数理解析研究所講究録 1397 巻 2004 年 118-125



# ALGEBROID STACKS AND WKB OPERATORS

ANDREA D'AGNOLO AND PIETRO POLESELLO

## INTRODUCTION

In microlocal analysis there are natural examples of sheaves of algebras which are locally defined in a natural way, but do not glue together as a globally defined sheaf. One example, addressed in Kashiwara [7], are the rings of microdifferential operators on a contact complex manifold. Another example, addressed in Polesello-Schapira [12], are the rings of WKB operators on a symplectic complex manifold. More generally, Kontsevich [10] considers rings of deformation-quantization on algebraic Poisson manifolds.

A useful notion introduced by Kontsevich in loc.cit. is that of an algebroid stack. This is on one hand a sheafified version of the categorical realization of an algebra (algebroid) as in [11]. On the other hand, it is the linear analogue of the notion of gerbe (groupoid stack) from algebraic geometry [6].

We start by defining what an algebroid stack is, and how it is locally described. We then discuss the case of microdifferential and WKB operators, and we conclude by a conjectural structure theorem for the algebroid stack of simple modules along involutive submanifolds.

## 1. ALGEBROID STACKS

We start here by recalling the categorical realization of an algebra as in [11], and we then sheafify that construction. We assume that the reader is familiar with the basic notions from the theory of stacks which are, roughly speaking, sheaves of categories. (The classical reference is [6], and a short presentation is given e.g. in [8, 4].)

This is a report on two talks given at the workshop *Microlocal Analysis and Asymptotic Analysis* held at RIMS on March 8–12, 2004, announcing a forthcoming joint paper. The warm hospitality at RIMS is gratefully acknowlged. The first author also gratefully acknowledges hospitality at Osaka University.

Let $R$ be a commutative ring. An $R$-linear category ($R$-category for short) is a category whose sets of morphisms are endowed with an $R$-module structure, so that composition is bilinear. An $R$-functor is a functor between $R$-categories which is linear at the level of morphisms.

If $A$ is an $R$-algebra, we denote by $A^+$ the $R$-category with a single object, and with $A$ as set of morphisms. This gives a fully faithful functor from $R$-algebras to $R$-categories. If $f, g\colon A \to B$ are $R$-algebra morphisms, then transformations $f^+ \Rightarrow g^+$ correspond to elements $b \in B$ such that $bf(a) = g(a)b$ for any $a \in A$. Note that the category $\mathsf{Mod}(A)$ of left $A$-modules is $R$-equivalent to the category $\mathsf{Hom}_R(A^+, \mathsf{Mod}(R))$ of $R$-functors from $A^+$ to $\mathsf{Mod}(R)$.

Let $X$ be a topological space, and $\mathcal{R}$ a (sheaf of) commutative algebra(s). As for categories, there are natural notions of $\mathcal{R}$-linear stacks ($\mathcal{R}$-stacks for short), and of $\mathcal{R}$-functor between $\mathcal{R}$-stacks.

If $\mathcal{A}$ is an $\mathcal{R}$-algebra, we denote by $\mathcal{A}^+$ the $\mathcal{R}$-stack associated with the separated prestack $U \mapsto \mathcal{A}(U)^+$. This gives a functor from $\mathcal{R}$-algebras to $\mathcal{R}$-categories which is faithful and locally full. If $f, g\colon \mathcal{A} \to \mathcal{B}$ are $\mathcal{R}$-algebra morphisms, transformations $f^+ \Rightarrow g^+$ are described as above.

As above, the stack $\mathfrak{Mod}(\mathcal{A})$ of left $\mathcal{A}$-modules is $\mathcal{R}$-equivalent to the stack of $\mathcal{R}$-functors $\mathfrak{Hom}_{\mathcal{R}}(\mathcal{A}^+, \mathfrak{Mod}(\mathcal{R}))$. The Yoneda embedding gives a fully faithful functor

$$\mathcal{A}^+ \to \mathfrak{Hom}_{\mathcal{R}}((\mathcal{A}^+)^{\mathrm{op}}, \mathfrak{Mod}(\mathcal{R})) \approx_{\mathcal{R}} \mathfrak{Mod}(\mathcal{A}^{\mathrm{op}}) \tag{1.1}$$

into the stack of right $\mathcal{A}$-modules. (Here $\approx_{\mathcal{R}}$ denotes $\mathcal{R}$-equivalence.) This identifies $\mathcal{A}^+$ with the full substack of locally free right $\mathcal{A}$-modules of rank one.

Recall that one says a stack $\mathfrak{A}$ is non-empty if $\mathfrak{A}(X)$ has at least one object; it is locally non-empty if there exists an open covering $U = \bigcup_i U_i$ such that $\mathfrak{A}|_{U_i}$ is non-empty; and it is locally connected by isomorphisms if for any open subset $U \subset X$ and any $F, G \in \mathfrak{A}(U)$ there exists an open covering $U = \bigcup_i U_i$ such that $F|_{U_i} \simeq G|_{U_i}$ in $\mathfrak{A}(U_i)$.

**Lemma 1.1.** *Let $\mathfrak{A}$ be an $\mathcal{R}$-stack. The following are equivalent*

(1) $\mathfrak{A} \approx_{\mathcal{R}} \mathcal{A}^+$ *for an $\mathcal{R}$-algebra $\mathcal{A}$,*

(2) $\mathfrak{A}$ *is non-empty and locally connected by isomorphisms.*

We are now ready to give a definition of algebroid stack, equivalent to that in Kontsevich [10].

**Definition 1.2.** (1) An $\mathcal{R}$-algebroid stack is an $\mathcal{R}$-stack which is locally non-empty and locally connected by isomorphisms.
(2) The stack of $\mathfrak{A}$-modules is $\mathfrak{Mod}_{\mathcal{R}}(\mathfrak{A}) = \mathfrak{Hom}_{\mathcal{R}}(\mathfrak{A}, \mathfrak{Mod}(\mathcal{R}))$.

## 2. A cocycle description

We will explain here how to recover an algebroid stack from local data. The parallel discussion for the case of gerbes can be found for example in [2, 3].

Let $\mathfrak{A}$ be an $\mathcal{R}$-algebroid stack. By definition, there exists an open covering $U = \bigcup_i U_i$ such that $\mathfrak{A}|_{U_i}$ is non-empty. By Lemma 1.1 there are $\mathcal{R}$-algebras $\mathcal{A}_i$ on $U_i$ such that $\mathfrak{A}|_{U_i} \approx_{\mathcal{R}} \mathcal{A}_i^+$. Let $\Phi_i\colon \mathfrak{A}|_{U_i} \to \mathcal{A}_i^+$ and $\Psi_i\colon \mathcal{A}_i^+ \to \mathfrak{A}|_{U_i}$ be quasi-inverse to each other. On double intersections $U_{ij} = U_i \cap U_j$ there are equivalences $\Phi_{ij} = \Phi_i \Psi_j\colon \mathcal{A}_j^+|_{U_{ij}} \to \mathcal{A}_i^+|_{U_{ij}}$. On triple intersections $U_{ijk}$ there are invertible transformations $\alpha_{ijk}\colon \Phi_{ij}\Phi_{jk} \Rightarrow \Phi_{ik}$ induced by $\Psi_j \Phi_j \Rightarrow \mathrm{id}$. On quadruple intersections $U_{ijkl}$ the following diagram commutes

$$\begin{array}{ccc} \Phi_{ij}\Phi_{jk}\Phi_{kl} & \overset{\alpha_{ijk}}{\Longrightarrow} & \Phi_{ik}\Phi_{kl} \\ \Big\Downarrow{\scriptstyle \alpha_{jkl}} & & \Big\Downarrow{\scriptstyle \alpha_{ikl}} \\ \Phi_{ij}\Phi_{jl} & \overset{\alpha_{ijl}}{\Longrightarrow} & \Phi_{il}. \end{array} \tag{2.1}$$

These data are enough to reconstruct $\mathfrak{A}$ (up to equivalence), and we will now describe them more explicitly.

On double intersections $U_{ij}$, the $\mathcal{R}$-functor $\Phi_{ij}\colon \mathcal{A}_j^+ \to \mathcal{A}_i^+$ is locally induced by $\mathcal{R}$-algebra isomorphisms. There thus exist an open covering $U_{ij} = \bigcup_\alpha U_{ij}^\alpha$ and isomorphisms of $\mathcal{R}$-algebras $f_{ij}^\alpha\colon \mathcal{A}_j \to \mathcal{A}_i$ on $U_{ij}^\alpha$ such that $(f_{ij}^\alpha)^+ = \Phi_{ij}|_{U_{ij}^\alpha}$.

On triple intersections $U_{ijk}^{\alpha\beta\gamma} = U_{ij}^\alpha \cap U_{ik}^\beta \cap U_{jk}^\gamma$, we have invertible transformations $\alpha_{ijk}|_{U_{ijk}^{\alpha\beta\gamma}}\colon (f_{ij}^\alpha)^+(f_{jk}^\gamma)^+ \Rightarrow (f_{ik}^\beta)^+$. There thus exist invertible sections $a_{ijk}^{\alpha\beta\gamma} \in \mathcal{A}_i^\times(U_{ijk}^{\alpha\beta\gamma})$ such that

$$f_{ij}^\alpha f_{jk}^\gamma = \mathrm{ad}(a_{ijk}^{\alpha\beta\gamma}) f_{ik}^\beta.$$

On quadruple intersections $U_{ijkl}^{\alpha\beta\gamma\delta\epsilon\varphi} = U_{ijk}^{\alpha\beta\gamma} \cap U_{ijl}^{\alpha\delta\epsilon} \cap U_{ikl}^{\beta\delta\varphi} \cap U_{jkl}^{\gamma\epsilon\varphi}$, the diagram (2.1) corresponds to the equalities

$$a_{ijk}^{\alpha\beta\gamma} a_{ikl}^{\beta\delta\varphi} = f_{ij}^\alpha (a_{jkl}^{\gamma\epsilon\varphi}) a_{ijl}^{\alpha\delta\epsilon}.$$

Indices of hypercoverings are quite cumbersome, and we will not write them explicitely anymore[1].

Let us summarize what we just obtained.

**Proposition 2.1.** *Up to equivalence, an $\mathcal{R}$-algebroid stack is given by the following data:*

(1) *an open covering $X = \bigcup_i U_i$,*
(2) *$\mathcal{R}$-algebras $\mathcal{A}_i$ on $U_i$,*
(3) *isomorphisms of $\mathcal{R}$-algebras $f_{ij} \colon \mathcal{A}_j \to \mathcal{A}_i$ on $U_{ij}$,*
(4) *invertible sections $a_{ijk} \in \mathcal{A}_i^\times(U_{ijk})$,*

*such that*

$$\begin{cases} f_{ij} f_{jk} = \mathrm{ad}(a_{ijk}) f_{ik}, & \text{as morphisms } \mathcal{A}_k \to \mathcal{A}_i \text{ on } U_{ijk}, \\ a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijl} & \text{in } \mathcal{A}_i(U_{ijkl}). \end{cases}$$

**Remark 2.2.** Neglecting the last equality, one still has

$$(f_{ij} f_{jk}) f_{kl} = \mathrm{ad}(a_{ijk}) f_{ik} f_{kl} = \mathrm{ad}(a_{ijk} a_{ikl}) f_{il},$$

and

$$f_{ij}(f_{jk} f_{kl}) = f_{ij}\, \mathrm{ad}(a_{jkl}) f_{kl} = \mathrm{ad}(f_{ij}(a_{jkl})) f_{ij} f_{kl} = \mathrm{ad}(f_{ij}(a_{jkl}) a_{ijl}) f_{il},$$

so that

$$\mathrm{ad}(a_{ijk} a_{ikl}) = \mathrm{ad}(f_{ij}(a_{jkl}) a_{ijl}).$$

Assuming for simplicity that the $\mathcal{R}$-algebras $\mathcal{A}_i$ are central, up to a refinement there are invertible sections $c_{ijkl} \in \mathcal{R}^\times(U_{ijkl})$ such that

$$a_{ijk} a_{ikl} = c_{ijkl} f_{ij}(a_{jkl}) a_{ijl}.$$

The cohomology class in $H^3(X; \mathcal{R}^\times)$ of the Cech 3-cocycle $\{c_{ijkl}\}$ corresponds to the *obstruction attachée à un lien* of Giraud [6, §VI.2]

**Example 2.3.** If $\mathcal{A}_i = \mathcal{R}|_{U_i}$, then $f_{ij} = \mathrm{id}$. Hence, $\mathcal{R}$-algebroid stacks locally $\mathcal{R}$-equivalent to $\mathcal{R}^+$ are determined by a 2-cocycle $c_{ijk} = a_{ijk} \in \mathcal{R}^\times(U_{ijk})$. One checks that two such stacks are (globally) $\mathcal{R}$-equivalent if and only if the corresponding cocycles give the same cohomology class in $H^2(X; \mathcal{R}^\times)$.

**Example 2.4.** Let $X$ be a complex manifold, and $\mathcal{O}_X$ its structural sheaf. A line bundle $\mathcal{L}$ on $X$ is determined (up to isomorphism) by its transition functions $f_{ij} \in \mathcal{O}_X^\times(U_{ij})$, where $X = \bigcup_i U_i$ is an open

[1] Recall that, on a paracompact space, usual coverings are cofinal among hypercoverings.

covering such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_j}$. Let $\lambda \in \mathbb{C}$, and choose determinations $g_{ij}$ of the multivalued functions $f_{ij}^{\lambda}$. Since $g_{ij}g_{jk}$ and $g_{ik}$ are both determinations of $f_{ik}^{\lambda}$, one has $g_{ij}g_{jk} = c_{ijk}g_{ik}$ for $c_{ijk} \in \mathbb{C}_X^{\times}(U_{ijk})$.

Let us denote by $\mathbb{C}_{\mathcal{L}^\lambda}$ the $\mathbb{C}$-algebroid stack associated with the cocycle $\{c_{ijk}\}$ as in the previous example. Then one has $\mathcal{L}^\lambda \in \mathfrak{Mod}_{\mathbb{C}}(\mathbb{C}_{\mathcal{L}^\lambda})$. Note that $\mathbb{C}_{\mathcal{L}^m} \approx_{\mathbb{C}} \mathbb{C}_X^+$ for $m \in \mathbb{Z}$.

## 3. Microdifferential operators

Refer to [13] for the theory of microdifferential operators.

Let $M$ be a complex manifold, and $\pi\colon P^*M \to M$ its projective cotangent bundle. Denote by $\mathcal{E}_M$ the sheaf of microdifferential operators on $P^*M$. Its twist by half-forms $\mathcal{E}_{M,*} = \pi^{-1}\Omega_M^{1/2} \otimes_{\pi^{-1}\mathcal{O}} \mathcal{E}_M \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\Omega_M^{-1/2}$ is endowed with a canonical anti-involution $*$.

Let $Y$ be a contact complex manifold of dimension $2n-1$. Then there are an open covering $Y = \bigcup_i U_i$ and contact embeddings $\psi_i\colon U_i \to P^*M$, for $M = \mathbb{C}^n$. Let $\mathcal{A}_i = \psi_i^{-1}\mathcal{E}_{M,*}$. Kashiwara [7] proved the existence of isomorphisms of $\mathbb{C}$-algebras $f_{ij}$ and invertible sections $a_{ijk}$ as in Proposition 2.1. Using the notion of algebroid stack, his result may be restated as

**Theorem 3.1.** *(cf [7]) On any contact complex manifold $Y$ there exist a canonical $\mathbb{C}$-algebroid stack $\mathfrak{E}_Y$ locally equivalent to $\mathcal{E}_{M,*}^+$.*

The stack of microdifferential modules on $Y$ is $\mathfrak{Mod}_{\mathbb{C}}(\mathfrak{E}_Y)$.

## 4. WKB operators

The relation between WKB operators and microdifferential operators is discussed in [1]. We follow here the presentation in [12].

Let $M$ be a complex manifold, and $\rho\colon J^1M \to T^*M$ the projection from the 1-jet bundle to the cotangent bundle. Let $(t;\tau)$ be the system of homogeneous symplectic coordinates on $T^*\mathbb{C}$, and recall that $J^1M$ is identified with the affine chart of $P^*(M \times \mathbb{C})$ given by $\tau \neq 0$. Denote by $\partial_t \in \mathcal{E}_{M\times\mathbb{C},*}$ the pull-back of the operator in $\mathcal{E}_{\mathbb{C}}$ with total symbol $\tau$, and consider the subring $\mathcal{E}_{M\times\mathbb{C},*,\hat{t}}$ of operators commuting with $\partial_t$. The ring of WKB operators (twisted by half-forms) is defined by

$$\mathcal{W}_{M,*} = \rho_*(\mathcal{E}_{M\times\mathbb{C},*,\hat{t}}|_{J^1M}).$$

It is endowed with a canonical anti-involution $*$, and its center is the subfield $k = \mathcal{W}_{\mathrm{pt}} \subset \mathbb{C}[\![\tau^{-1}, \tau]$ of WKB operators over a point.

Let $X$ be a symplectic complex manifold of dimension $2n$. Then there are an open covering $X = \bigcup_i U_i$ and symplectic embeddings $\Phi_i\colon U_i \to T^*M$, for $M = \mathbb{C}^n$. Let $\mathcal{A}_i = \Phi_i^{-1}\mathcal{W}_{M,*}$. Adapting Kashiwara's construction, Polesello-Schapira [12] proved that there exist isomorphisms of $k$-algebras $f_{ij}$ and invertible sections $a_{ijk}$ as in Proposition 2.1. Their result may thus be restated as

**Theorem 4.1.** *(cf [12]) On any symplectic complex manifold $X$ there exist a canonical $k$-algebroid stack $\mathfrak{W}_X$ locally equivalent to $\mathcal{W}^+_{M,*}$.*

The stack of WKB modules on $X$ is $\mathfrak{Mod}_k(\mathfrak{W}_X)$.

## 5. Simple WKB modules

For the notion of simple microdifferential modules refer to [9].

Let $M$ be a complex manifold, $\rho\colon P^*(X \times \mathbb{C}) \supset J^1M \to T^*M$ the projection, and $V \subset T^*M$ an involutive submanifold. The $\rho^{-1}V$-filtration on $\mathcal{E}_{M\times\mathbb{C},*}$ induces a $V$-filtration on $\mathcal{W}_{M,*}$. A simple WKB-module along $V$ is a coherent $\mathcal{W}_{M,*}$-module $\mathcal{M}$ locally endowed with a good $V$-filtration $F_k\mathcal{M}$ such that $F_0\mathcal{M}/F_{-1}\mathcal{M} \simeq \mathcal{O}_V$.

Let $X$ be a symplectic complex manifold of dimension $2n$, and $V \subset X$ an involutive submanifold. The notion of simple module along $V$ still makes sense in $\mathfrak{Mod}_k(\mathfrak{W}_X)$, and we denote by $\mathfrak{Simp}_V(\mathfrak{W}_X)$ its full substack of simple objects. Being locally non-empty and locally connected by isomorphisms, $\mathfrak{Simp}_V(\mathfrak{W}_X)$ is a $k$-algebroid stack on $V$.

For simplicity, assume there exist a symplectic complex manifold $Z$ and a map $q\colon V \to Z$ whose fibers are the bicharacteristic leaves of $V$. Here is our conjectural structure theorem for the $k$-algebroid stack of simple WKB modules.

**Conjecture 5.1.** *There is an equivalence of $k$-algebroid stacks*

$$\mathfrak{Simp}_V(\mathfrak{W}_X) \approx_k \mathbb{C}_{\Omega_V^{1/2}} \otimes_{\mathbb{C}} q^{-1}\mathfrak{W}_Z^{\mathrm{op}}.$$

If $V = X$, then $q = \mathrm{id}$. Denoting by $\omega$ the symplectic form, $\omega^n$ is a nowhere vanishing section of $\Omega_X$, so that $\Omega_X \simeq \mathcal{O}_X$ and $\mathbb{C}_{\Omega_X^{1/2}} \approx_{\mathbb{C}} \mathbb{C}_X^+$. The conjecture thus reduces to the equivalence

$$\mathfrak{Simp}_X(\mathfrak{W}_X) \approx_k \mathfrak{W}_X^{\mathrm{op}}.$$

By definition, simple modules along $X$ are locally free $\mathfrak{W}_X$-modules of rank one, so that the above equivalence is a corollary of Yoneda embedding (1.1).

If $V = \Lambda$ is Lagrangian, then $Z = \mathrm{pt}$. Since $q^{-1}\mathfrak{W}^{\mathrm{op}}_{\mathrm{p}t} \approx_k k^+_\Lambda$, the conjecture reduces to the equivalence

$$\mathfrak{Simp}_\Lambda(\mathfrak{W}_X) \approx_k \mathbb{C}_{\Omega_V^{1/2}} \otimes_{\mathbb{C}} k^+_\Lambda.$$

This is a reformulation of a result of D'Agnolo-Schapira [5], obtained by adapting a similar theorem of Kashiwara [7] for microdifferential operators.

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A. D'AGNOLO AND P. POLESELLO

UNIVERSITÀ DI PADOVA, DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA, VIA G. BELZONI, 7, 35131 PADOVA, ITALY
*E-mail address*: dagnolo@math.unipd.it
*URL*: www.math.unipd.it/~dagnolo/

UNIVERSITÀ DI PADOVA, DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA, VIA G. BELZONI, 7, 35131 PADOVA, ITALY
*E-mail address*: pietro@math.unipd.it