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On the complete description of the Stokes geometry for the first Painlevé hierarchy [1]

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We dedicate this paper to

Professor Louis Boutet de Monvel

with our sincerest congratulations on his being awarded Prix de l'Etat (Academie des Science).

One of the central issues of this article is the introduction of the notion of virtual turning points for higher order Painlevé equations, and two of the authors (Kawai and Takei), together with T. Aoki, fondly remember the stimulating and comfortable conference (Algebraic analysis of singular perturbations, 1991), which Professor Boutet de Monvel, together with Professor M. Sato, organized, and where the notion of a virtual turning point for linear ordinary differential equations was first made public (under the modest name "a new turning point"). The notion of virtual turning points is one of the most important gifts to the exact WKB analysis from microlocal analysis, and hence we believe this article to be most appropriate to dedicate to Professor Boutet de Monvel, who has made substantial contributions to the development of microlocal analysis and asymptotic analysis.

1 Introduction

As was first discovered numerically by Nishikawa [N1, N2], Stokes curves of higher order Painlevé equations cross in general and some degeneracy of Stokes geometry of the underlying Lax pair is often observed along a curved ray emanating from such a crossing point of Stokes curves ("Nishikawa phenomenon"). To analyze this intriguing phenomenon we investigated in [KKNT] several properties of the curved

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

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ray, which we named a "new Stokes curve", by making full use of the underlying Lax pair. The analysis done in [KKNT] tells us that introduction of new Stokes curves is inevitable to obtain a complete description of the global Stokes geometry of higher order Painlevé equations. In this report, using the results of [KKNT], we discuss how to obtain the "complete Stokes geometry" of higher order Painlevé equations.

Similar phenomena, that is, crossing of Stokes curves and the necessity of introducing new Stokes curves, were first observed by Berk-Nevins-Roberts [BNR] for a third order linear ordinary differential equation. Later Aoki-Kawai-Takei [AKT] pointed out that such a new Stokes curve for a higher order linear equation can be interpreted as a Stokes curve emanating from a "virtual turning point" (it was called a "new turning point" in [AKT]). In this report we introduce the notion of a virtual turning point for a higher order Painlevé equation and, using virtual turning points and new Stokes curves emanating from them, we present an explicit procedure for determining the complete Stokes geometry of higher order Painlevé equations.

For the sake of definiteness we restrict our consideration here to the first Painlevé hierarchy ("Painlevé-I hierarchy" or "P_{I}-hierarchy"): We recall the formulation of the P_{I}-hierarchy in §2 and review the definition of its Stokes geometry in §3. In §4 we explain (an example of) the Nishikawa phenomenon in the case of the fourth order Painlevé-I equation. After these preparations we define a virtual turning point in §5 and finally in §6 we discuss the complete description of the Stokes geometry for the P_{I}-hierarchy.

2 P_{I}-hierarchy

The P_{I}-hierarchy with a large parameter η is the following family of systems of first order nonlinear differential equations:

Definition 1. (P_{I}-hierarchy with a large parameter η)

$$(P_{\mathrm{I}})_m \qquad \begin{cases} \dfrac{du_j}{dt} = 2\eta v_j & \text{(1.a)} \\[2mm] \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) & \text{(1.b)} \end{cases}$$

($j = 1, \ldots, m$), where u_j and v_j are unknown functions (we conventionally assume $u_{m+1} \equiv 0$) and w_j is a polynomial of u_k and v_l ($1 \leq k, l \leq j$) determined by the following recursion formula:

$$(2) \qquad w_j = \frac{1}{2}\left(\sum_{k=1}^{j} u_k u_{j+1-k}\right) + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\left(\sum_{k=1}^{j-1} v_k v_{j-k}\right) + c_j + \delta_{jm} t$$

($j = 1, \ldots, m$). Here c_j is a constant and δ_{jm} stands for Kronecker's delta.

The P_{I}-hierarchy was first introduced by Kudryashov ([K], [KS]) through the reduction of the KdV hierarchy, and studied by Gordoa and Pickering ([GP]) and by Shimomura ([S1, S2, S3]) from different points of view respectively. The above expression is a slight modification of the formulation of Shimomura [S2, S3], where the P_{I}-hierarchy is derived from the most degenerate Garnier system.

Remark 1. (i) $(P_{\mathrm{I}})_1$ is equivalent to the following equation that u_1 satisfies:

$$u_1'' = \eta^2(6u_1^2 + 4c_1 + 4t). \tag{3}$$

Thus $(P_{\mathrm{I}})_1$ can be reduced to the traditional Painlevé I equation with a large parameter η (in the notation of [KT1, KT2] etc.).
(ii) $(P_{\mathrm{I}})_2$ is equivalent to

$$u_1'''' = \eta^2(20u_1u_1'' + 10(u_1')^2) - \eta^4(40u_1^3 + 16c_1u_1 - 16c_2 - 16t). \tag{4}$$

(iii) $(P_{\mathrm{I}})_3$ is equivalent to

$$\begin{aligned} u_1^{(6)} &= \eta^2(28u_1u_1^{(4)} + 56u_1'u_1^{(3)} + 42(u_1'')^2) - \eta^4(280u_1^2u_1'' + 280u_1(u_1')^2 \\ &\quad + 16c_1u_1'') + \eta^6(280u_1^4 + 96c_1u_1^2 - 64c_2u_1 - 32c_1^2 + 64c_3 + 64t). \end{aligned} \tag{5}$$

As is confirmed in [KKNT], $(P_{\mathrm{I}})_m$ describes the compatibility condition of the following 2×2 system of linear differential equations ("Lax pair"):

$$(L_{\mathrm{I}})_m \qquad \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\psi = 0, & (6.a) \\[2ex] \left(\dfrac{\partial}{\partial t} - \eta B\right)\psi = 0, & (6.b) \end{cases}$$

with

$$A = \begin{pmatrix} V(x)/2 & U(x) \\ (2x^{m+1} - xU(x) + 2W(x))/4 & -V(x)/2 \end{pmatrix}, \tag{7}$$

$$B = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 & 0 \end{pmatrix}. \tag{8}$$

Here $U(x)$ etc. denote the following polynomials in x with coefficients u_j etc.

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \tag{9}$$

$$V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \tag{10}$$

$$W(x) = \sum_{j=1}^{m} w_j x^{m-j}. \tag{11}$$

3 Stokes geometry of $(P_{\mathrm{I}})_m$

Each member $(P_{\mathrm{I}})_m$ of the Painlevé-I hierarchy admits the following formal solution $(\hat{u}_j, \hat{v}_j)$ called "0-parameter solution":

$$\hat{u}_j(t,\eta) = \hat{u}_{j,0}(t) + \eta^{-1}\hat{u}_{j,1}(t) + \cdots, \tag{12}$$

$$\hat{v}_j(t,\eta) = \hat{v}_{j,0}(t) + \eta^{-1}\hat{v}_{j,1}(t) + \cdots, \tag{13}$$

where $\hat{v}_{j,0} \equiv 0$ $(1 \le j \le m)$, $\hat{u}_{1,0}$ is algebraically determined, and the other $\hat{u}_{j,k}$'s ($k = 0$ and $2 \le j \le m$, or $k \ge 1$) and $\hat{v}_{j,k}$'s ($k \ge 1$) are uniquely determined in a recursive manner once (the branch of) $\hat{u}_{1,0}$ is fixed. (See [KKNT] for the details.) Using this 0-parameter solution, we define the Stokes geometry (i.e., a turning point and a Stokes curve) of $(P_{\mathrm{I}})_m$ in the following way (cf. [KKNT, Section 2.1]): We first consider the linearized equation of $(P_{\mathrm{I}})_m$ at $(\hat{u}_j, \hat{v}_j)$ (sometimes called "Fréchet derivative" for short), that is, the linear part in $(\Delta u_j, \Delta v_j)$ after the substitution $u_j = \hat{u}_j + \Delta u_j$ and $v_j = \hat{v}_j + \Delta v_j$ in $(P_{\mathrm{I}})_m$.

$$(\Delta P_{\mathrm{I}})_m \quad \begin{cases} \dfrac{d}{dt}\Delta u_j = 2\eta\Delta v_j, & (14.\mathrm{a}) \\[2mm] \dfrac{d}{dt}\Delta v_j = 2\eta(\Delta u_{j+1} + \hat{u}_1\Delta u_j + \hat{u}_j\Delta u_1 + \Delta w_j), & (14.\mathrm{b}) \end{cases}$$

$(j = 1, \ldots, m)$, where Δw_j denotes

$$\Delta w_j = \sum_{k=1}^{j}\left(\frac{\partial w_j}{\partial u_k}\bigg|_{u=\hat{u},v=\hat{v}}\Delta u_k + \frac{\partial w_j}{\partial v_k}\bigg|_{u=\hat{u},v=\hat{v}}\Delta v_k\right). \tag{15}$$

Note that $(\Delta P_{\mathrm{I}})_m$ is a system of first order linear ordinary differential equations for $(\Delta u_j, \Delta v_j)$. The Stokes geometry of $(P_{\mathrm{I}})_m$ is then defined as follows:

Definition 2. A turning point (resp., Stokes curve) of $(P_{\mathrm{I}})_m$ is, by definition, a turning point (resp., Stokes curve) of $(\Delta P_{\mathrm{I}})_m$.

If we write $(\Delta P_{\mathrm{I}})_m$ as

$$\frac{d}{dt}\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix} = \eta C(t,\eta)\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix} \tag{16}$$

(where $\Delta u = {}^t(\Delta u_1, \ldots, \Delta u_m)$ and Δv are m-vectors and $C(t,\eta)$ is a formal power series (in η^{-1}) with coefficients of $(2m) \times (2m)$ matrices), and if we let $C_0(t)$ denote the top order part (i.e., the part of order 0 in η) of $C(t,\eta)$, Definition 2 means that a turning point of $(P_{\mathrm{I}})_m$ is a zero of the discriminant of the characteristic equation $\det(\nu - C_0(t)) = 0$, i.e., a turning point is a point where two characteristic roots $\nu_k(t)$ and $\nu_{k'}(t)$ of $C_0(t)$ merge, and that a Stokes curve of $(P_{\mathrm{I}})_m$ emanating from a turning point τ is given by

$$\operatorname{Im}\int_{\tau}^{t}(\nu_k - \nu_{k'})dt = 0 \tag{17}$$

where $\nu_k(t)$ and $\nu_{k'}(t)$ are two characteristic roots of $C_0(t)$ that merge at $t = \tau$. To specify which characteristic roots are relevant, we sometimes call a Stokes curve defined by (17) "Stokes curve of type (k, k')" and, furthermore, it is called "of type $k > k'$" when $\operatorname{Re}\int_\tau^t (\nu_k - \nu_{k'})dt > 0$ holds on it.

It is proved in [KKNT, Proposition 2.1.3] that the characteristic equation $\det(\nu - C_0(t)) = 0$ is always a polynomial of ν^2 in ν, i.e., it is of the form $f(\nu^2, t)$ where $f = f(z,t)$ is a polynomial of degree m in z. Hence there are two kinds of turning points for $(P_{\mathrm{I}})_m$:

(i) A turning point where the degree 0 part (in z) of f vanishes.

(ii) A turning point where the discriminant (with respect to z) of f vanishes.

We call the former one a "turning point of the first kind", and the latter one a "turning point of the second kind".

As is verified in [KKNT, Section 2.1], the Stokes geometry of $(P_{\mathrm{I}})_m$ thus defined has close relationship with that of its underlying Lax pair $(L_{\mathrm{I}})_m$ (particularly of its first equation (6.a)). Since this relationship between the two Stokes geometries plays a crucially important role in the following discussions, let us review its core part here.

We first substitute a 0-parameter solution $(\hat{u}_j, \hat{v}_j)$ of $(P_{\mathrm{I}})_m$ into the coefficients A and B of its underlying Lax pair $(L_{\mathrm{I}})_m$. Then they are accordingly expanded in powers of η^{-1} like

$$A = A_0 + \eta^{-1}A_1 + \cdots, \tag{18}$$

$$B = B_0 + \eta^{-1}B_1 + \cdots. \tag{19}$$

Similarly $U(x)$, $V(x)$ and $W(x)$ given respectively by (9), (10) and (11) are also expanded in powers of η^{-1}; we let $U_l(x,t)$, $V_l(x,t)$ and $W_l(x,t)$ denote the coefficients of η^{-l} in the expansion. After substituting the 0-parameter solution we now consider the Stokes geometry of the underlying Lax pair $(L_{\mathrm{I}})_m$, which is defined in terms of the top order parts of these expansions. In particular, for the Stokes geometry of the first equation (6.a) of $(L_{\mathrm{I}})_m$ we find the following

Proposition 1. (Cf. [KKNT, Proposition 2.1.1])
If we write the characteristic equation of A_0 as $\det(\lambda - A_0) = \lambda^2 - Q_0(x,t)$, then the following holds

$$Q_0(x,t)(= -\det A_0) = \frac{1}{4}(x + 2\hat{u}_{1,0}(t))U_0(x,t)^2. \tag{20}$$

Proposition 1 implies that (the first equation of) the Lax pair $(L_{\mathrm{I}})_m$ has the following two types of turning points;

- one simple turning point $x = -2\hat{u}_{1,0}(t)$, which will be denoted by $x = a(t)$ in what follows,

- m double turning points given by roots of $U_0(x,t) = x^m - \hat{u}_{1,0}(t)x^{m-1} - \cdots - \hat{u}_{m,0}(t) = 0$, which will be denoted by $x = b_1(t), \ldots, x = b_m(t)$ in what follows.

These turning points $x = a(t)$ and $x = b_j(t)$ of $(L_{\mathrm{I}})_m$ relate its Stokes geometry to that of $(P_{\mathrm{I}})_m$ in the following manner: First, we can verify that $\pm 2\sqrt{x + 2\hat{u}_{1,0}(t)}\Big|_{x=b_j(t)}$ gives a characteristic root of C_0, the top order part of the coefficient matrix of $(\Delta P_{\mathrm{I}})_m$, for $j = 1, \ldots, m$ (cf. [KKNT, Proposition 2.1.3]). In what follows we label the characteristic roots of C_0 by $(j, \pm)$, i.e., a combination of the index j and the sign, so that the relations

$$\nu_{j,\pm} = \pm 2\sqrt{x + 2\hat{u}_{1,0}(t)}\Big|_{x=b_j(t)} \tag{21}$$

may be satisfied. Note that $\nu_{j,+} + \nu_{j,-} = 0$ holds for every j. Then the main relations between the two Stokes geometries can be stated in the following propositions.

Proposition 2. ([KKNT, Proposition 2.1.4])
(i) *Let $t = \tau^{\mathrm{I}}$ be a turning point of the first kind of $(P_{\mathrm{I}})_m$. Then at $t = \tau^{\mathrm{I}}$ a double turning point $x = b_j(t)$ merges with the simple turning point $x = a(t)$ in the Stokes geometry of* (6.a). *Consequently the two characteristic roots $\nu_{j,\pm}$ of C_0 merge and vanish at $t = \tau^{\mathrm{I}}$. Furthermore the following relation holds:*

$$\frac{1}{2}\int_{\tau^{\mathrm{I}}}^{t} (\nu_{j,+} - \nu_{j,-})dt = 2\int_{a(t)}^{b_j(t)} \sqrt{Q_0(x,t)}dx. \tag{22}$$

(ii) *Let $t = \tau^{\mathrm{II}}$ be a turning point of the second kind of $(P_{\mathrm{I}})_m$. Then at $t = \tau^{\mathrm{II}}$ a double turning point $x = b_j(t)$ merges with another double turning point $x = b_{j'}(t)$. Consequently two characteristic roots $\nu_{j,+}$ and $\nu_{j',+}$ of C_0 merge at $t = \tau^{\mathrm{II}}$, and so do $\nu_{j,-}$ and $\nu_{j',-}$. Furthermore the following relation holds:*

$$\int_{\tau^{\mathrm{II}}}^{t} (\nu_{j,+} - \nu_{j',+})dt = -\int_{\tau^{\mathrm{II}}}^{t} (\nu_{j,-} - \nu_{j',-})dt = 2\int_{b_{j'}(t)}^{b_j(t)} \sqrt{Q_0(x,t)}dx. \tag{23}$$

As an immediate consequence of the relations (22) and (23) we also obtain

Proposition 3. ([KKNT, Proposition 2.1.5])
If t lies on a Stokes curve of $(P_{\mathrm{I}})_m$ emanating from a turning point $t = \tau^{\mathrm{I}}$ (resp. $t = \tau^{\mathrm{II}}$) of the first (resp. second) kind, two turning points $x = b_j(t)$ and $x = a(t)$ (resp. $x = b_j(t)$ and $x = b_{j'}(t)$) are connected by a Stokes curve of (6.a).

4 Nishikawa phenomena and new Stokes curves

In this section, taking the fourth order Painlevé-I equation $(P_{\mathrm{I}})_2$ as an example, we review the Nishikawa phenomena.

Example 1. (4th order Painlevé-I equation)

$$(P_{\mathrm{I}})_2 \qquad u'''' = \eta^2(20uu'' + 10(u')^2) - \eta^4(40u^3 + 16cu - 16t).$$

(In (4) we put $c_2 = 0$ and omit the suffix of u_1 and c_1 for the sake of simplicity.) In this case the Fréchet derivative is given by

$$(\Delta P_{\mathrm{I}})_2 \qquad (\Delta u)'''' = 20\eta^2(\hat{u}(\Delta u)'' + \hat{u}'(\Delta u)' + \hat{u}''\Delta u) - \eta^4(120\hat{u}^2 + 16c)\Delta u.$$

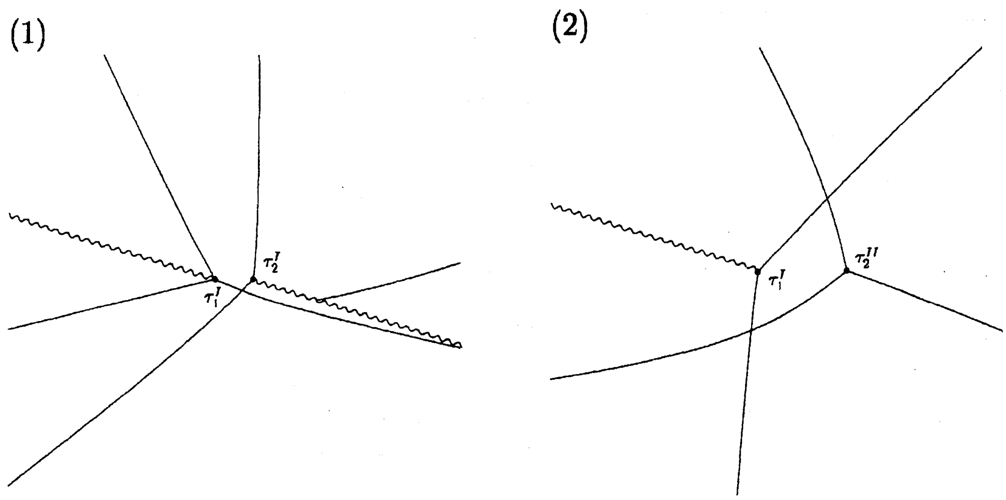
Hence the characteristic equation (of the top order part with respect to η^{-1}) of $(\Delta P_{\mathrm{I}})_2$ becomes

$$\nu^4 - 20\hat{u}_0\nu^2 + (120\hat{u}_0^2 + 16c) = 0 \tag{24}$$

where $\hat{u}_0$ satisfies an algebraic equation

$$40\hat{u}_0^3 + 16c\hat{u}_0 - 16t = 0. \tag{25}$$

Turning points and Stokes curves of $(P_{\mathrm{I}})_2$ can be computed by using (24) and (25) with the aid of a computer. Figure 1 describes the configuration of Stokes curves of $(P_{\mathrm{I}})_2$ for $c = 1 - 1.7i$. Note that the coefficients of (24) contain the algebraic function $\hat{u}_0$ and hence such configuration should be drawn on the Riemann surface $\mathcal{R}$ of $\hat{u}_0$: Figure $1(j)$ $(j = 1, 2, 3)$ shows the configuration on the j-th sheet of $\mathcal{R}$. (The wiggly lines in Figure 1 designate the cuts to describe the global structure of $\mathcal{R}$. The branch points of $\mathcal{R}$ are coincident with the turning points of the first kind, since both of them are given as the zeros of the discriminant of (25).)



(3)

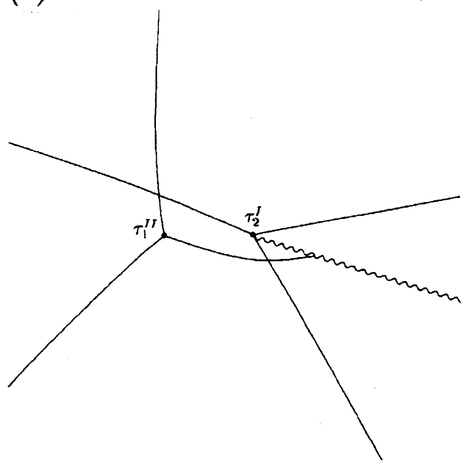


Figure 1 : Stokes curves of $(P_{\mathrm{I}})_2$ on the first sheet (1), on the second sheet (2), and on the third sheet (3) of $\mathcal{R}$.

In this case, if we take $u = \hat{u}_0$ itself as a local parameter of $\mathcal{R}$, we then readily find that this choice of parameters globally uniformizes $\mathcal{R}$ (cf. [NT]). Thus all of the three figures Figure $1(j)$ $(j = 1, 2, 3)$ can be drawn just in one sheet, i.e., in the u-plane: Figure 2 describes the configuration of Stokes curves of $(P_{\mathrm{I}})_2$ in the u-plane.

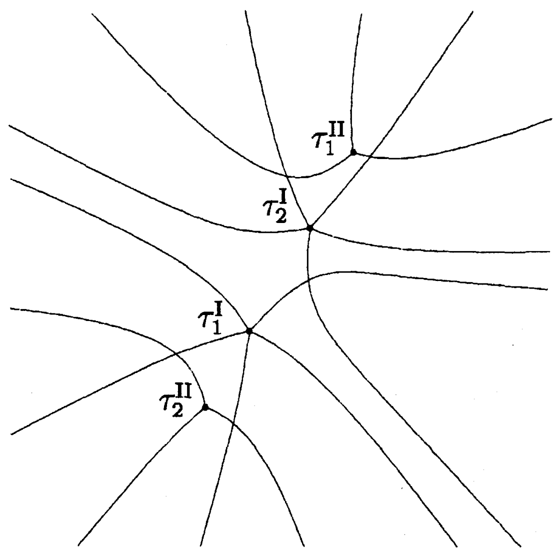


Figure 2 : Stokes curves of $(P_{\mathrm{I}})_2$ in the u-plane.

One can observe that there are several crossing points of Stokes curves in Figure 1 (or equivalently in Figure 2). As is discussed in [KKNT, Sections 3 and 4], a new Stokes curve emanates from each crossing point of Stokes curves (since in the case of $(P_{\mathrm{I}})_2$ every crossing point is "Lax-adjacent" in the terminology of [KKNT]): This is the Nishikawa phenomena for $(P_{\mathrm{I}})_2$.

In [KKNT] we interpreted the Nishikawa phenomenon as the occurrence of degeneracy of Stokes geometry of the underlying Lax pair on the new Stokes curve in question. For example, let us take a crossing point T of a Stokes curve emanating from τ_1^{I} with another Stokes curve emanating from τ_2^{II} in Figure 1(2), i.e., on the second sheet of $\mathcal{R}$ (cf. Figure 3).

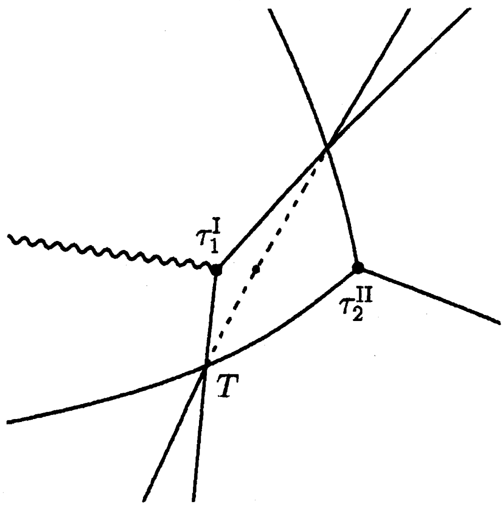


Figure 3 : Crossing point $t = T$ of two Stokes curves on the second sheet and a new Stokes curve emanating from T.

Here the Stokes curve emanating from τ_1^{I} is of type $(1,+) > (1,-)$ and defined by

$$\mathrm{Im} \int_{\tau_1^{\mathrm{I}}}^{t} (\nu_{1,+} - \nu_{1,-}) dt = 0, \tag{26}$$

and the Stokes curve emanating from τ_2^{II} is of type $(2,+) > (1,+)$ and $(1,-) > (2,-)$, defined by

$$\mathrm{Im} \int_{\tau_2^{\mathrm{II}}}^{t} (\nu_{2,+} - \nu_{1,+}) dt = \mathrm{Im} \int_{\tau_2^{\mathrm{II}}}^{t} (\nu_{1,-} - \nu_{2,-}) dt = 0. \tag{27}$$

(Concerning Stokes curves emanating from a turning point of the second kind, two Stokes curves sit on one and the same curve in general.) Since $t = T$ lies on the Stokes curve (26) and

$$2\,\mathrm{Im} \int_{a(t)}^{b_1(t)} \sqrt{Q_0(x,t)}\,dx = \frac{1}{2}\,\mathrm{Im} \int_{\tau_1^{\mathrm{I}}}^{t} (\nu_{1,+} - \nu_{1,-}) dt = 0 \tag{28}$$

holds there thanks to (22), we find a simple turning point $x = a(t)$ and a double turning point $x = b_1(t)$ are connected by a Stokes curve of $(L_{\mathrm{I}})_2$ at $t = T$. Similarly, since $t = T$ lies on the Stokes curve (27) and

$$
\begin{aligned}
(29) \qquad & 2\,\mathrm{Im} \int_{b_1(t)}^{b_2(t)} \sqrt{Q_0(x,t)}dx \\
&= \mathrm{Im} \int_{\tau_2^{\mathrm{II}}}^{t} (\nu_{2,+} - \nu_{1,+})dt = \mathrm{Im} \int_{\tau_2^{\mathrm{II}}}^{t} (\nu_{1,-} - \nu_{2,-})dt \\
&= 0
\end{aligned}
$$

holds there, the double turning point $x = b_1(t)$ and another double turning point $x = b_2(t)$ are connected by a Stokes curve at $t = T$. Thus, if we draw the configuration of Stokes geometry of (the first equation (6.a) of) the underlying Lax pair $(L_{\mathrm{I}})_2$ at $t = T$, we should find that the three turning points $x = a(t)$, $x = b_1(t)$ and $x = b_2(t)$ are simultaneously connected by Stokes curves of $(L_{\mathrm{I}})_2$. Actually, with the help of a computer, we find the following Figure 4 which describes the configuration of Stokes curves of $(L_{\mathrm{I}})_2$ at $t = T$. The new Stokes curve emanating from T is then defined

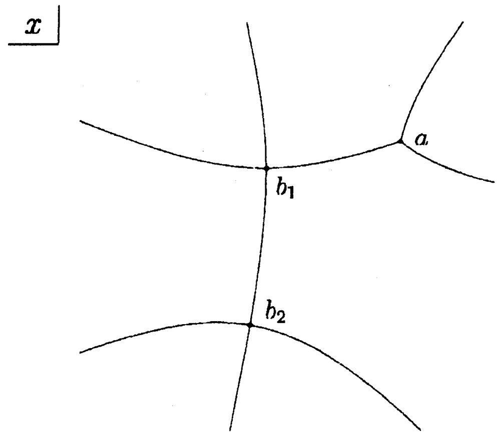


Figure 4 : Stokes curves of $(L_{\mathrm{I}})_2$ at $t = T$.

as a curve on which the two 'distant' turning points $x = a(t)$ and $x = b_2(t)$ are connected by a Stokes curve of $(L_{\mathrm{I}})_2$. As a matter of fact, the relation

$$
(30) \qquad 2\,\mathrm{Im} \int_{a(t)}^{b_2(t)} \sqrt{Q_0(x,t)}dx = \frac{1}{2}\,\mathrm{Im} \int_{T}^{t} (\nu_{2,+} - \nu_{2,-})dt
$$

holds (cf. [KKNT, Theorem 4.1]) and hence on the new Stokes curve in question we have

$$
(31) \qquad \mathrm{Im} \int_{a(t)}^{b_2(t)} \sqrt{Q_0(x,t)}dx = 0,
$$

as the definition of the new Stokes curve is given by vanishing of the right-hand side of (30). (See [KKNT, Section 4] for details.)

5 Virtual turning points

In this section we discuss a new Stokes curve from the viewpoint of virtual turning points; we first introduce the notion of a "virtual turning point" for $(P_{\mathrm{I}})_m$ and consider a new Stokes curve as a Stokes curve emanating from a virtual turning point.

For the illustration of our discussion let us continue discussing the fourth order Painlevé-I equation $(P_{\mathrm{I}})_2$ and particularly the new Stokes curve of it passing through the crossing point T of Stokes curves on the second sheet. We first recall that, as was explained in the preceding section, the new Stokes curve in question is defined by (31). Note that (22) and (23) (cf. (28) and (29) also) lead to

$$
\begin{aligned}
(32)\qquad 2\int_{a(t)}^{b_2(t)}\sqrt{Q_0(x,t)}dx &= 2\int_{a(t)}^{b_1(t)}\sqrt{Q_0(x,t)}dx + 2\int_{b_1(t)}^{b_2(t)}\sqrt{Q_0(x,t)}dx \\
&= \frac{1}{2}\int_{\tau_1^{\mathrm{I}}}^{t}(\nu_{1,+}-\nu_{1,-})dt + \int_{\tau_2^{\mathrm{II}}}^{t}(\nu_{2,+}-\nu_{1,+})dt \\
&= \frac{1}{2}\left(\int_{\tau_2^{\mathrm{II}}}^{t}(\nu_{2,+}-\nu_{1,+})dt + \int_{\tau_1^{\mathrm{I}}}^{t}(\nu_{1,+}-\nu_{1,-})dt + \int_{\tau_2^{\mathrm{II}}}^{t}(\nu_{1,-}-\nu_{2,-})dt\right).
\end{aligned}
$$

Letting $I(t)$ denote the quantity in the most right-hand side of (32), we now pick up a point $t=\omega$ satisfying

$$
(33)\qquad 2\int_{a(\omega)}^{b_2(\omega)}\sqrt{Q_0(x,\omega)}dx = I(\omega) = 0.
$$

(The existence of such a point $t=\omega$ has been already discussed in [KKNT, Remark 4.1].) Then we obtain

$$
\begin{aligned}
(34)\qquad & 2\int_{a(t)}^{b_2(t)}\sqrt{Q_0(x,t)}dx \\
&= I(t) - I(\omega) \\
&= \frac{1}{2}\left(\int_{\omega}^{t}(\nu_{2,+}-\nu_{1,+})dt + \int_{\omega}^{t}(\nu_{1,+}-\nu_{1,-})dt + \int_{\omega}^{t}(\nu_{1,-}-\nu_{2,-})dt\right). \\
&= \frac{1}{2}\int_{\omega}^{t}(\nu_{2,+}-\nu_{2,-})dt.
\end{aligned}
$$

Hence the new Stokes curve passing through T can be described also by

$$
(35)\qquad \operatorname{Im}\int_{\omega}^{t}(\nu_{2,+}-\nu_{2,-})dt = 0,
$$

showing that it is a Stokes curve emanating from $t=\omega$!

Remark 2. The point $t = \omega$ can be regarded as a virtual turning point of the Fréchet derivative $(\Delta P_{\mathrm{I}})_2$ in the following sense:

For a higher order linear ordinary differential operator with a large parameter η a virtual turning point is defined as (the projection onto the space of independent variable, i.e., the t-space in the case of $(\Delta P_{\mathrm{I}})_2$, of) a self-intersection point of the bicharacteristic curve of its Borel transform with respect to the large parameter η (cf. [AKT]). In the case of $(\Delta P_{\mathrm{I}})_2$, if we ignore the singularities in the lower order terms of $(\Delta P_{\mathrm{I}})_2$, the bicharacteristic curve is locally given by

$$\{(t,y) \in \mathbb{C}^2 \,;\, y = \int^t \nu_{j,*} dt\} \qquad (j = 1, 2 \text{ and } * = \pm) \tag{36}$$

(cf. [T, Section 3.2]). Note that the y-component of the bicharacteristic curve is determined only up to an additive constant. Now at the turning point τ_1^{I} two branches

$$(t, \int_{\tau_1^{\mathrm{I}}}^t \nu_{1,+} dt) \quad \text{and} \quad (t, \int_{\tau_1^{\mathrm{I}}}^t \nu_{1,-} dt) \tag{37}$$

of the bicharacteristic curve meet as τ_1^{I} is a simple turning point of $(\Delta P_{\mathrm{I}})_2$. (To be more precise, the two branches form a cusp near τ_1^{I}, while the bicharacteristic strip, i.e., the lift of the bicharacteristic curve to the cotangent space, defines a smooth curve there.) These two branches (37) are prolonged to a neighborhood of the other turning point τ_2^{II} and have the following expression there:

$$(t, \int_{\tau_1^{\mathrm{I}}}^{\tau_2^{\mathrm{II}}} \nu_{1,+} dt + \int_{\tau_2^{\mathrm{II}}}^t \nu_{1,+} dt) \quad \text{and} \quad (t, \int_{\tau_1^{\mathrm{I}}}^{\tau_2^{\mathrm{II}}} \nu_{1,-} dt + \int_{\tau_2^{\mathrm{II}}}^t \nu_{1,-} dt). \tag{38}$$

Then, by a similar reasoning, we find that the two branches (38) respectively meet the following branches at τ_2^{II}:

$$(t, \int_{\tau_1^{\mathrm{I}}}^{\tau_2^{\mathrm{II}}} \nu_{1,+} dt + \int_{\tau_2^{\mathrm{II}}}^t \nu_{2,+} dt) \quad \text{and} \quad (t, \int_{\tau_1^{\mathrm{I}}}^{\tau_2^{\mathrm{II}}} \nu_{1,-} dt + \int_{\tau_2^{\mathrm{II}}}^t \nu_{2,-} dt). \tag{39}$$

Now we consider a crossing point of the two branches given by (39) (cf. Figure 5). Such a crossing point (i.e., a self-intersection point of the bicharacteristic curve) is determined by

$$\int_{\tau_1^{\mathrm{I}}}^{\tau_2^{\mathrm{II}}} \nu_{1,+} dt + \int_{\tau_2^{\mathrm{II}}}^t \nu_{2,+} dt = \int_{\tau_1^{\mathrm{I}}}^{\tau_2^{\mathrm{II}}} \nu_{1,-} dt + \int_{\tau_2^{\mathrm{II}}}^t \nu_{2,-} dt, \tag{40}$$

which is equivalent to $I(t) = 0$. Hence the point $t = \omega$ can be regarded as a a virtual turning point of $(\Delta P_{\mathrm{I}})_2$.

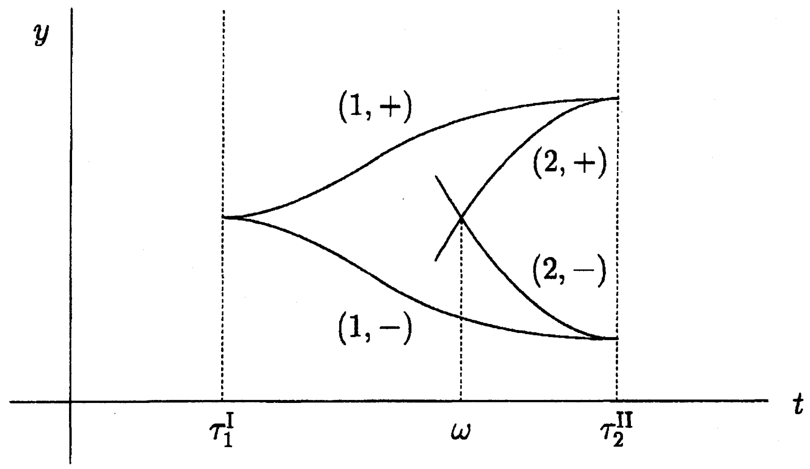


Figure 5 : Schematic illustration of the bicharacteristic curve of $(\Delta P_{\mathrm{I}})_2$. (The symbol $(1,+)$ etc. designate the branch given by (36) with $(j,*) = (1,+)$ etc.)

In view of Remark 2, it is appropriate to call the point $t = \omega$ to be a "virtual turning point" of $(P_{\mathrm{I}})_2$. One important point here is that $t = \omega$ is defined not only by $I(\omega) = 0$ but also by the equation $2\int_{a(\omega)}^{b_2(\omega)} \sqrt{Q_0(x,\omega)}dx = 0$ which is equivalent to $I(\omega) = 0$, that is, $t = \omega$ is defined in terms of the integral associated with the underlying Lax pair $(L_{\mathrm{I}})_2$. Having this fact in mind, we define a virtual turning point of $(P_{\mathrm{I}})_m$ by using the underlying Lax pair $(L_{\mathrm{I}})_m$ in the following manner:

Definition 3. Let $*_k(t)$ $(k = 1, 2)$ be arbitrarily chosen two turning points of (the first equation (6.a) of) the Lax pair $(L_{\mathrm{I}})_m$ (i.e., $*_k(t) = a(t)$ or $b_j(t)$), and let C_t be an arbitrarily chosen path (in the x-space) connecting $*_1(t)$ and $*_2(t)$. Then a point $t = \omega$ satisfying

$$\int_{*_2(\omega),\ \text{along } C_\omega}^{*_1(\omega)} \sqrt{Q_0(x,\omega)}dx = 0 \tag{41}$$

is called a virtual turning point of $(P_{\mathrm{I}})_m$.

Remark 3. In the case of the Painlevé-I hierarchy, as there exists only one simple turning point, the number of possible paths C_t in (41) is finite. Furthermore, since $\sqrt{Q_0(x,t)}$ can be explicitly integrated (with respect to x) like

$$\int^x \sqrt{Q_0}dx = (\text{a polynomial in } x) \times \sqrt{x - a(t)} \tag{42}$$

in view of (20), for each choice of C_t (the square of) (41) becomes of the form

$$F(b_j, \hat{u}_{1,0}) = 0 \quad \text{or} \quad G(b_j, b_{j'}, \hat{u}_{1,0}) = 0 \tag{43}$$

according as $(*_1(t), *_2(t)) = (b_j(t), a(t))$ or $(b_j(t), b_{j'}(t))$, where $F(X, u)$ and $G(X, Y, u)$ are polynomials of (X, u) and (X, Y, u) respectively. Recalling that $b_j(t)$ is a root of

$U_0(x,t)=0$ (or, more precisely, a root of $U_0(x,\hat{u}_{1,0})=0$ since the t-dependence of U_0 comes only from its $\hat{u}_{j,0}$-dependence), we thus find that, in order to seek for a virtual turning point of $(P_{\mathrm{I}})_m$, it is sufficient to solve the following system of algebraic equations

$$F(X,u)=U_0(X,u)=0 \quad \text{or} \quad G(X,Y,u)=U_0(X,u)=U_0(Y,u)=0 \tag{44}$$

(where we put $X=b_j$, $Y=b_{j'}$ and $u=\hat{u}_{1,0}$). Such a system can be algebraically solved by using the resultant. Hence we conclude that there exist finitely many virtual turning points in the case of $(P_{\mathrm{I}})_m$.

We also define a new Stokes curve emanating from a virtual turning point as follows:

Definition 4. Let $t=\omega$ be a virtual turning point of $(P_{\mathrm{I}})_m$.
(i) When ω is defined by (41) with $*_1(t)=b_j(t)$ and $*_2(t)=a(t)$, we define a new Stokes curve emanating from ω by

$$\mathrm{Im}\int_\omega^t(\nu_{j,+}-\nu_{j,-})dt=0. \tag{45}$$

(ii) When ω is defined by (41) with $*_1(t)=b_j(t)$ and $*_2(t)=b_{j'}(t)$ $(j\neq j')$, we define a new Stokes curve emanating from ω by

$$\mathrm{Im}\int_\omega^t(\nu_{j,+}-\nu_{j',\pm})dt=\mathrm{Im}\int_\omega^t(\nu_{j',\mp}-\nu_{j,-})dt=0, \tag{46}$$

where we take $+$ sign in the first term and $-$ sign in the second term (resp. $-$ sign in the first term and $+$ sign in the second term) if C_ω does not cross (resp. does cross) a cut to define $\sqrt{Q_0(x,\omega)}$.

A new Stokes curve defined by (45) (resp. (46)) is called a new Stokes curve of type $(j,+;j,-)$ (resp. of type $(j,+;j',\pm)$ and $(j,-;j',\mp)$). (The type of a virtual turning point is defined in a similar manner.)

In parallel with Proposition 2, we then obtain the following Proposition 4, a counterpart of the relations (22) and (23), for a virtual turning point and a new Stokes curve emanating from it:

Proposition 4. *Let $t=\omega$ be a virtual turning point of $(P_{\mathrm{I}})_m$. Then we have the following relations:*
(i) *When ω is defined by (41) with $*_1(t)=b_j(t)$ and $*_2(t)=a(t)$,*

$$\frac{1}{2}\int_\omega^t(\nu_{j,+}-\nu_{j,-})dt=2\int_{a(t)}^{b_j(t)}\sqrt{Q_0(x,t)}dx \tag{47}$$

holds, where the integral of the right-hand side of (47) *should be taken along the path* C_t *that appears in the definition of* ω.
(ii) *When* ω *is defined by* (41) *with* $*_1(t) = b_j(t)$ *and* $*_2(t) = b_{j'}(t)$ $(j \neq j')$,

$$\int_\omega^t (\nu_{j,+} - \nu_{j',\pm})dt = \int_\omega^t (\nu_{j',\mp} - \nu_{j,-})dt = 2\int_{b_{j'}(t)}^{b_j(t)} \sqrt{Q_0(x,t)}dx, \tag{48}$$

holds, where the integral of the right-hand side should be taken along the path C_t *as in (i), and the sign* $\pm$ *and* $\mp$ *are chosen in the same way as in Definition 4, (ii).*

Proof. The proof is essentially the same as that of Proposition 2 (cf. [KKNT, Proposition 2.1.4]); to prove (i), let us consider the t-derivative of the right-hand side of (47). Since both endpoints $x = b_j(t)$ and $x = a(t)$ of the integral are zeros of Q_0, we find

$$\frac{\partial}{\partial t}\left(2\int_{a(t)}^{b_j(t)} \sqrt{Q_0(x,t)}dx\right) = 2\int_{a(t)}^{b_j(t)} \frac{\partial}{\partial t}\sqrt{Q_0(x,t)}dx. \tag{49}$$

Here, as is proved in [KKNT, Proposition 2.1.2],

$$\frac{\partial}{\partial t}\sqrt{Q_0(x,t)} = \frac{\partial}{\partial x}\sqrt{x + 2\hat{u}_{1,0}} \tag{50}$$

holds. It then follows from (21) that

$$\begin{aligned} \frac{\partial}{\partial t}\left(2\int_{a(t)}^{b_j(t)} \sqrt{Q_0(x,t)}dx\right) &= 2\int_{a(t)}^{b_j(t)} \frac{\partial}{\partial x}\sqrt{x + 2\hat{u}_{1,0}}dx \\ &= 2\sqrt{x + 2\hat{u}_{1,0}}\Big|_{x=b_j(t)} \\ &= \nu_{j,+} = \frac{1}{2}(\nu_{j,+} - \nu_{j,-}). \end{aligned} \tag{51}$$

As $\int_{a(\omega)}^{b_j(\omega)} \sqrt{Q_0}dx = 0$ holds by the definition of ω, integrating (51) from ω to t verifies (47). In a similar manner we can prove (ii) also. □

Remark 4. In labeling the characteristic roots $\nu_{j,\pm}$ of C_0, we implicitly used the Riemann surface of $\sqrt{Q_0}$, or cuts to define $\sqrt{Q_0}$. (See (21) and compare it with (20).) Intuitively speaking, each characteristic root $\nu_{j,\pm}$ is attached to a double turning point $x = b_j(t)$ on the Riemann surface of $\sqrt{Q_0}$. (Thus it may be better to use the notation "$x = b_{j,\pm}(t)$" from this viewpoint.) If in Definition 4 (ii) we consider C_ω to be a path on the Riemann surface of $\sqrt{Q_0}$ connecting two such double turning points $b_{j,\pm}$ and $b_{j',\pm}$, the choice of the sign in (46) is consistent with this identification between $\nu_{j,\pm}$ and $b_{j,\pm}$; for example, if C_ω does not cross a cut to define $\sqrt{Q_0}$, it connects $b_{j,+}(t)$ and $b_{j',+}(t)$ (and simultaneously $b_{j,-}(t)$ and $b_{j',-}(t)$). Then the choice of the sign in (46) immediately follows from the above identification. Note that, from this point of view, the 'true' path in Definition 4 (i) is not C_ω, but rather its double cover $\tilde{C}_\omega$, i.e., double cover (on the Riemann surface of $\sqrt{Q_0}$) of C_ω that connects $b_{j,+}(\omega)$ and $b_{j,-}(\omega)$ and passes through $a(\omega)$.

6 Complete description of the Stokes geometry

In the preceding section we gave the definition of virtual turning points and new Stokes curves emanating from them. As is exemplified by $(P_{\mathrm{I}})_2$, we have to take such virtual turning points and new Stokes curves into account to obtain the correct global Stokes geometry for $(P_{\mathrm{I}})_m$. On the other hand, Definitions 3 and 4 have given us sufficiently many virtual turning points and new Stokes curves in the following sense: If we add all the virtual turning points and new Stokes curves given by Definitions 3 and 4 to the ordinary turning points and Stokes curves, we then obtain a "saturated Stokes geometry". Here we say that a Stokes geometry, i.e., the collection of (ordinary and virtual) turning points and (ordinary and new) Stokes curves, is saturated (in the sense that all the possibilities are exhausted) if every crossing point of (ordinary and/or new) Stokes curves in the Stokes geometry in question belongs to one of the following three types : (In the description below the indices j, k, l and m are assumed to be mutually distinct.)

Type (a) ("two 1st & two 2nd")

A type (a) crossing point (or a "two 1st & two 2nd crossing point") is a crossing point of the following four Stokes curves:

$$\left(\begin{array}{l}\text{a 1st kind Stokes curve of type } (j,+;j,-),\\ \text{a 1st kind Stokes curve of type } (k,+;k,-),\\ \text{a 2nd kind Stokes curve of type } (j,+;k,+) \text{ and } (j,-;k,-),\\ \text{a 2nd kind Stokes curve of type } (j,+;k,-) \text{ and } (j,-;k,+).\end{array}\right.$$

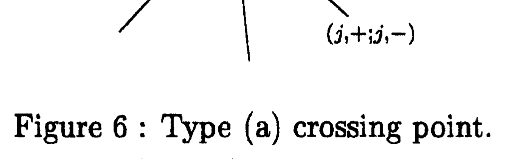


Figure 6 : Type (a) crossing point.

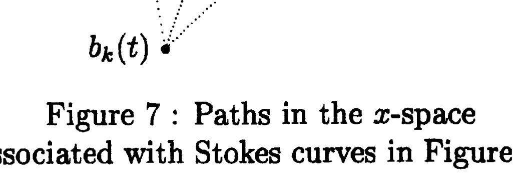


Figure 7 : Paths in the x-space associated with Stokes curves in Figure 6.

Type (b) ("three 2nd")

A type (b) crossing point (or a "three 2nd crossing point") is a crossing point of the following three Stokes curves:

$$\left(\begin{array}{l}\text{a 2nd kind Stokes curve of type } (j,+;k,+) \text{ and } (j,-;k,-),\\ \text{a 2nd kind Stokes curve of type } (k,+;l,+) \text{ and } (k,-;l,-),\\ \text{a 2nd kind Stokes curve of type } (j,+;l,+) \text{ and } (j,-;l,-),\end{array}\right.$$

(or the pattern where the sign $\pm$ associated with an index, say, l is interchanged like "type $(j,+;k,+)$ and $(j,-;k,-)$, type $(k,+;l,-)$ and $(k,-;l,+)$, and type $(j,+;l,-)$ and $(j,-;l,+)$").

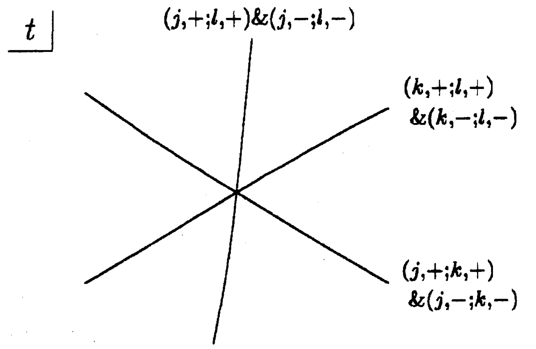


Figure 8 : Type (b) crossing point.

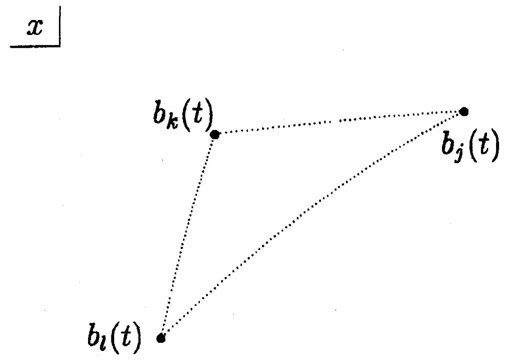


Figure 9 : Paths in the x-space associated with Stokes curves in Figure 8.

<u>Type (c)</u> ("disjoint")

A type (c) crossing point (or a "disjoint crossing point") is a crossing point of the following two Stokes curves:

<u>Type (c-1)</u>

$$\begin{pmatrix} \text{a 1st kind Stokes curve of type } (j,+;j,-), \\ \text{a 2nd kind Stokes curve of type } (k,+;l,+) \text{ and } (k,-;l,-), \end{pmatrix}$$

(or a 2nd kind Stokes curve of type $(k,+;l,-)$ and $(k,-;l,+)$).

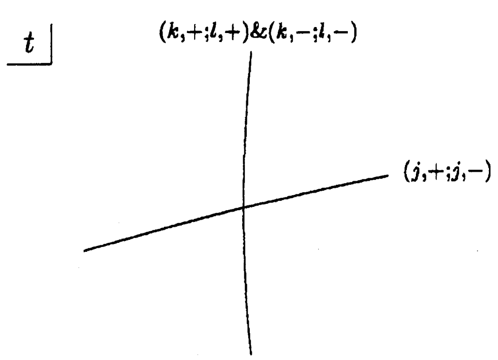


Figure 10 : Type (c-1) crossing point.

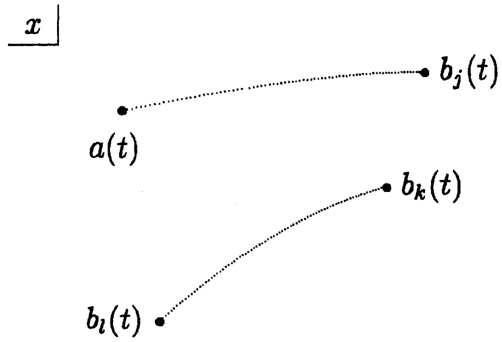


Figure 11 : Paths associated with Stokes curves in Figure 10.

<u>Type (c-2)</u>

$$\begin{pmatrix} \text{a 2nd kind Stokes curve of type } (j,+;k,+) \text{ and } (j,-;k,-), \\ \text{a 2nd kind Stokes curve of type } (l,+;m,+) \text{ and } (m,-;l,-), \end{pmatrix}$$

(or the pattern(s) where the sign $\pm$ associated with k and/or m is interchanged).

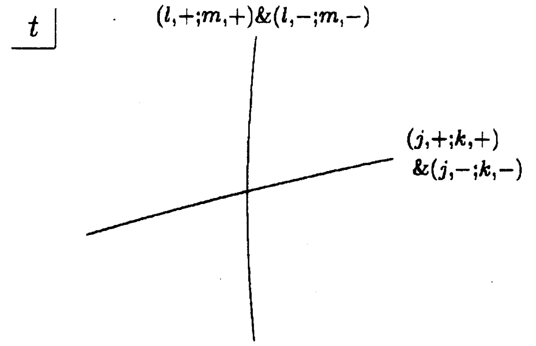


Figure 12 : Type (c-2) crossing point.

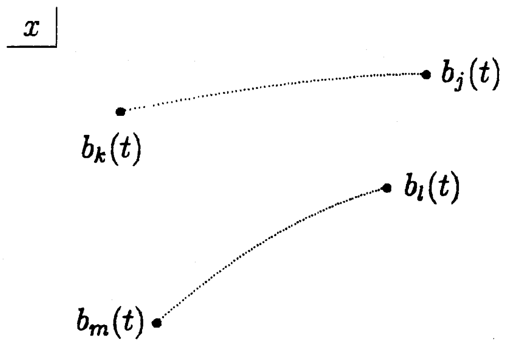


Figure 13 : Paths associated with Stokes curves in Figure 12.

Let us explain the reason why we obtain a saturated Stokes geometry if we consider all the virtual turning points and new Stokes curves together with the ordinary turning points and Stokes curves. A key point is that the above local data near a crossing point of Stokes curves in the t-space can be translated into the global data in the x-space. In fact, as is claimed in Proposition 2 or 4, a path (in the x-space) connecting two turning points of the underlying Lax pair $(L_{\mathrm{I}})_m$ is associated with each (ordinary or new) Stokes curve of $(P_{\mathrm{I}})_m$ via the integral relations like (22), (23), (47) or (48). Thus, if we take a crossing point of two (ordinary and/or new) Stokes curves, we are given two such paths in the x-space associated with them and four turning points of $(L_{\mathrm{I}})_m$ being endpoints of these two paths. Then, concerning the combination of the four endpoints, we have the following three cases:

(1) The two paths share one endpoint and consequently we are given three endpoints, among which a simple turning point is included.

(2) The two paths share one endpoint and consequently we are given three endpoints, all of which are double turning points.

(3) The four endpoints (turning points) are mutually disjoint.

The case (1) corresponds to a type (a) crossing point: At such a crossing point three turning points $a(t)$, $b_j(t)$ and $b_k(t)$ are relevant in the x-space. Then, as a path of integration for $\sqrt{Q_0}$ connecting two of them, we may consider four possible paths, as is shown in Figure 7. (Note that $\sqrt{Q_0}$ is holomorphic at a double turning point, while a simple turning point is a square-root type singular point of $\sqrt{Q_0}$.) Since the imaginary part of the integral of $\sqrt{Q_0}$ along two of such four possible paths vanishes by the assumption that the point in question is a crossing point of two Stokes curves, the imaginary part of the integral along all of these four paths should vanish. As we have exhaustively taken into account all the possible paths in defining virtual turning points and new Stokes curves, this means that four Stokes curves must cross at the point in question and hence we conclude that such a crossing point is a type (a) crossing point. By a similar argument we can also confirm that the case (2) (resp. (3)) corresponds to a type (b) (resp. (c)) crossing point. Thus, if we consider all the (ordinary and virtual) turning points and (ordinary and new) Stokes curves, only the three types of crossing points of Stokes curves may appear.

Remark 5. For $(P_{\mathrm{I}})_2$ neither type (b) nor type (c) crossing points appear, since the underlying Lax pair $(L_{\mathrm{I}})_2$ has just two double turning points. Similarly type (c-2) crossing points do not appear for $(P_{\mathrm{I}})_3$.

In this way, by adding virtual turning points and new Stokes curves, we obtain a saturated Stokes geometry of $(P_{\mathrm{I}})_m$. However, to obtain a "complete Stokes geometry" of $(P_{\mathrm{I}})_m$, i.e., its correct global Stokes geometry, we still need to discuss the "effectiveness" or "activity" of Stokes curves. That is, on each portion of Stokes curves we have to check whether the degeneracy of Stokes geometry of the underlying Lax pair $(L_{\mathrm{I}})_m$ does really occur or not. (On each Stokes curve we have the relation

$$\mathrm{Im} \int_{*_1(t)}^{*_2(t)} \sqrt{Q_0}\, dx = 0 \tag{52}$$

with some turning points $*_1(t)$ and $*_2(t)$ of $(L_{\mathrm{I}})_m$, but (52) does not necessarily imply the degeneracy of Stokes geometry of $(L_{\mathrm{I}})_m$. See [AKT, p.80] and [KKNT, Remark 4.1].)

Concerning the problem of activity of Stokes curves, we first note the following

Proposition 5. *A new Stokes curve is not active near a virtual turning point, that is, no degeneracy of the Stokes geometry of* $(L_{\mathrm{I}})_m$ *occurs on a new Stokes curve near a virtual turning point.*

Proof. Assume that a virtual turning point $t = \omega$ is not an ordinary turning point and that it is defined by

$$\int_{*_2}^{*_1} \sqrt{Q_0}\, dx = 0 \tag{53}$$

with some turning points $*_1$ and $*_2$ of $(L_{\mathrm{I}})_m$. If the degeneracy of Stokes geometry of $(L_{\mathrm{I}})_m$ were to occur on a new Stokes curve emanating from $t = \omega$, the turning points $*_1$ and $*_2$ should be connected by a Stokes curve γ of $(L_{\mathrm{I}})_m$ at $t = \omega$. Since

$$\int_{*_2}^{x} \sqrt{Q_0}\, dx \tag{54}$$

is a real-valued monotone function (of x) on γ, it then follows from (53) that $*_1$ should coincide with $*_2$. This means that $t = \omega$ should be an ordinary turning point, contradicting the assumption. □

Hence, as in the case of higher order linear equations, the portion of a new Stokes curve of $(P_{\mathrm{I}})_m$ containing a virtual turning point should be ignored in the

Stokes geometry (i.e., be drawn by a dotted line). On the other hand, in view of Proposition 2, we should keep solid the portion of an ordinary Stokes curve near an ordinary turning point. Thus the activity of Stokes curves is completely determined near turning points.

Note that the degeneracy of Stokes geometry of $(L_{\mathrm{I}})_m$, i.e., the existence of a Stokes curve connecting two turning points, may be resolved only when another turning point of $(L_{\mathrm{I}})_m$ comes across the Stokes curve in question. Since such a phenomenon occurs only at a crossing point of Stokes curves of $(P_{\mathrm{I}})_m$, we find that the activity of a Stokes curve of $(P_{\mathrm{I}})_m$ changes only at a crossing point of Stokes curves. Thus, from now on, we consider classification of all the 'admissible' patterns for the activity of Stokes curves at each type of crossing points. Let us first discuss a type (a) (i.e., two 1st & two 2nd) crossing point of Stokes curves. At such a crossing point three turning points $a(t)$, $b_j(t)$ and $b_k(t)$ of $(L_{\mathrm{I}})_m$ are relevant in the x-space (cf. Figure 7). Concerning the occurrence of degeneracy of the Stokes geometry of $(L_{\mathrm{I}})_m$, we have the following three cases:

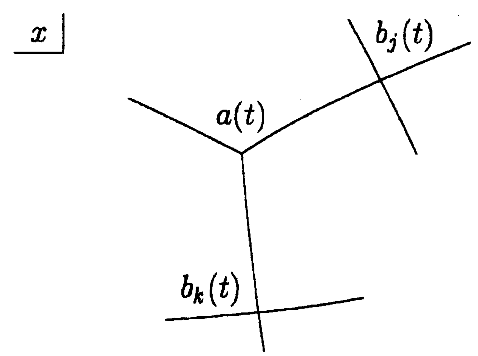
(i) No pair of the three turning points is connected by a Stokes curve of $(L_{\mathrm{I}})_m$.

(ii) Only two of them are connected by a Stokes curve of $(L_{\mathrm{I}})_m$.

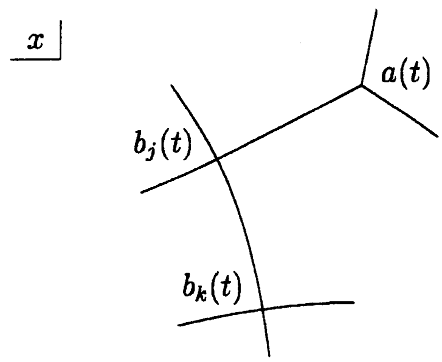
(iii) All of them are connected by (two) Stokes curves of $(L_{\mathrm{I}})_m$.

In Case (i) all Stokes curves of $(P_{\mathrm{I}})_m$ passing through the crossing point in question are inactive (i.e., should be drawn by a dotted line), while only one Stokes curve is active and the others are inactive in Case (ii). Case (iii) can be further classified into the following three subcases:

Case (iii-1)



Case (iii-2)



Case (iii-3)

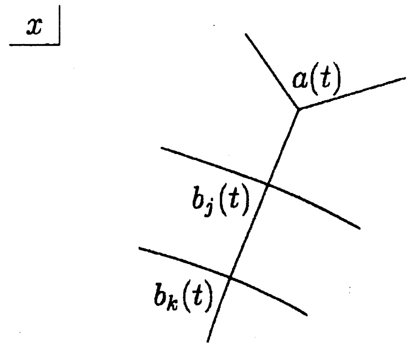


Figure 14 : Stokes geometry of $(L_{\mathrm{I}})_m$ in Case (iii).

All of these three subcases have already been discussed in [KKNT, Section 4]; Cases (iii-1) and (iii-2) are Lax-adjacent crossing points and Case (iii-3) is non-Lax-adjacent. The corresponding admissible patterns for the activity of Stokes curves of $(P_{\mathrm{I}})_m$ will be given in Figure 15 below. Thus the classification of all the admissible patterns at a type (a) crossing point is now completed.

In a similar manner we can classify all the admissible patterns also at type (b) and (c) crossing points. The following is a list of all the admissible patterns for the activity of Stokes curves at each type of crossing points:

List of the admissible patterns for the activity of Stokes curves at each type of crossing points

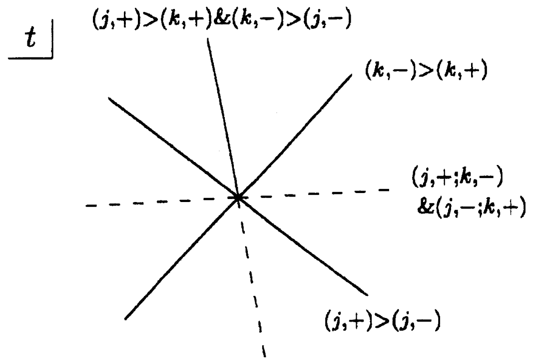
Type (a) ("two 1st & two 2nd")

(i) All curves are dotted.

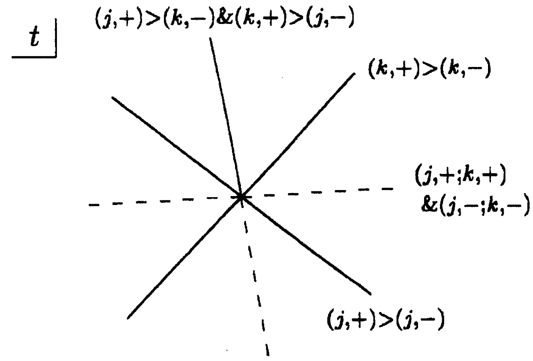
(ii) Only one curve is solid, the others are dotted.

(iii) (See below.)

Case (iii-1)



Case (iii-1)′



Case (iii-2) Case (iii-3)

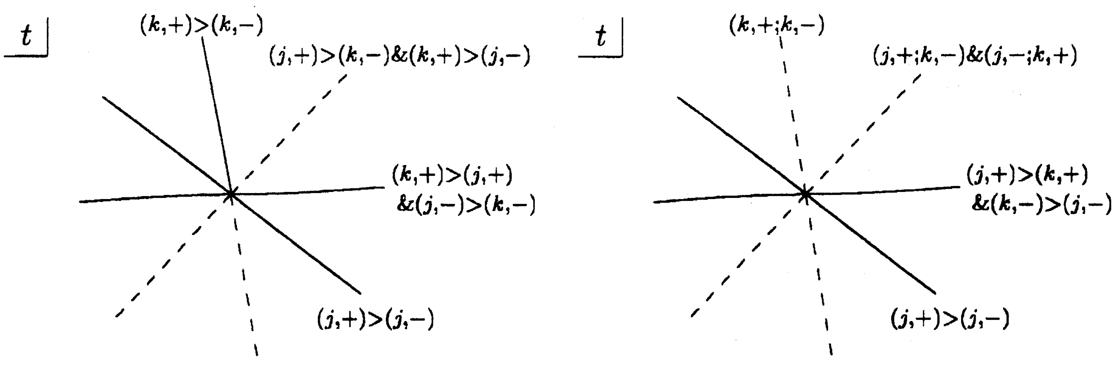


Figure 15 : Admissible patterns at a type (a) crossing point (in Case (iii)).

Type (b) ("three 2nd")

(i) All curves are dotted.

(ii) Only one curve is solid, the others are dotted.

(iii) (See below.)

Case (iii-1) Case (iii-2)

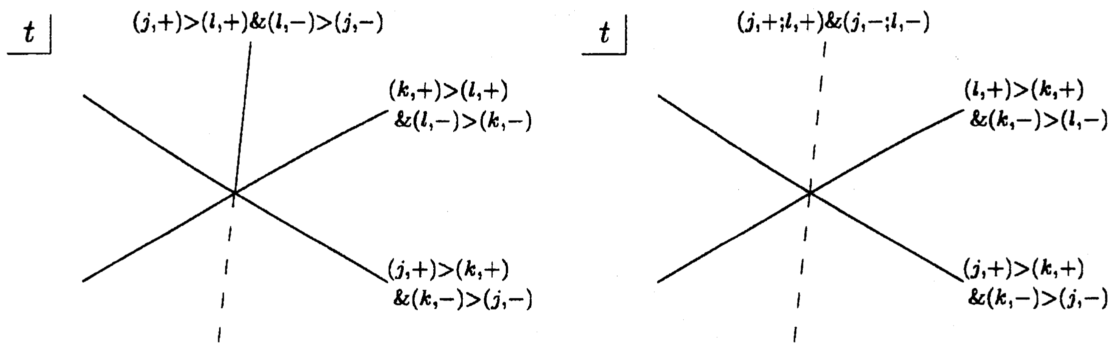


Figure 16 : Admissible patterns at a type (b) crossing point (in Case (iii)).

Type (c) ("disjoint")

(i) All curves are dotted.

(ii) Only one curve is solid, the others are dotted.

(iii) Both curves are solid.

Remark 6. As in the description of each type of crossing points, i.e., as in Figures 6, 8, 10 and 12, the combination of the types of Stokes curves is not completely listed and some interchange of the sign $\pm$ is allowed in Figures 15 and 16. In these figures the placement of Stokes curves is not specified, either.

By the same reasoning as in [KKNT, Remark 4.1] we can verify that a point $t = t_0$ where the degeneracy of Stokes geometry of $(L_{\mathrm{I}})_m$ is observed should lie on an ordinary or new Stokes curve of $(P_{\rm I})_m$. Hence the "complete Stokes geometry" of $(P_{\rm I})_m$, i.e., the collection of points where the degeneracy of Stokes geometry of the underlying Lax pair $(L_{\rm I})_m$ is observed, should consist of the (ordinary and virtual) turning points and the (ordinary and new) Stokes curves. Furthermore, in the complete Stokes geometry the pattern of the activity of Stokes curves at each crossing point should necessarily belong to the above list. Thus, as the procedure for determining the complete Stokes geometry of $(P_{\rm I})_m$, we can propose the following:

Procedure for determining the complete Stokes geometry

1°) Draw the Stokes curves emanating from ordinary turning points.

2°) Locate all the virtual turning points and draw the new Stokes curves emanating from them.

3°) The portion of a new Stokes curve containing a virtual turning point should be ignored in the Stokes geometry (i.e., be drawn by a dotted line).

4°) The portion of an ordinary Stokes curve adjacent to an ordinary turning point should be kept solid.

5°) We determine the activity of each portion of Stokes curves so that, in addition to 3°) and 4°), the pattern of the activity at every (type (a), (b) or (c)) crossing point of Stokes curves may belong to the above list.

6°) The complete Stokes geometry is then given by the collection of the turning points and solid (active) portions of Stokes curves determined by 5°).

If the activity of each portion of Stokes curves is uniquely determined in a globally consistent manner by 5°), the Stokes geometry thus obtained is nothing but the complete Stokes geometry. For example, as we shall see in what follows, we can obtain the complete Stokes geometry of $(P_{\rm I})_2$ and that of $(P_{\rm I})_3$ by following the above Procedure.

Example 1 (revisited). In the case of the 4th order Painlevé-I equation $(P_{\rm I})_2$, if we add virtual turning points and new Stokes curves to ordinary turning points and ordinary Stokes curves, we obtain Figure 17. (In Figure 17 (and in Figures 18, 20 and 21 below as well) virtual turning points are denoted by small dots, while ordinary turning points are denoted by large dots.) Furthermore, using 3°), 4°) and 5°) of the above Procedure, we can uniquely determine the activity of each portion of Stokes curves, as is shown in Figure 18. Thus Figure 18 gives a complete description of the global Stokes geometry for $(P_{\rm I})_2$.

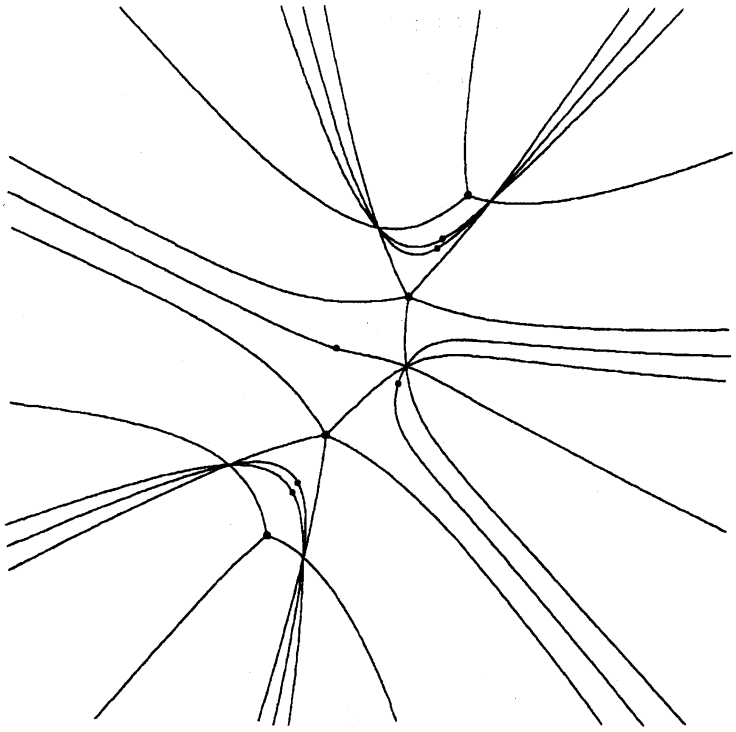


Figure 17 : Saturated Stokes geometry of $(P_{\mathrm{I}})_2$ in the u-plane.

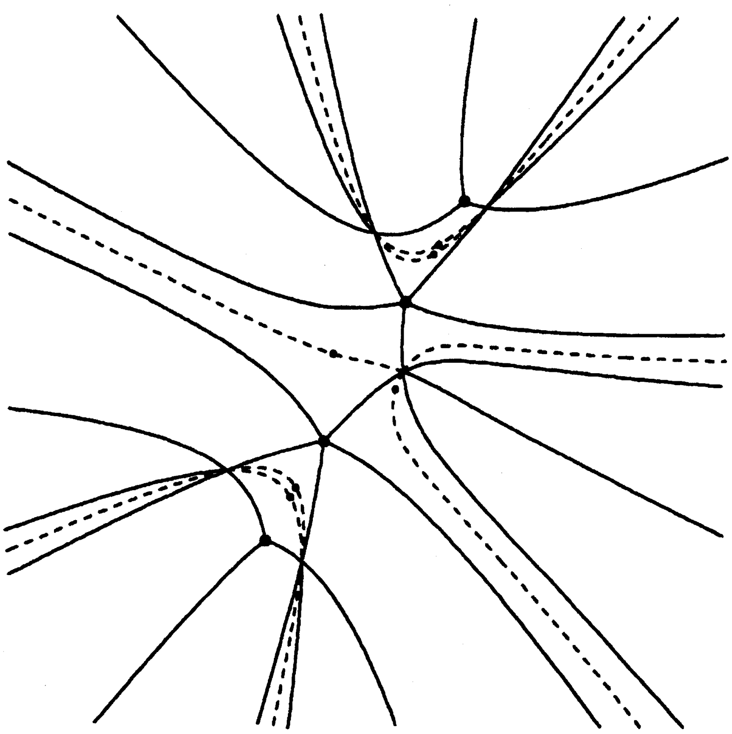


Figure 18 : Complete Stokes geometry of $(P_{\mathrm{I}})_2$ in the u-plane.

Example 2. (6th order Painlevé-I equation)

$$(P_{\mathrm{I}})_3 . \qquad u^{(6)} = \eta^2(28uu^{(4)} + 56u'u^{(3)} + 42(u'')^2) - \eta^4(280u^2u'' + 280u(u')^2 + 16c_1u'') + \eta^6(280u^4 + 96c_1u^2 - 64c_2u - 32c_1^2 + 64t).$$

Similarly to the case of $(P_{\mathrm{I}})_2$ we can take $u = \hat{u}_0$ as a globally uniformizing parameter of its Riemann surface $\mathcal{R}$ (cf. [NT]). Figure 19 describes the configuration of ordinary Stokes curves of $(P_{\mathrm{I}})_3$ in the u-plane.

Just like $(P_{\mathrm{I}})_2$, adding virtual turning points and new Stokes curves to Figure 19 and using 3°), 4°) and 5°) of the above Procedure to determine the activity of each portion of Stokes curves, we obtain Figure 20 and Figure 21. Thus Figure 21 gives a complete description of the global Stokes geometry for $(P_{\mathrm{I}})_3$.

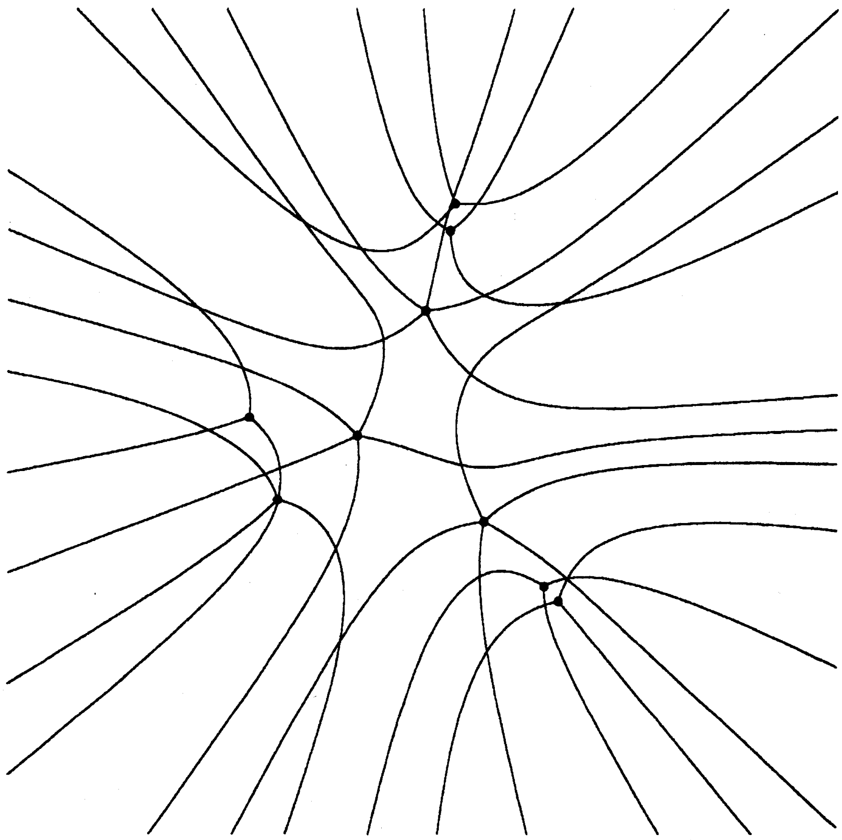


Figure 19 : Stokes curves of $(P_{\mathrm{I}})_3$ (in the u-plane).

Remark 7. The procedure for determining the complete Stokes geometry can be applied in principle to other (hierarchies of) higher order Painlevé equations, as long as their underlying Lax pairs are 2×2 linear systems. There are, however, some difficulties to obtain a complete description of the global Stokes geometry for a higher order Painlevé equation by this method: One difficulty is that the above

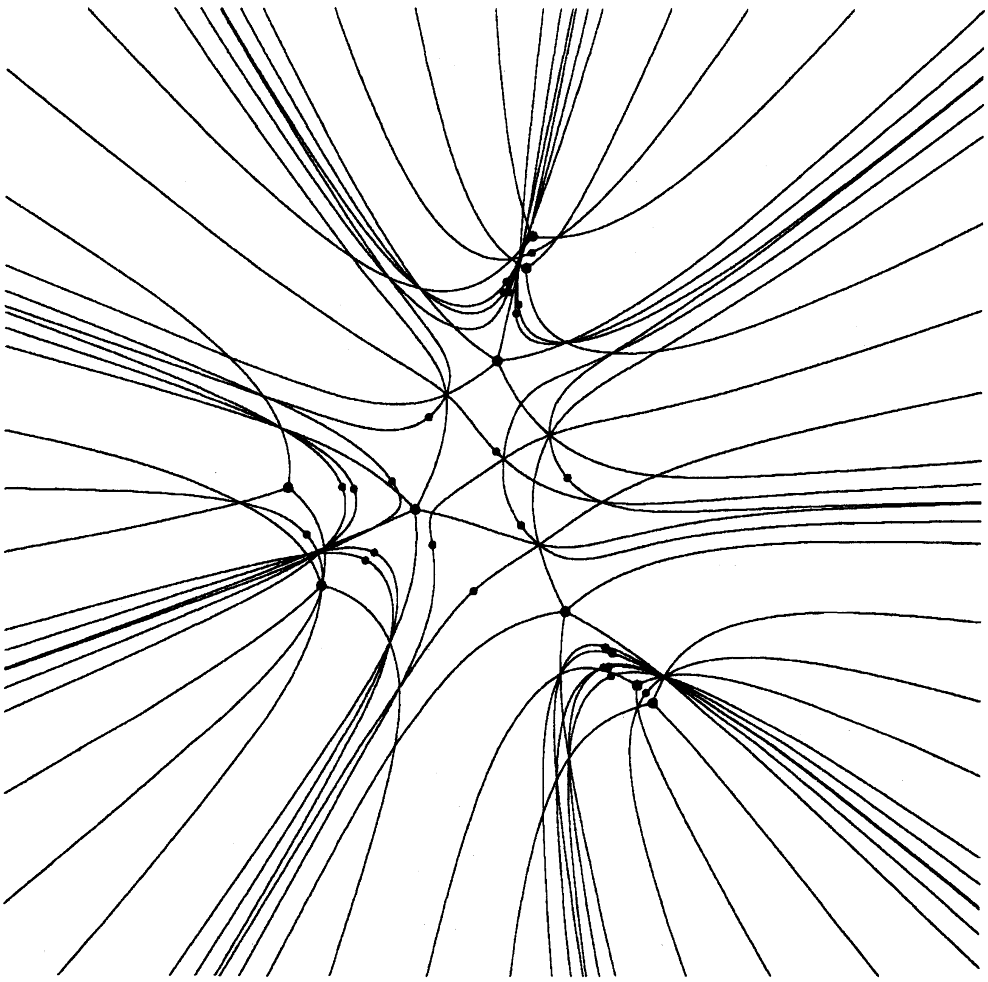


Figure 20 : Saturated Stokes geometry of $(P_{\mathrm{I}})_3$.

classification (of the types) of crossing points of Stokes curves in a saturated Stokes geometry may not be complete in general, and another one is that infinitely many virtual turning points may appear for higher order Painlevé equations (except for the Painlevé-I hierarchy). Both difficulties originate from the fact that the underlying Lax pair has several simple turning points and consequently there exist nontrivial period integrals $\oint \sqrt{Q_0}dx$. Among them the second difficulty is more serious; as in the case of higher order linear equations, how to deal with infinitely many redundant virtual turning points is an important open problem.

Acknowledgement

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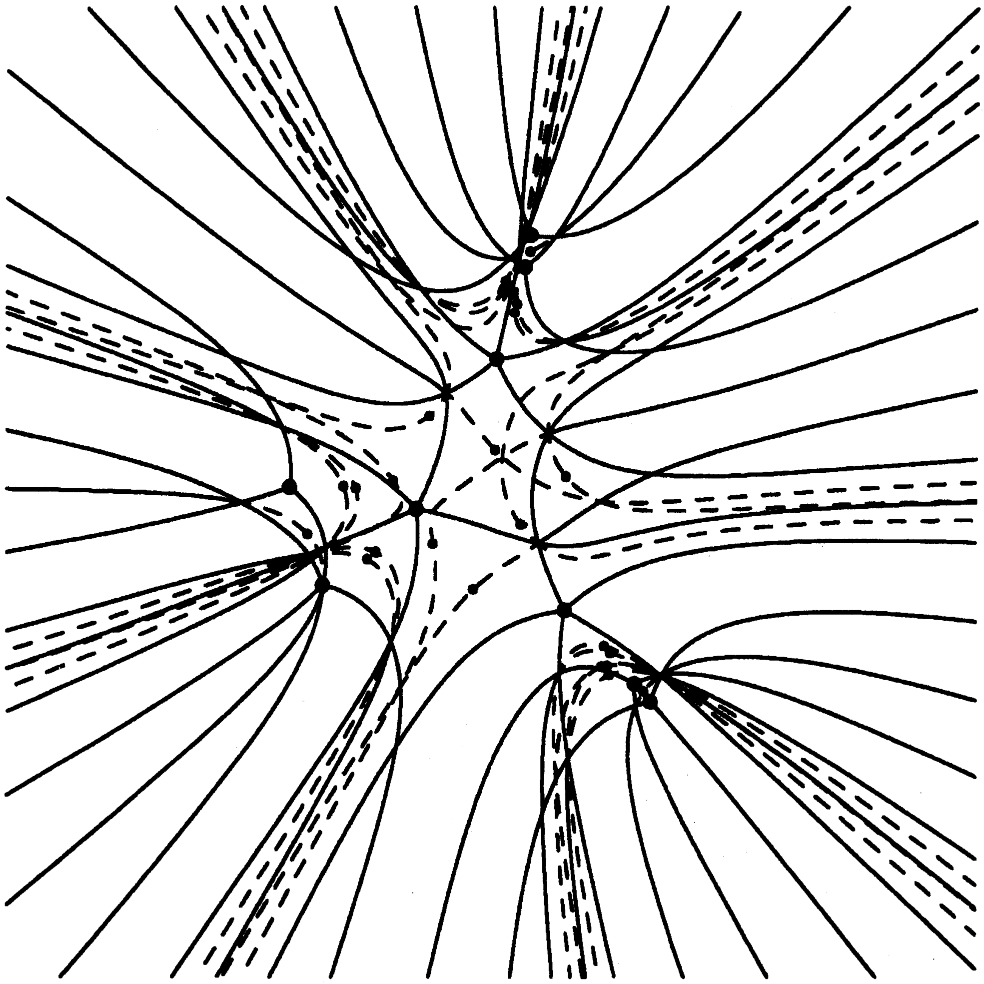

Figure 21 : Complete Stokes geometry of $(P_{\mathrm{I}})_3$.

References

[AKT] T. Aoki, T. Kawai and Y. Takei: New turning points in the exact WKB analysis for higher order ordinary differential equations, Analyse algébrique des perturbations singulières, I; Méthodes résurgentes, Hermann, 1994, pp. 69–84.

[BNR] H. L. Berk, W. M. Nevins and K. V. Roberts: New Stokes' line in WKB theory, J. Math. Phys., **23**(1982), 988–1002.

[GP] P.R. Gordoa and A. Pickering: Nonisospectral scattering problems: A key to integrable hierarchies, J. Math. Phys., **40**(1999), 5749–5786.

[KKNT] T. Kawai, T. Koike, Y. Nishikawa and Y. Takei: On the Stokes geometry of higher order Painlevé equations, RIMS Preprint No. 1443, 2004.

[KT1] T. Kawai and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. I, Adv. Math., **118**(1996), 1–33.

[KT2] ———: Algebraic Analysis of Singular Perturbations, Iwanami, Tokyo, 1998. (In Japanese. An English translation is to be published by A.M.S.)

[K] N. A. Kudryashov: The first and second Painlevé equations of higher order and some relations between them, Phys. Lett. A, **224**(1997), 353–360.

[KS] N. A. Kudryashov and M. B. Soukharev: Uniformization and transcendence of solutions for the first and second Painlevé hierarchies, Phys. Lett. A, **237**(1998), 206–216.

[N1] Y. Nishikawa: WKB analysis of P_{II}-P_{IV} hierarchies, Master Thesis, Kyoto Univ., 2003. (In Japanese.)

[N2] ———: Towards the exact WKB analysis of P_{II}-P_{IV} hierarchies. Preprint.

[NT] Y. Nishikawa and Y. Takei: On the structure of the Riemann surface in the Painlevé hierarchies. In Prep.

[S1] S. Shimomura: Painlevé property of a degenerate Garnier system of (9/2)-type and of a certain fourth order non-linear ordinary differential equation, Ann. Scuola Norm. Sup. Pisa, **29**(2000), 1–17.

[S2] ———: On the Painlevé I hierarchy, RIMS Kôkyûroku, No. 1203, 2001, pp. 46–50.

[S3] ———: A certain expression of the first Painlevé hierarchy, preprint.

[T] Y. Takei: Exact WKB analysis, and exact steepest descent method. — A sequel to “algebraic analysis of singular perturbations” —, Sûgaku, **55**(2003), 350–367. (In Japanese. Its English translation will appear in Sûgaku Expositions.)