Propagation of microlocal solutions near a hyperbolic fixed point (Microlocal Analysis and Asymptotic Analysis)

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Propagation of microlocal solutions near a hyperbolic fixed point

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Joint work with Jean-François Bony, Thierry Ramond and Maher Zerzeri

1 Introduction

This is a partial report of the work in progress with Jean-François Bony, Thierry Ramond and Maher Zerzeri about the quantum monodromy operator associated to a homoclinic trajectory. A major part of the results here was already reported by one of the collaborators in [3].

The notion of monodromy operator was introduced by J. Sjöstrand and M. Zworski in [4] for a periodic trajectory. It consists in continuing microlocal solutions of the semiclassical Schrödinger equation

\[-h^2 \Delta u + V(x)u = Eu\]  

along a Hamilton flow \( H_p \) on \( p^{-1}(E) \) of the corresponding classical mechanics:

\[H_p = \sum_{j=1}^{d} \left( \frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} \right), \quad p(x, \xi) = \xi^2 + V(x).\]  

Recall briefly the notion of microlocal solution according to [4]. If \( dp \neq 0 \) at a point \( (x^0, \xi^0) \in p^{-1}(E) \), there exists a local canonical transformation \( \kappa \) defined in a neighborhood of \( (x^0, \xi^0) \) with \( \kappa(x^0, \xi^0) = (0, 0) \), and a semiclassical microlocal Fourier integral operator \( U \) associated to \( \kappa \), such that \( p = \kappa^* \xi_1 \) and \( UP^{-1}U^{-1} = hD_{x_1} \). We can then define the space of microlocal solution at \( (x^0, \xi^0) \) by

\[ \ker_{(x^0, \xi^0)}(P) = U^{-1}(\ker(hD_{x_1})), \quad \ker(hD_{x_1}) = \{ u \in \mathcal{D}'(\mathbb{R}^d) : hD_{x_1} u = 0 \} \]
Since \( \ker(hD_{x}) \) is identified with \( \mathcal{D}'(\mathbb{R}^{d-1}) \), so is \( \ker_{(x^{0}, \xi^{0})}(P) \). If \( (x^{1}, \xi^{1}) = \exp TH_{p}(x^{0}, \xi^{0}) \) is another point on this flow, we can naturally define the propagator of microlocal solutions from \( \ker_{(x^{0}, \xi^{0})}(P) \) to \( \ker_{(x^{1}, \xi^{1})}(P) \) as operator on \( \mathcal{D}'(\mathbb{R}^{d-1}) \).

Here we study the case where \( \exp tH_{p}(x^{0}, \xi^{0}) \) tends to a hyperbolic fixed point \( (0, 0) \) as \( t \) tends to \( +\infty \). To such a point associate the stable and unstable Lagrangian manifolds \( \Lambda_- \) and \( \Lambda_+ \), on which Hamilton flows tend to \( (0, 0) \) as \( t \) tends to \( +\infty \) and \( -\infty \) respectively. Moreover, any point close to \( \Lambda_+ \) comes from a point close to \( \Lambda_- \). We expect, therefore, that a microlocal solution at a point on \( \Lambda_+ \) is determined by that on \( \Lambda_- \).

The purpose of this report is to study this correspondence of microlocal solutions from \( \Lambda_- \) to \( \Lambda_+ \). After preparing the geometrical setting in section 2, we state a uniqueness theorem in section 3, which says that if a solution to (1) is microlocally exponentially small on \( \Lambda_- \), it is also microlocally exponentially small on \( \Lambda_+ \) for \( E \) away from a discrete subset \( \Gamma(h) \). In section 4, based on an idea in [2], we construct a solution with a given microlocal data at a point \( (x^{0}, \xi^{0}) \) on \( \Lambda_- \), as superposition of time-dependent WKB solutions via Fourier transform with respect to \( E \), and formally calculate its microlocal output at the corresponding point \( (x^{0}, \xi^{0}_+) \) on \( \Lambda_+ \). Section 5 is an appendix about the notion of *expandible symbol*, which is used repeatedly for the study of the large time behavior of both classical and quantum objects.

2 Symplectic geometry

Let \( p(x, \xi) = \xi^2 + V(x) \) be the Hamiltonian associated to the semiclassical Schrödinger operator \(-h^2\Delta + V(x)\) in \( \mathbb{R}^d \). Here, we use the following notations:

\[
  x = (x_1, \ldots, x_d), \quad \xi = (\xi_1, \ldots, \xi_d), \quad \xi^2 = \sum_{j=1}^{d} \xi_j^2, \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}
\]

Suppose that the potential \( V(x) \) is real and analytic in a neighborhood of \( x = 0 \), and that \( x = 0 \) is a non-degenerate minimum of \( V(x) \), so that \( (x, \xi) = (0, 0) \) is a saddle point of the Hamiltonian \( p(x, \xi) \). After a change of variables, we can assume that \( p(x, \xi) \) is of the form

\[
  p(x, \xi) = \xi^2 - \sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 + O(|x|^3), \quad (x \to 0),
\]
where \(\{\lambda_j\}_{j=1}^{d}\) are positive numbers which we assume \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d\). Let \(H_p\) be the Hamilton vector field associated to \(p\). In the \((x, \xi)\) coordinates, the linearized vector field \(F_p\) of \(H_p\) at \((0, 0)\) is simply

\[
F_p = d_{(0, 0)}H_p = \begin{pmatrix} 0 & I \\ L^2 & 0 \end{pmatrix},
\]

where \(L\) is the \(d \times d\) matrix defined as \(L = \text{diag}(\lambda_1, \ldots, \lambda_d)\). The eigenvalues of \(F_p\) are the \(\lambda_j\)'s and the \(-\lambda_j\)'s.

Associated to the hyperbolic fixed point, we have thus a natural decomposition of \(T_{(0,0)}^*\mathbb{R}^d = \mathbb{R}^d\) in a direct sum of two linear subspaces \(\Lambda_+^0\) and \(\Lambda_-^0\), of dimension \(d\), associated respectively to the positive and negative eigenvalues of \(F_p\). These spaces \(\Lambda_\pm^0\) are given by

\[
\Lambda_\pm^0 = \{ (x, \xi); \xi_j = \pm \frac{\lambda_j}{2} x_j, j = 1, \ldots, d \}.
\]

The stable/unstable manifold theorem gives us the existence of two Lagrangian manifolds \(\Lambda_+\) and \(\Lambda_-\), defined in a vicinity \(\Omega\) of \((0,0)\), which are stable under the \(H_p\) flow and whose tangent space at \((0,0)\) are precisely \(\Lambda_+^0\) and \(\Lambda_-^0\). In particular, we see that these manifolds can be written as

\[
\Lambda_\pm = \{ (x, \xi); \xi = \nabla \phi_\pm(x) \},
\]

for some smooth functions \(\phi_+\) and \(\phi_-\), which can be chosen so that

\[
\phi_\pm(x) = \pm \sum_{j=1}^{d} \frac{\lambda_j}{4} x_j^2 + o(x^2).
\]

We shall say that \(\Lambda_+\) is the outgoing Lagrangian manifold and \(\Lambda_-\) the incoming Lagrangian manifold associated to the hyperbolic fixed point. Indeed \(\Lambda_+\) (resp. \(\Lambda_-\)) can be characterized as the set of points \((x, \xi) \in \Omega\) such that \(\exp tH_p(x, \xi) \to (0,0)\) as \(t \to -\infty\) (resp. as \(t \to +\infty\)): Take a point \(x^0 \in \mathbb{R}^d\) near \(0\). Then there exist unique \(\xi_+^0 \in \mathbb{R}^d\) and \(\xi_-^0 \in \mathbb{R}^d\) such that \((x^0, \xi_\pm^0) \in \Lambda_\pm\). Let \(\gamma_\pm(t) = \exp tH_p(x^0, \xi_\pm^0)\) be the Hamilton flow emanating from \((x^0, \xi_\pm^0)\). Then, we know from Proposition 10 in Appendix that \(\gamma_\pm(t)\) are expandible, i.e.

\[
\gamma_\pm(t) \sim \sum_{k=1}^{\infty} e^{\pm \mu_k t} \gamma_{\pm, k}(t), \quad t \to \mp \infty,
\]

where \(\{\lambda_j\}_{j=1}^{d}\) are positive numbers which we assume \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d\). Let \(H_p\) be the Hamilton vector field associated to \(p\). In the \((x, \xi)\) coordinates, the linearized vector field \(F_p\) of \(H_p\) at \((0, 0)\) is simply

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\[
\gamma_\pm(t) \sim \sum_{k=1}^{\infty} e^{\pm \mu_k t} \gamma_{\pm, k}(t), \quad t \to \mp \infty,
\]
where $\gamma_{\pm}(t)$ are vectors whose elements are polynomials in $t$ ($\gamma_{\pm,1}$ is constant) and $0 < \mu_1 < \mu_2 < \cdots$ are the various non-vanishing linear combinations over $\mathbb{N}$ of the $\lambda_j$'s. In particular, $\mu_1 = \lambda_1$. If we assume

(A1) $\lambda_1 < \lambda_2$,

then there exists a constant $\gamma_1 = \gamma_1(x^0)$ such that

$$
\gamma_{\pm}(t) = \gamma_1 e^{\pm \lambda_1 t} x^t(1,0,\ldots,0,\pm \lambda_1/2,0,\ldots,0) + O(e^{\pm \mu_2 t}), \quad (t \to \mp \infty). \quad (8)
$$

We see that $\gamma_{\pm}(t)$ is tangential to the $(x_1, \xi_1)$-plane if $c \neq 0$.

### 3 Uniqueness

We begin this section by introducing the notion of microsupport of solutions.

For $u \in L^2(\mathbb{R}^n)$, the Bargman transform (or global FBI transform) is defined by

$$
Tu(x, \xi; h) = c_d(h) \int_{\mathbb{R}^d} e^{i(x-y)\cdot \xi/h - (x-y)^2/2h} u(y; h) dy.
$$

$Tu(x, \xi; h)$ belongs to $L^2(\mathbb{R}^{2d}_{x,\xi})$ and $c_d(h)$ is taken so that $T$ be an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$. It is seen that by this transform, the function $u$ is localized in $x$ by a Gaussian up to $O(\sqrt{h})$ when $h$ is small. Moreover, it is localized also in $\xi$ up to $O(\sqrt{h})$. Indeed we have an identity

$$
Tu(x, \xi; h) = e^{ix\cdot \xi/h} \hat{T}u(\xi, -x; h),
$$

where $\hat{T}u(x, \xi; h)$ is the semiclassical Fourier transform

$$
\hat{T}u(\xi) = (2\pi h)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot \xi/h} u(x) dx. \quad (9)
$$

A ($h$-dependent) function $u \in L^2$ is said to be zero at a point $(x^0, \xi^0)$ in the phase space iff there exists a neighborhood $U$ of $(x^0, \xi^0)$ and a positive number $\epsilon$ such that

$$
Tu(x, \xi; h) = O(e^{-\epsilon/h})
$$
as $h \to 0$ uniformly in $U$. The complement of such points is called microsupport of $u$ and denoted by $MS(u)$. Microsupport is a closed set. Two functions $u$ and $v$ are identified near $(x^0, \xi^0)$ if $(x^0, \xi^0) \notin MS(u - v)$.

Microsupport has the following properties: Let $u$ be a solution of $Pu = E(h)u$ in a domain $\Omega \subset \mathbb{R}^n$, where $E(h) = O(h)$, and assume that $||u||_{L^2(\Omega)} \leq 1$. 
• The microsupport of \( u \) is included in the energy surface \( p^{-1}(0) \).

• The microsupport of \( u \) propagates along a simple Hamilton flow in \( p^{-1}(0) \).

• The microsupport of a WKB solution \( u = e^{i\psi(x)/h}b(x,h) \), \( b(x,h) = O(h^{-N}) \) for some \( N \in \mathbb{R} \) as \( h \) tends to 0, is included in the Lagrangian submanifold \( \{(x,\xi);\xi = \partial_x \psi(x)\} \).

Now we come back to our problem near the hyperbolic fixed point. Let \( \Gamma(h) \) be the discrete subset of \( \mathbb{C} \) defined by
\[
\Gamma(h) = \{E_{\alpha} = -ih \sum_{j=1}^{d} \lambda_j (\alpha_j + \frac{1}{2}); \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d\}.
\]

Notice that for \( E = E_{\alpha} \), the functions
\[
u_{\alpha} = \Pi_{j=1}^{d} H_{\alpha_j} \left( e^{-\pi i/4} \frac{\sqrt{\lambda_j}}{\sqrt{2h}} x_j \right) \exp \left( i \sum_{j=1}^{m} \frac{\lambda_j}{4h} x_j^2 \right),
\]
where \( H_n \) is the Hermite polynomial, satisfy the equation
\[-h^2 \Delta u_{\alpha} - \sum_{j=1}^{m} \frac{\lambda_j}{4} x_j^2 u_{\alpha} = E_{\alpha} u_{\alpha}.
\]

These functions are of WKB form and, by the above third property, the microsupport of \( u_{\alpha} \) is \( \Lambda_{+}^0 \).

Let us assume

(A2) \( |E(h)| \leq Ch \) in \( \mathbb{C} \) with \( C > 0 \), and there exists \( \delta > 0 \) such that \( d(E(h),\Gamma(h)) > \delta h \) for all small \( h \).

The following theorem says that the solution of (1) is uniquely determined microlocally in a neighborhood of \( (0,0) \), modulo microlocally small functions, by its data on \( \Lambda_{-}\backslash (0,0) \) if \( E(h) \) is away from the exceptional set \( \Gamma(h) \).

**Theorem 1** Assume (A2). If an \( h \)-dependent function \( u \in L^2(\mathbb{R}^d) \) with \( \|u\|_{L^2} \leq 1 \) satisfies
\[
MS((P - E(h))u) = \emptyset, \quad MS(u) \cap \{\Lambda_{-} \backslash (0,0)\} = \emptyset,
\]
in a neighborhood of \( (0,0) \), then \( (0,0) \notin MS(u) \).
4 Integral representation of the solution

In order to study the correspondence of microlocal solutions from $\Lambda_-$ to $\Lambda_+$, we fix a point $(x^0, \xi_-^0)$ on $\Lambda_-$ sufficiently close to the origin and consider solutions of \((1)\) whose microsupport on $\Lambda_-$ is included in a neighborhood of $\exp tH_p(x^0, \xi_-^0)$ (recall that the microsupport is invariant by the Hamilton flow). Then, under the assumption (A2), the solution $u$ is uniquely determined in a full neighborhood of the origin, in particular on $\Lambda_+$, if a microlocal data $u_0$ is given at $(x^0, \xi_0^0)$. We study in this section the map $\mathcal{I}_\delta$ which associates $u_0$ to the microlocal solution $u$ at $(x^0, \xi_+^0)$, which we call here propagator (it is called singular part of the monodromy operator in [3]).

The symbol $p$ is of principal type at $(x^0, \xi_-^0) \in \Lambda_-$ and the space of microlocal solutions $\ker_{(x^0, \xi_-^0)}(P)$ is identified with $\mathcal{D}'(\mathbb{R}^{d-1})$. If we assume

\[(A3) \quad \gamma_1(x^0) \neq 0,\]

where $\gamma_1(x^0)$ is defined in (8), a microlocal solution $u_0 \in \ker_{(x^0, \xi_-^0)}(P)$ can be considered as distribution on

\[H_0 = \{x \in \mathbb{R}^d; x_1 = x_1^0\},\]

since (the projection of) the Hamilton flows are tangential to the $x_1$ axis at the origin.

Let $u_0(x') \in \mathcal{D}'(\mathbb{R}^{d-1})$ be such that $\hat{u}_0(\eta)$, the semiclassical Fourier transform of $u_0$ (see (9)), is supported in a small neighborhood of $\xi_0'$.

Following an idea of Helffer and Sjöstrand [2], we write the solution $u$ in the form

\[u(x, h) = \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^{d-1}} \int_{0}^{+\infty} e^{i\phi(t,x,\eta)/h} a(t, x, \eta, h) \hat{u}_0(\eta) dt d\eta, \quad \text{(11)}\]

with

\[\left\{ \frac{h}{i} \frac{\partial}{\partial t} + P(x, hD) - E(h) \right\}(e^{i\phi/h} a) = O(h^\infty).\]

If $a$ and the energy $E(h)$ have classical asymptotic expansions with respect to $h$:

\[a(t, x, \eta, h) \sim \sum_{l=0}^{\infty} a_l(t, x, \eta) h^l, \quad E(h) \sim \sum_{l=0}^{\infty} E_l h^{l+1},\]

}\]
It will be shown that, for $x$ close to $x^0$, there exists a unique critical point $t = t(x, \eta)$. On the other hand, the Lagrangian manifold $\Lambda^\eta_t$ tends to $\Lambda_+$ as $t \rightarrow +\infty$, which means that $\partial_t \phi$ tends to $\phi_+$. Thus we will have microlocally

$$
\int_0^{+\infty} e^{i\phi(t,x,\eta)/h} a(t,x,\eta,h) dt \sim \begin{cases} e^{i\psi(x,\eta)} b(x,\eta,h) & \text{near } (x,\xi) = (x^0,\xi^0), \\ e^{i\theta(x,\eta)} c(x,\eta,h) & \text{near } (x,\xi) = (x^0,\xi^+_0), \end{cases}
$$

with $\psi(x,\eta) = \phi(t(x,\eta),x,\eta)$ and $\theta(x,\eta) = \phi_+(x) + \tilde{\psi}(\eta)$ for some $\tilde{\psi}$.

We require that $u$ is equal to $u_0$ on $H_0$ microlocally near $(x^0,\xi^0)$, which is satisfied if

$$\psi(x,\eta) = x' \cdot \eta, \quad b(x,\eta,h) = 1 \quad \text{on } H_0. \quad (15)$$

We will see in the following that it is possible to construct $\phi$ and $a$ so that $\psi$ and $b$ satisfy the condition (15) and to calculate $\theta$ and $c$. Then we will write $I_S$ as Fourier integral operator.

## 4.1 The phase function

Since $\gamma_-$ is a simple characteristic for the operator $p$, by the usual Hamilton-Jacobi theory we have first the

**Lemma 2** For all $\eta \in \mathbb{R}^{d-1}$ close enough to $\xi^0$, there is a unique function $\psi_\eta = \psi(x,\eta)$, defined in a neighborhood $\omega_0$ of $x_0$ such that

$$
\begin{cases} p(x,\nabla \psi_\eta(x)) = 0 & \text{in } \omega_0, \\ \psi_\eta(x) = x' \cdot \eta & \text{on } H_0 \cap \omega_0. \end{cases}
$$
We denote by $\Lambda_\psi^n$ the corresponding Lagrangian manifold
\[ \Lambda_\psi^n = \{(x, \xi) \in T^*\mathbb{R}^d, \ x \in \omega_0, \ \xi = \nabla \psi(x)\}. \] (16)

**Lemma 3** The Lagrangian manifolds $\Lambda_-$ and $\Lambda_\psi^n$ intersect along an integral curve $\gamma^n$ for $H_p$, and the intersection is clean. In particular, $\gamma^{0'} = \gamma_-$. Let $(x^0(\eta), \xi^0(\eta))$ be the intersection of $\gamma^n$ and $H_0 \times \mathbb{R}^d$. The curve $\gamma^n$ is parametrized as $\gamma^n(t) = \exp t H_p(x^0(\eta), \xi^0(\eta))$, and it has the asymptotic property like (8);
\[ \gamma^n(t) \sim \gamma_1(\eta)e^{-\lambda_1 t} \times t(1,0,\ldots,0,-\lambda_1/2,0,\ldots,0) \ (t \to +\infty), \] (17)
with a non vanishing constant $\gamma_1(\eta)$ for $\eta$ close to $\xi^{0'}$.

Let $\Gamma_0^n$ be the level set of $\psi$, passing by $x^0(\eta)$:
\[ \Gamma_0^n = \{(x, \xi) \in \Lambda_\psi^n, \ \psi_\eta(x) = \psi(x^0(\eta))\}. \] (18)

**Lemma 4** For any $\eta$ close enough to $\xi^{0'}$, one can find a Lagrangian manifold $\Lambda_0^n$ such that

1. $\Lambda_0^n$ intersects cleanly with $\Lambda_\psi^n$ along $\Gamma_0^n$,
2. for any $t \geq 0$, the projection $\Pi : \Lambda_0^n = \exp t H_p(\Lambda_0^n) \to \mathbb{R}^d$ is a diffeomorphism in a neighborhood of $\gamma^n(t) \in \Lambda_\psi^n$.

The Lagrangian manifold $\Lambda_0^n = \exp t H_p(\Lambda_0^n)$ is then represented by a generating function $\phi(t, x, \eta)$:
\[ \Lambda_0^n = \{(x, \xi); \ \xi = \nabla_x \phi(t, x, \eta)\}. \] (19)
and $\phi(t, x, \eta)$ satisfies the eikonal equation (12) for every $\eta$.

Now we fix $\eta$ and define
\[ \Gamma_t^n = \Lambda_0^n \cap \Lambda_\psi^n \ (= \exp(t H_p) \Gamma_0^n). \] (20)
If $(x, \xi) \in \Gamma_t^n$, then $\xi = \nabla_x \phi(t, x, \eta)$ and $p(x, \xi) = 0$ ($\Lambda_\psi^n \subset p^{-1}(0)$). Together with (12), we get that $t$ is a critical point for the function $t \mapsto \phi(t, x, \eta)$ if and only if $x \in \Pi_x \Gamma_t^n$. More precisely, we have
Proposition 5 For each $x$ close enough to $\gamma^n$, there is a unique time $t = t(x, \eta)$ such that $x \in \Pi_x \Gamma^n_t$. Moreover, it is the only critical point for the function $t \mapsto \phi(t, x, \eta)$ and it is non-degenerate, $\partial^2_t \phi(t(x), x, \eta) > 0$.

As a consequence, we obtain
\[ \nabla_x \psi_n(x) = \nabla_x (\phi(t(x, \eta), x)), \tag{21} \]
so that $x \mapsto \psi_n(x)$ and $x \mapsto \phi(t(x), x)$ are equal up to constant. We choose $\phi$ so that
\[ \phi(t(x, \eta), x, \eta) = \psi_n(x). \tag{22} \]

Finally we observe the asymptotic behavior of the phase function $\phi(t, x, \eta)$ when $t$ tends to $+\infty$.

Proposition 6 The phase function $(t, x) \mapsto \phi(t, x, \eta)$ is expandible uniformly with respect to $\eta$:
\[ \phi(t, x, \eta) - (\phi_+(x) + \tilde{\psi}(\eta)) \sim \sum_{j \geq 1} e^{-\mu_j t} \phi_j(t, x, \eta). \tag{23} \]
Here $\tilde{\psi}$ is a generating function of the $d-1$ dimensional Lagrangian submanifold $\Lambda_- \cap (H_0 \times \mathbb{R}_t^d)$, i.e.
\[ \{(y', \eta) \in T^*\mathbb{R}^{d-1}; \eta = \nabla_y \phi_-(x_1^0, y')\} = \{(y', \eta) \in T^*\mathbb{R}^{d-1}; y' = \nabla_\eta \tilde{\psi}(\eta)\}, \]
and so
\[ \tilde{\psi}(\eta) \sim -\sum_{j=2}^d \frac{1}{\lambda_j} \eta_j^2, \quad (\eta \to 0). \]
Moreover, the function $\phi_1$ does not depend on $t$, and
\[ \phi_1(x, \eta) = -2\lambda_1 \gamma_1(\eta) x_1 + O(x^2), \tag{24} \]
where $\gamma_1(\eta)$ is defined in (17).
4.2 Transport equations

We study the transport equations (13), (14), using the informations about the phase function $\phi(t, x, \eta)$ obtained in the previous subsection. We want to solve these equations under the condition

$$a(t(x, \eta), x, \eta, h)|_{H_0} = e^{-\pi i/4} \sqrt{\partial_t^2 \phi(t(x, \eta), x, \eta)},$$

(25)

so that the right hand side of (11), after the stationary phase method applied to the integration with respect to $t$ at the critical point $t = t(x, \eta)$, reduces to $u_0$ on $H_0$. Notice that the initial condition (25) determines uniquely the solutions of (13), (14) on the hypersurface $\{(t, x); t = t(x, \eta)\}$, since this hypersurface is invariant under the flow of the vector field $\partial_t + 2\nabla_x \phi \cdot \nabla_\eta$.

As for the asymptotic behavior as $t \to +\infty$, we recall that $\phi$ is expandible and

$$\nabla_x \phi \cdot \nabla_x = \sum_{j=1}^d \left( \frac{\lambda_j}{2} x_j + O(x^2) \right) \frac{\partial}{\partial x_j}, \quad \Delta \phi = \sum_{j=1}^d \frac{\lambda_j}{2} + O(x) \quad (x \to 0).$$

Then again by Proposition 10 applied to $e^{St} a_j$, where

$$S = \frac{1}{2} \sum_{j=1}^d \lambda_j - iE_0,$$

we have the following asymptotic expansion.

**Proposition 7** For each $l$, $a_l(t, x, \eta)$ is expandible and has an asymptotic expansion as $t \to \infty$

$$a_l(t, x, \eta) \sim e^{-St} \sum_{k=0}^\infty a_{l,k}(t, x, \eta) e^{-\mu_k t},$$

(26)

which is uniform with respect to $\eta$. Here $\mu_0$ is defined to be 0, and $a_{0,0}$ is independent of $t$.

4.3 Asymptotics of the propagator

Let us fix $\eta$ close to $\xi^{0'}$ and $x$ close to $\gamma$. Then there are two $t$'s which contribute in the semiclassical limit to the integration with respect to $t$ of
the expression (11). One is \( t = t(x, \eta) \), which is the unique critical point, and the other is \( t = +\infty \). They correspond to the Lagrangian manifolds \( \Lambda_{t(x, \eta)}^\eta \) and \( \Lambda_+ \) respectively.

Since the contribution from \( t = t(x, \eta) \) reproduces the given data \( u_0(x') \) on \( H_0 \) after integration with respect to \( \eta \), we will obtain the propagator \( T_S \) in the form of Fourier integral operator after calculating the contribution from \( t = +\infty \).

**Lemma 8** Suppose \( b \in \mathbb{R} \), \( \lambda > 0 \) and \( \rho > 0 \). Then as \( h \to 0 \), we have

\[
\int_0^\infty \exp\{ibe^{-\lambda t}/h - \rho t\} dt - \frac{1}{\lambda} \left( \frac{ih}{b} \right)^{\rho/\lambda} \Gamma\left( \frac{\rho}{\lambda} \right)
\sim \frac{e^{ib/h}}{\lambda} \sum_{n=0}^\infty \left( \frac{\rho}{\lambda} - \frac{1}{n+1} \right) n! \left( \frac{ih}{b} \right)^{n+1}
\]

Let us compute the contribution from \( t = \infty \) of the integral

\[
\int_0^\infty e^{i\phi/(t, x, \eta)/h} a(t, x, \eta, h) dt.
\]

If we substitute \( \phi_+(x) + \tilde{\psi}(\eta) + e^{-\lambda_1 t} \phi_1(x, \eta) \) to \( \phi(t, x, \eta) \) and \( a_{0,0}(x, \eta)e^{-St} \) to \( a(t, x, \eta, h) \) according to (23), (26), we get

\[
\int_0^\infty e^{i\phi/h} dt = e^{i(\phi_++\tilde{\psi})/h} a_{0,0} \int_0^\infty \exp\{i\phi_1 e^{-\lambda_1 t}/h - St\} dt
\]

Applying Lemma 8 with \( b = \phi_1 \), \( \lambda = \lambda_1 \) and \( \rho = S \), we get

\[
\int_0^\infty e^{i\phi/h} dt \sim e^{i(\phi_++\tilde{\psi})/h} a_{0,0}
\times \left\{ \frac{1}{\lambda_1} \Gamma\left( \frac{S}{\lambda_1} \right) \left( \frac{ih}{\phi_1} \right)^{S/\lambda_1} + \frac{e^{i\phi_1/h} ih}{\lambda_1 \phi_1} + O(h^2) \right\} (h \to 0).
\]

The leading term of the left hand side changes according to the real part of \( S/\lambda_1 \):

\[
\text{Re } S/\lambda_1 > 1 \iff \text{Im } E_0 > \left( \lambda_1 - \sum_{j=2}^d \lambda_j \right) / 2.
\]
Theorem 9 The propagator $I_S$ can be written in the form

$$\frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}^{d-1}} e^{i\theta(x,\eta)} c(x,\eta,h) \hat{u}_0(\eta) d\eta,$$

microlocally near $(x^0, \xi^0_+)$ with

$$\theta(x,\eta) = \phi_+(x) + \tilde{\psi}(\eta),$$

and if $\text{Im} \ E_0 < (\lambda_1 - \sum_{2}^{d} \lambda_j)/2$

$$c(x,\eta,h) \sim \frac{1}{\sqrt{2\pi h \lambda_1}} \left( \frac{S}{\lambda_1} \right)^{S/\lambda_1} \left( \frac{ih}{\phi_1(x)} \right) a_{0,0}(x,\eta),$$

if $\text{Im} \ E_0 > (\lambda_1 - \sum_{2}^{d} \lambda_j)/2$

$$c(x,\eta,h) \sim \frac{1}{\sqrt{2\pi h \lambda_1}} e^{i\phi_1(x)/h} \frac{ih}{\phi_1(x)} a_{0,0}(x,\eta),$$

and if $\text{Im} \ E_0 = (\lambda_1 - \sum_{2}^{d} \lambda_j)/2$

$$c(x,\eta,h) \sim \frac{1}{\sqrt{2\pi h \lambda_1}} \left( \Gamma \left( \frac{S}{\lambda_1} \right) \left( \frac{ih}{\phi_1(x)} \right)^{S/\lambda_1} + e^{i\phi_1(x)/h} \frac{ih}{\phi_1(x)} \right) a_{0,0}(x,\eta),$$

where $\tilde{\psi}(\eta)$ and $\phi_1(x)$ are given in Proposition 6 and $a_{0,0}$ is given in Proposition 7.

5 Appendix - Expandible symbols

Here we recall from [2] the notion of expandible symbol.

We denote by $(\mu_j)_{j \geq 0}$ the strictly growing sequence of linear combinations over $\mathbb{N}$ of the $\lambda_j$'s. We have for example $\mu_0 = 0$, $\mu_1 = \lambda_1$ and $\mu_2 = 2\lambda_1$ or $\mu_2 = \lambda_2$, whether $2\lambda_1 < \lambda_2$ or not.

First we introduce a convenient notation for error terms. We shall write, with $\mu \in \mathbb{R}^+$, $M \in \mathbb{N}$,

$$w(t,x) = \tilde{O}(e^{-\mu t|x|^M})$$

if, for every $\epsilon > 0$, we have

$$w(t,x) = O(e^{-(\mu-\epsilon)t|x|^M}).$$
Definition 1 ([2], Definition 3.1) Let $\omega$ be a neighborhood of 0 in $\mathbb{R}^d$. A smooth function $u : [0, +\infty) \times \omega \to \mathbb{R}$ is expandible if there exists a sequence $(u_k)$ of smooth functions on $[0, +\infty) \times \omega$, which are polynomials in $t$, such that for any $n, N \in \mathbb{N}, \alpha \in \mathbb{N}^d$

$$\partial_t^n \partial_x^\alpha \left( u(t, x) - \sum_{j=0}^{N} u_k(t, x)e^{-\mu jt} \right) = \tilde{O}(e^{-\mu N + t})$$

(29)

If (29) holds, we write simply

$$u(t, x) \sim \sum_{k \geq 0} u_k(t, x)e^{-\mu kt}.$$ 

(30)

Proposition 10 ([2], Theorem 3.8) Let $A(t, x)$ be a real smooth expandible matrix with $A(0, 0) = \text{diag}(\lambda_1, \ldots, \lambda_d)$. Then, if $v(t, x)$ is expandible, the solution $u(t, x)$ to the problem

$$\begin{cases}
\partial_t u + A(t, x)x \cdot \partial_x u = v, & t \geq 0, x \in \omega, \\
u|_{t=0} = 0,
\end{cases}$$

(31)

is expandible.

References


