z \}.$$
Then Odd graph $O_{m+1}$ is a distance-regular graph of diameter $m$ with:

**Case**: $m = 2t$,

\[
\begin{array}{ccccccccccc}
* & 1 & 1 & 2 & 2 & \cdots & t & t & \{ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & t+1 \\
2t+1 & 2t & 2t & 2t-1 & 2t-1 & \cdots & t+1 & * \\
\end{array} \}
\end{array}
\]

**Case**: $m = 2t - 1$,

\[
\begin{array}{ccccccccccc}
* & 1 & 1 & 2 & 2 & \cdots & t-1 & t & \{ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & t \\
2t & 2t-1 & 2t-1 & 2t-2 & 2t-2 & \cdots & t+1 & * \\
\end{array} \}
\end{array}
\]

The doubled Odd graph $2O_{m+1}$ is distance-regular of diameter $2m + 1$ with:

\[
\begin{array}{ccccccccccc}
* & 1 & 1 & 2 & 2 & \cdots & m-1 & m & m+1 & \{ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
m+1 & m & m-1 & m-1 & \cdots & 2 & 1 & 1 & * \\
\end{array} \}
\end{array}
\]

More information for these graphs can be found in [3, §9.1.D].

These graphs had been characterized by some of their intersection numbers. In particular, Ray-Chaudhuri and Sprague [8], Cuypers [4, Theorems 4.6-4.7] and Koolen [7, Theorem 16] proved the following result.

**Theorem 1** Let $\Gamma$ be a distance-regular graph of diameter $d \geq 5$ such that $c_1 = c_2 = 1$, $c_3 = c_4 = 2$ and $a_1 = \cdots = a_4 = 0$. Then $\Gamma$ is either the Odd graph or the doubled Odd graph.
3 Some Inequalities for distance-regular graphs

The following are well known basic properties of the intersection numbers.

(1) \( 1 = c_1 \leq c_2 \leq \cdots \leq c_{d-1} \leq c_d \leq k \).

(2) \( k = b_0 \geq b_1 \geq \cdots \geq b_{d-2} \geq b_{d-1} \geq 1 \).

(3) \( c_i \leq b_j \) if \( i + j \leq d \).

We recall the proof of these properties:

Let \( (x_0, x_1, \ldots, x_d) \) be a path of length \( d \) such that \( \partial_T(x_0, x_d) = d \). Then we have

\[
\begin{align*}
C_1(x_1, x_0) \subseteq C_2(x_2, x_0) \subseteq \cdots \subseteq C_{d-1}(x_{d-1}, x_0) \subseteq C_d(x_d, x_0) \subseteq \Gamma_1(x_0),
B_0(x_0, x_0) \supseteq B_1(x_1, x_0) \supseteq \cdots \supseteq B_{d-2}(x_{d-2}, x_0) \supseteq B_{d-1}(x_{d-1}, x_0) \supseteq \{x_1\}
\end{align*}
\]

and

\[
C_i(x_0, x_i) \subseteq B_j(x_{i+j}, x_i).
\]

The idea of this proof is: "Let \( X \subseteq Y \) be subsets of \( VT \). Then \( \#X \leq \#Y \)."

We prove several inequalities for intersection numbers by applying this idea. For example:

**Theorem 2** Suppose \( 1 \leq c_{t-1} < c_t \). Then \( c_{2t-1} \neq c_t \).

**Sketch of the Proof.**
Suppose \( c_{2t-1} = c_t \) and derive a contradiction. Let \( u, x, y \in VT \) such that

\[
\partial_T(u, y) = 2t - 1, \quad \partial_T(u, x) = t - 1 \quad \text{and} \quad \partial_T(x, y) = t.
\]

Then there exist \( u \in C_{2t-1}(y, u) \setminus C_{t-1}(x, u) \) and \( u' \in \Gamma_1(v) \cap \Gamma_{t-1}(x) \) such that

\[
B_{t-1}(u', x) \subseteq B_{t-1}(u, x) \setminus C_{t-1}(y, x).
\]

Then this implies \( b_{t-1} \leq b_{t-1} - c_{t-1} \) which is a contradiction.
Theorem 3 Let \( \Gamma \) be a distance-regular graph with:
\[
\begin{array}{cccccccc}
1 & 1 & \cdots & 1 & 2 & \cdots & 2 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
k & k & \cdots & k & k & \cdots & k \\
r & & & & s & & \\
\end{array}
\]

Then \( s \leq \frac{r+4}{3} \).

Sketch of the Proof.
Suppose \( s > \frac{r+4}{3} \) and derive a contradiction. Let \( u, v, x \in V \Gamma \) such that
\[
\begin{align*}
\partial(r, u, v) &= r + 1, \\
\partial(r, u, x) &= 1, \\
\partial(r, x, v) &= r.
\end{align*}
\]
Then we can take a path \((u = y_0, y_1, y_2, \ldots, y_{r+1} = y)\) of length \( r + 1 \) such that
\[
C_{r+1}(v, y) \subseteq C_{r+1}(u, y) \setminus C_{r}(x, v).
\]
This is a contradiction.

We have the following result by Theorem 1 and 3.

Corollary 4 Let \( \Gamma \) be a distance-regular graph as in theorem. If \( s = r \geq 2 \), then \( r = 2 \) and \( \Gamma \) is either the Odd graph or the doubled Odd graph.

Theorem 5 Let \( \Gamma \) be a distance-regular graph with
\[
\begin{array}{cccccccc}
* & 1 & \cdots & 1 & c & \cdots & c \\
0 & a_1 & \cdots & a_1 & a & \cdots & a \\
k & b_1 & \cdots & b_1 & b & \cdots & b \\
r & & & & s & & \\
\end{array}
\]

Suppose \( s = r \geq 2 \). Then \( a = a_1 = 0 \) and \( r = 2 \). In particular, \( \Gamma \) is either the Odd graph, the doubled Odd graph or the doubled Grassmann graph.
Proposition 6  Let $\Gamma$ be a distance-regular graph of diameter $d \geq 2$ and $k \geq 3$. Let $q, h$ be positive integers with $q + h \leq d$.

1. If $c_q < c_{q+1}$ and $a_q = 0$, then $c_h \leq c_{q+h} - c_m$ and $b_{q+h} \leq b_h - c_q$.
2. If $b_{q+h} < b_{q+h-1}$ and $a_{q+h} = 0$, then $c_h \leq b_q - b_{q+h}$.

Sketch of the Proof of (1).
Let $u, x, y \in V\Gamma$ such that
\[ \partial_{\Gamma}(u, x) = h, \quad \partial_{\Gamma}(x, y) = q \quad \text{and} \quad \partial_{\Gamma}(u, y) = q + h. \]
Define
\[ W := \bigcup_{z \in C_h(u, x)} \{ C_{q+1}(z, y) \setminus C_q(x, y) \}. \]
Then
\[ \#C_h(u, x) \leq \#W \quad \text{and} \quad W \subseteq C_{q+h}(u, y) \setminus C_q(x, y). \]
Hence $c_h \leq c_{q+h} - c_q$.

4  A characterization of $O_k$ and $2O_k$

Theorem 7  Let $\Gamma$ be a distance-regular graph of diameter $d \geq 5$ valency $k \geq 3$ and $r = \max\{i \mid (c_i, b_i) = (c_1, b_1)\} \geq 2$. Suppose one of the following conditions holds. Then $\Gamma$ is either the Odd graph $O_k$, or the doubled Odd graph $2O_k$.

(i) $a_{m+r} = 0$ and $1 + c_m = c_{m+r} \leq k - 2$ hold for some $m$ with $r \leq m \leq d - r - 1$.
(ii) $a_m = 0$ and $2 \leq b_{m+r} = b_m - 1$ hold for some $m$ with $r \leq m \leq d - r - 1$.

Sketch of the Proof of (i).
We have $a_{m+r} = \cdots = a_1 = 0$ and $c_{r+1} > c_r$. Since $c_m = c_{m+r} - c_r$, the equality holds in Proposition 6. Then we obtain that $c_{m-1} = c_{m+r-1} - c_r$ holds.
Inductively, we have $c_j = c_{j+r} - c_r$ for all $j < m$. This implies that
\[ c_{r+1} = \cdots = c_2r = 2. \]
The desired result is proved by Corollary 3.

The reader is referred to [2, 5, 6] for more detailed proofs of the results.
References


