Strong Convergence Theorem
by the Hybrid and Extragradient Method
for Nonexpansive Mappings
and Monotone Mappings

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Abstract

In this paper we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz-continuous mapping. The iterative process is based on two well-known methods - hybrid and extragradient. We obtain a strong convergence theorem for three sequences generated by this process.

1 Introduction

Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $P_C$ be the metric projection of $H$ onto $C$. A mapping $A$ of $C$ into $H$ is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0$$

for all $u, v \in C$. The variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. A mapping $A$ of $C$ into $H$ is called $\alpha$-inverse-strongly-monotone if there exists a positive real number $\alpha$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$; see [1], [4]. It is obvious that an $\alpha$-inverse-strongly-monotone mapping $A$ is monotone and Lipschitz-continuous. A mapping $S$ of $C$ into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|$$

for all $u, v \in C$; see [8]. We denote by $F(S)$ the set of fixed points of $S$. For finding an element of $VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex and a mapping $A$ of $C$ into $H$ is $\alpha$-inverse-strongly-monotone, Iiduka, Takahashi and Toyoda [2] introduced the following iterative scheme by a hybrid method:

$$\begin{align*}
x_0 &= x \in C \\
y_n &= P_C(x_n - \lambda_n Ax_n) \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}$$
for every $n = 0, 1, 2, ..., \text{ where } \lambda_n \subset [a, b] \text{ for some } a, b \in (0, 2\alpha)$. They showed that if $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$, generated by this iterative process, converges strongly to $P_{VI(C, A)}x$. On the other hand, for solving the variational inequality problem in a finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex and a mapping $A$ of $C$ into $\mathbb{R}^n$ is monotone and $k$-Lipschitz-continuous, Korpelevich [3] introduced the following so-called extragradient method:

\begin{equation}
\begin{cases}
x_0 = x \in \mathbb{R}^n \\
x_n = P_C(x_n - \lambda Ax_n) \\
x_{n+1} = P_C(x_n - \lambda Ax_n)
\end{cases}
\end{equation}

for every $n = 0, 1, 2, ..., \text{ where } \lambda \in (0, 1/k)$. He showed that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{x_n\}$, generated by (1), converge to the same point $z \in VI(C, A)$.

In this paper, by an idea of combining hybrid and extragradient methods, we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping in a real Hilbert space. Then we obtain a strong convergence theorem for three sequences generated by this process.

2 Preliminaries

Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ and let $C$ be a closed convex subset of $H$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to $x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to $x$. For every point $x \in H$ there exists a unique nearest point in $C$, denoted by $P_Cx$, such that $\|x - P_Cx\| \leq \|x - y\|$ for all $y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is a nonexpansive mapping of $H$ onto $C$. It is also known that $P_C$ is characterized by the following properties: $P_Cx \in C$ and

\begin{equation}
(x - P_Cx, P_Cx - y) \geq 0;
\end{equation}

\begin{equation}
\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2
\end{equation}

for all $x \in H$, $y \in C$; see [8] for more details. Let $A$ be a monotone mapping of $C$ into $H$. In the context of variational inequality problem this implies

$$u \in VI(C, A) \iff u = P_C(u - \lambda Au), \quad \forall \lambda > 0.$$ 

It is also known that $H$ satisfies Opial’s condition [6], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $(x - y, f - g) \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x, f) \in H \times H$, $(x - y, f - g) \geq 0 \forall (y, g) \in G(T)$ implies $f \in Tx$. Let $A$ be a monotone, $k$-Lipschitz-continuous mapping of $C$ into $H$ and $N_Cv$ be the normal cone to $C$ at $v \in C$, i.e. $N_Cv = \{w \in H : (v - u, w) \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} 
Au + N_Cv, & \text{if } v \in C, \\
\emptyset, & \text{if } v \notin C.
\end{cases}$$

Then $T$ is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [7].
3 Strong Convergence Theorem

In this section we prove a strong convergence theorem by a combined hybrid-extragradient method for nonexpansive mappings and monotone, k-Lipschitz-continuous mappings.

Theorem 3.1 Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ and $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$
\begin{align*}
\begin{cases}
\quad x_0 = x \in C \\
\quad y_n = P_C(x_n - \lambda_n Ax_n) \\
\quad z_n = SP_C(x_n - \lambda_n Ay_n) \\
\quad C_n = \{z \in C : ||z - x|| \leq ||x_n - z||\} \\
\quad Q_n = \{z \in C : (x_n - z, x - x_n) \geq 0\} \\
\quad x_{n+1} = P_{C_n \cap Q_n}x
\end{cases}
\end{align*}
$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $F_{F(S) \cap VI(C,A)}x$.

Proof. It is obvious that $C_n$ is closed and $Q_n$ is closed and convex for every $n = 0, 1, 2, \ldots$. As $C_n = \{z \in C : ||z - x_n||^2 + 2(x_n - x_n, z - x) \leq 0\}$, we also have $C_n$ is convex for every $n = 0, 1, 2, \ldots$. Put $t_n = P_C(x_n - \lambda_n Ay_n)$ for every $n = 0, 1, 2, \ldots$. Let $u \in F(S) \cap VI(C,A)$. From (3), monotonicity of $A$ and $u \in VI(C,A)$, we have

$$
||t_n - u||^2 \leq ||x_n - \lambda_n Ay_n - u||^2 - ||x_n - \lambda_n Ay_n - t_n||^2 \\
= ||x_n - u||^2 - ||x_n - t_n||^2 + 2\lambda_n (Ay_n, u - t_n) \\
= ||x_n - u||^2 - ||x_n - t_n||^2 + 2\lambda_n ((Ay_n, u - y_n) + (Ay_n, u - y_n)) \\
\leq ||x_n - u||^2 - ||x_n - t_n||^2 + 2\lambda_n (Ay_n, y_n - t_n) \\
= ||x_n - u||^2 - ||x_n - y_n||^2 - 2(x_n - y_n, y_n - t_n) - ||y_n - t_n||^2 + 2\lambda_n (Ay_n, y_n - t_n) \\
= ||x_n - u||^2 - ||x_n - y_n||^2 - ||y_n - t_n||^2 + 2(x_n - \lambda_n Ay_n - y_n, t_n - y_n).
$$

Further, since $y_n = P_C(x_n - \lambda_n Ax_n)$ and $A$ is $k$-Lipschitz-continuous, we have

$$
\begin{align*}
\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n \rangle \\
= \langle x_n - \lambda_n Ax_n - y_n, t_n - y_n \rangle + \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n \rangle \\
\leq \lambda_n k ||x_n - y_n|| ||t_n - y_n|| \\
\leq \lambda_n k ||x_n - y_n|| ||t_n - y_n||.
\end{align*}
$$

So, we have

$$
\begin{align*}
||t_n - u||^2 &\leq ||x_n - u||^2 - ||x_n - y_n||^2 - ||y_n - t_n||^2 + 2\lambda_n k ||x_n - y_n|| ||t_n - y_n|| \\
&\leq ||x_n - u||^2 - ||x_n - y_n||^2 - ||y_n - t_n||^2 + \lambda_n^2 k^2 ||x_n - y_n||^2 + ||y_n - t_n||^2 \\
&\leq ||x_n - u||^2 + (\lambda_n^2 k^2 - 1) ||x_n - y_n||^2 \tag{4}
\end{align*}
$$

Therefore from $x_n = St_n$ and $u = Su$, we have

$$
||x_n - u|| = ||St_n - Su|| \leq ||t_n - u|| \leq ||x_n - u|| \tag{5}
$$

for every $n = 0, 1, 2, \ldots$ and hence $u \in C_n$. So, $F(S) \cap VI(C,A) \subset C_n$ for every $n = 0, 1, 2, \ldots$. Next, let us show by mathematical induction that $\{x_n\}$ is well-defined and $F(S) \cap VI(C,A) \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \ldots$. For $n = 0$ we have $Q_0 = C$. Hence we obtain $F(S) \cap VI(C,A) \subset C_0 \cap Q_0$. Suppose that $x_k$ is given and $F(S) \cap VI(C,A) \subset C_k \cap Q_k$ for some $k \in N$. Since $F(S) \cap VI(C,A)$ is nonempty,
$C_k \cap Q_k$ is a nonempty closed convex subset of $C$. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for every $z \in C_k \cap Q_k$. Since $F(S) \cap VI(C, A) \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for $z \in F(S) \cap VI(C, A)$ and hence $F(S) \cap VI(C, A) \subset C_{k+1}$. Therefore, we obtain $F(S) \cap VI(C, A) \subset C_{k+1} \cap Q_{k+1}$. Let $t_0 = P_{F(S) \cap VI(C, A)} x$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and $t_0 \in F(S) \cap VI(C, A) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq ||t_0 - x|| \quad (6)$$

for every $n = 0, 1, 2, \ldots$. Therefore, \{x_n\} is bounded. We also have

$$\|x_n - u\| = \|St_n - Su\| \leq ||t_n - u|| \leq ||x_n - u||$$

for some $u \in F(S) \cap VI(C, A)$. So, \{x_n\} and \{t_n\} are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n} x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for every $n = 0, 1, 2, \ldots$. Therefore, there exists $c = \lim_{n \to \infty} \|x_n - x\|$. Since $x_n = P_{Q_n} x$ and $x_{n+1} \in Q_n$, we have

$$\|x_{n+1} - x_n\|^2 = \|x_{n+1} - x\|^2 + \|x_n - x\|^2 + 2 \langle x_{n+1} - x, x - x_n \rangle$$

$$= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 \langle x_n - x_{n+1}, x - x_n \rangle$$

$$\leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 0, 1, 2, \ldots$. This implies that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have $\|x_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$ and hence

$$\|x_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2 \|x_{n+1} - x_n\|$$

for every $n = 0, 1, 2, \ldots$. From $\|x_{n+1} - x_n\| \to 0$, we have $\|x_n - x_n\| \to 0$.

For $u \in F(S) \cap VI(C, A)$, from (4) and (5) we obtain

$$\|z_n - u\|^2 \leq \|t_n - u\|^2 \leq \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|x_n - y_n\|^2.$$

Therefore, we have

$$\|x_n - y_n\|^2 \leq \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\|^2 - \|x_n - u\|^2\right)$$

$$= \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| - \|x_n - u\|\right) (\|x_n - u\| + \|x_n - u\|)$$

$$\leq \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\| + \|x_n - u\|) \|x_n - x_n\|.$$

Since $\|x_n - x_n\| \to 0$, we obtain $x_n - y_n \to 0$. From (4) and (5) we also have

$$\|t_n - u\|^2 \leq \|x_n - u\|^2$$

$$\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2 \lambda_n k \|x_n - y_n\| \|t_n - y_n\|$$

$$\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \|x_n - y_n\|^2 + \lambda_n^2 k^2 \|y_n - t_n\|^2$$

$$\leq \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1) \|y_n - t_n\|^2.$$

Therefore we have

$$\|t_n - y_n\|^2 \leq \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\|^2 - \|x_n - u\|^2\right)$$

$$= \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| - \|x_n - u\|\right) (\|x_n - u\| + \|x_n - u\|)$$

$$\leq \frac{1}{1 - \lambda_n^2 k^2} (\|x_n - u\| + \|x_n - u\|) \|x_n - y_n\|.\]
Since \( \|x_n - z_n\| \to 0 \), we obtain \( t_n - y_n \to 0 \). Since \( A \) is \( k \)-Lipschitz-continuous, we have \( Ay_n - At_n \to 0 \). From \( \|x_n - t_n\| \leq \|x_n - y_n\| + \|y_n - t_n\| \) we also have \( x_n - t_n \to 0 \). Since

\[
\|t_n - St_n\| = \|t_n - x_n\| \leq \|t_n - x_n\| + \|x_n - z_n\|
\]

we have \( \|t_n - St_n\| \to 0 \).

As \( \{x_n\} \) is bounded, there is a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( \{x_{n_i}\} \) converges weakly to some \( u \). We can obtain that \( u \in F(S) \cap VI(C, A) \). First, we show \( u \in VI(C, A) \). Since \( x_n - t_n \to 0 \) and \( x_n - y_n \to 0 \), we have \( \{t_{n_i}\} \to u \) and \( \{x_{n_i}\} \to u \). Let

\[
T_u = \begin{cases} 
Av + NCv, & \text{if } v \in C, \\
\emptyset, & \text{if } v \notin C.
\end{cases}
\]

Then \( T \) is maximal monotone and \( 0 \in T_u \) if and only if \( v \in VI(C, A) \); see [7]. Let \((v, w) \in G(T) \). Then, we have \( w \in T_u = Av + NCv \) and hence \( w - Av \in NCv \). So, we have \( \langle v - t_{n_i} - Av \rangle \geq 0 \) for all \( t_i \in C \). On the other hand, from \( t_{n_i} = PC(x_n - \lambda_{n_i}Ay_{n_i}) \) and \( v \in C \) we have

\[
\langle x_{n_i} - \lambda_{n_i}Ay_{n_i} - t_{n_i} - v \rangle \geq 0
\]

and hence

\[
\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \rangle \geq 0.
\]

Therefore from \( w - Av \in NCv \) and \( t_{n_i} \in C \), we have

\[
\langle v - t_{n_i}, w \rangle \geq \langle v - t_{n_i}, Av \rangle \\
\geq \langle v - t_{n_i}, Av \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \rangle \\
= \langle v - t_{n_i}, Av - At_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle \\
\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle.
\]

Hence, we obtain \( \langle v - u, w \rangle \geq 0 \) as \( i \to \infty \). Since \( T \) is maximal monotone, we have \( u \in T^{-1}0 \) and hence \( u \in VI(C, A) \).

Let us show \( u \in F(S) \). Assume \( u \notin F(S) \). From Opial's condition, we have

\[
\liminf_{i \to \infty} \|t_{n_i} - u\| < \liminf_{i \to \infty} \|t_{n_i} - Su\| = \liminf_{i \to \infty} \|t_{n_i} - St_{n_i} + St_{n_i} - Su\| \\
\leq \liminf_{i \to \infty} \|St_{n_i} - Su\| \\
\leq \liminf_{i \to \infty} \|t_{n_i} - u\|.
\]

This is a contradiction. So, we obtain \( u \in F(S) \). This implies \( u \in F(S) \cap VI(C, A) \).

From \( t_0 = PC(x) \in F(S) \cap VI(C, A) \), we have

\[
|t_0 - x| \leq |u - x| \leq \liminf_{i \to \infty} \|x_{n_i} - x\| \leq \limsup_{i \to \infty} \|x_{n_i} - x\| \leq |t_0 - x|.
\]

So, we obtain

\[
\lim_{i \to \infty} \|x_{n_i} - x\| = |u - x|.
\]

From \( x_{n_i} - x \to u - x \) we have \( x_{n_i} - x \to u - x \) and hence \( x_{n_i} \to u \). Since \( x_n \in P_{Q_n} x \) and \( t_0 \in F(S) \cap VI(C, A) \subset C_n \cap Q_n \subset Q_n \), we have

\[
-t_0 - x_{n_i} = (t_0 - x_{n_i} - x_{n_i}) + (t_0 - x_{n_i} - x_{n_i}) \geq (t_0 - x_{n_i} - x_{n_i} - x_{n_i}).
\]

As \( i \to \infty \), we obtain \( |t_0 - u| \leq |t_0 - u| \to 0 \) by \( t_0 = PC(x) \cap VI(C, A) \) and \( u \in F(S) \cap VI(C, A) \).

Hence we have \( u = t_0 \). This implies that \( x_n \to t_0 \). It is easy to see \( y_n \to t_0 \), \( x_n \to t_0 \).
4 Applications.

Using Theorem 3.1, we prove some theorems in a real Hilbert space.

**Theorem 4.1** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ such that $VI(C,A)$ is nonempty. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{align*}
x_0 &= x \in C \\
y_n &= P_C (x_n - \lambda_n Ax_n) \\
z_n &= P_C (x_n - \lambda_n Ay_n) \\
C_n &= \{ z \in C : \|z_n - z\| \leq \|x_n - z\| \} \\
Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \} \\
z_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{VI(C,A)} x$.

**Proof.** Putting $S = I$, by Theorem 3.1, we obtain the desired result.

**Remark.** See Iiduka, Takahashi and Toyoda [2] for the case when $A$ is $\alpha$-inverse-strongly-monotone.

**Theorem 4.2** Let $C$ be a closed convex subset of a real Hilbert space $H$ and $S$ be a nonexpansive mapping of $C$ into itself such that $F(S)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{align*}
x_0 &= x \in C \\
y_n &= S x_n \\
C_n &= \{ z \in C : \|y_n - z\| \leq \|x_n - z\| \} \\
Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}$$

for every $n = 0, 1, 2, \ldots$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_F(S)x$.

**Proof.** Putting $A = 0$, by Theorem 3.1, we obtain the desired result.

**Remark.** See also Nakajo and Takahashi [5] for more general result.

**Theorem 4.3** Let $H$ be a real Hilbert space. Let $A$ be a monotone, $k$-Lipschitz-continuous mapping of $H$ into itself and $S$ be a nonexpansive mapping of $H$ into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{align*}
x_0 &= x \in C \\
y_n &= S (x_n - \lambda_n A (x_n - \lambda_n Ax_n)) \\
C_n &= \{ z \in C : \|y_n - z\| \leq \|x_n - z\| \} \\
Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(S) \cap A^{-1}0}x$.

**Proof.** We have $A^{-1}0 = VI(H, A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result.

**Remark.** Notice that $F(S) \cap A^{-1}0 \subset VI(F(S), A)$. See also Yamada [9] for the case when $A$ is a strongly monotone and Lipschitz continuous mapping of a real Hilbert space $H$ into itself and $S$ is a nonexpansive mapping of $H$ into itself.

**Theorem 4.4** Let $H$ be a real Hilbert space. Let $A$ be a monotone, $k$-Lipschitz-continuous mapping of $H$ into itself and $B : H \rightarrow 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let $J_B^r$ be the resolvent of $B$ for each $r > 0$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{align*}
x_0 &= x \in C \\
y_n &= J_B^r (x_n - \lambda_n A (x_n - \lambda_n Ax_n)) \\
C_n &= \{ z \in C : \|y_n - z\| \leq \|x_n - z\| \} \\
Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{align*}$$
for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{A^{-1}0 \cap B^{-1}0} x$.

Proof. We have $A^{-1}0 = VI(H, A)$ and $F(J_{r}^{B}) = B^{-1}0$. Putting $P_{H} = I$, by Theorem 3.1 we obtain the desired result.

References


